The American Mathematical Monthly

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NOTICE TO AUTHORS

The *Monthly* publishes articles, notes, and other features about mathematics and the profession. The readership of the *Monthly* is intended to include everybody who is mathematically inclined, including of course professional mathematicians and students of mathematics at all collegiate levels. While no single article or feature is likely to appeal to everyone, material should interest and be accessible to a large number of readers. This is the most important criterion for acceptance.

Articles may be expositions of old results or presentations of new ones. They may concern all of mathematics or one small area, a broad development or a single application, historical reminiscences or one important event. While some articles may contain the author's new research, the novelty of material and generality of the results is far less important than the clarity of exposition and general interest. Discussing one illuminating case of a well known result is far better than providing all the details of an obscure but new proposition. Articles in the *Monthly* are supposed to inform and to entertain; they are meant to be read rather than archived.

Notes are short and possibly informal articles. A note may concern a clever new proof of an old theorem, a novel way to present tired material, or a lively discussion of a philosophical (but still mathematical) issue. Also any topic is suitable, so long as it is related to mathematics. Because a note is short, the first few sentences are the most important part: They should explain the purpose and invite the reader in. Photographs or diagrams often will attract the reader's attention.

All articles and notes should be sent to the editor:

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Please send 3 copies, typewritten on only one side of the paper. Illustrations should be carefully drawn on separate sheets of paper in black ink; the original should be without lettering and two copies should have appropriate captions and lettering indicated.

Proposed problems or solutions should be sent to:

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COMMENTS

New editors always fuss with a journal; they move the contents, change the headings, narrow the page. Most readers hardly notice such changes. Most readers want to read rather than to dissect a journal.

Observant readers *will* notice, however, some important changes in this issue of the *Monthly*. The Notes are now incorporated into the main section; articles on the Teaching of Mathematics are not separated from articles on mathematics; the problems are no longer classified into "Elementary" and "Advanced." The *mathematical* principle motivating these changes is the belief that mathematics ought to be viewed as a unified field, both horizontally and vertically. Articles on the mathematics of computers belong next to articles on Riemann surfaces; comments on teaching Calculus ought to be read with as much enthusiasm as comments on representations of Lie groups; elementary problems are often as inviting (and as difficult) as advanced. The *journalistic* principle motivating such changes is much simpler—variety makes more lively reading.

What kinds of articles and notes will we publish? There are few rules. Articles may be expositions of old results or presentations of new ones. They may concern all of mathematics or one small area, a broad development or a single application, historical reminiscences or one important event. While some articles may refer to the author's research, the novelty of material and generality of the results is far less important than the clarity of exposition and general interest. Discussing one illuminating case of a well known result is far more interesting than providing all the details of an obscure but new proposition. Articles in the *Monthly* are supposed to inform and to entertain.

Notes are short and possibly informal articles. A note may concern a clever new proof of an old theorem, a novel way to present tired material, or a lively discussion of a philosophical (but still mathematical) issue. Almost any topic is suitable, so long as it is related to mathematics.

How does one write such masterpieces? If I knew, I'd be out there writing rather than in here editing. As an ideal goal, we want *all* articles to be inviting to *most* readers. That doesn't mean the level of the material is necessarily elementary; even advanced mathematics can be inviting. Making articles inviting usually means aiming them at the right audience, which ought to be reasonable mathematicians who are novices in the particular subject. If authors write about analysis, they should think what they would say (on the way to lunch) to the algebraist down the hall—the one who knows complicated things about noncommutative syzygies but only remembers Measure Theory from last week's colloquium. When one speaks, it's always important to know who's listening. Writing is no different.

The *Monthly* has a long tradition of publishing high-quality exposition of mathematics; we will not change that tradition. During the next two years, however, we *will* add flexibility to the *Monthly*, gradually expanding both the scope and the style of the material we publish. We hope readers will help us to make the adjustment. Please write with irate criticism, with profound suggestions, or with friendly observations.

John Ewing

The Car and the Goats

Leonard Gillman

1. THE PROBLEM. *A TV host shows you three numbered doors, one hiding a car (all three equally likely) and the other two hiding goats. You get to pick a door, winning whatever is behind it. You choose door #1, say. The host, who knows where the car is, then opens one of the other two doors to reveal a goat, and invites you to switch your choice if you so wish. Assume he opens door #3. Should you switch to #2?*

I'll call this Game I. It appeared in the *Ask Marilyn* column in *Parade* (a Sunday supplement) [**4**(a)]. Marilyn asserted that you should switch, arguing that the probability of winning, originally 1/3, had now gone up to 2/3. ("Marilyn" is standard terminology.) This led to an uproar featuring "thousands" of letters, nine-tenths of them insisting that with door # 3 now eliminated, #1 and #2 were equally likely; even the responses from college faculty voted her down two to one [**4**(b, c), **3**]. There is no denying that the problem is tricky (even though, technically speaking, it involves only undergraduate mathematics). The purpose of this article is to unravel it all.

2. GAME II. Marilyn's solution goes like this. The chance is 1/3 that the car is actually at #1, and in that case you lose when you switch. The chance is 2/3 that the car is either at #2 (in which case the host perforce opens #3) or at #3 (in which case he perforce opens #2)—and in these cases, the host's revelation of a goat shows you how to switch and win.

This is an elegant proof, but it does not address the problem posed, in which the host has shown you a goat at #3. Instead it is still considering the possibility that the car is at #3—whence the host cannot have already opened that door (much less to reveal a goat). In this game—Game II—you have to announce *before a door has been opened* whether you plan to switch.

3. GAME I. Game I is more complicated: What is the probability P that you win if you switch, *given that the host has opened door* #3? This is a *conditional* probability, which takes account of this extra condition. When the car is actually at #2, the host will open #3. But when it is at #1, he may open either #2 or #3. The answer to the question just asked *depends on his selection strategy when he has this choice* —on the *probability* q that he will then open door #3. (Marilyn did not address this question.)

In any case, it still pays you to switch (except in one extreme case, where it's fifty-fifty). The host has opened #3. It was *certain* he would do that if the car is at #2, but less than certain (except in the extreme case) if it is at #1. This gives the edge to #2. This argument is well known in the game of bridge as the "principle of restricted choice." A player holding both Queen and Jack of a suit will play them at random so as not to betray her holding. Hence when West plays the Queen, the Jack is now more likely to be with East, since if West had it she could have played

it. (This ignores other information that may have come to light in the course of the play.)

We are interested in the following events:

$$C_i\text{: the car is at door } i;\ H_j\text{: the host opens door } j.$$

In this notation, the probability that you will win if you switch is the conditional probability

$$P = \mathbf{P}(C_2|H_3);$$

as noted, its value depends on the conditional probability

$$q = \mathbf{P}(H_3|C_1).$$

It turns out that *P can be any number between* $1/2$ *and* 1. (So the critics are still quite wrong.)

4. EXAMPLES. In the extreme case $q = 1$, the host's opening of #3 gives you no information, and $P = 1/2$. At the other extreme, $q = 0$, the host opens #3 only when the car is at #2, and $P = 1$.

When $q = 1/2$ the host is not differentiating between the two available doors, and you are essentially playing Game II. In fact, when the car is at #1 he opens #3 one time in two, but if it is at #2 he opens it two times in two. So when he actually does open #3, the car is at #2 two times out of three: $P = 2/3$. Similarly, if $q = m/n$ then $P = n/(n + m)$; thus,

$$P = \frac{1}{1 + q} \tag{1}$$

for any rational q. By Bayes's rule (Section 6), (1) holds for all real q, $0 \leqslant q \leqslant 1$. Note that these inequalities imply $1 \geqslant P \geqslant 1/2$.

To illustrate that the solution to Game I is consistent with that of Game II, consider the extreme case $q = 0$. Here the host would actually open #3, giving you the sure shot, only $1/3$ of the time. The remaining $2/3$ of the time, when he opens #2, your win probability is only $1/2$. Your net probability is $1/3$ in each case, for a total of $2/3$.

5. NOTATION AND TERMINOLOGY. Let

$$\mathbf{P}(C_i) = \textit{a priori} \text{ probability that the car is at door } i,$$

a priori referring to the state of our knowledge before any doors have been opened. It is given that $\mathbf{P}(C_i) = 1/3$ $(i = 1, 2, 3)$.

The host's choice of which door to open is made in response to the actual location of the car. We say, picturesquely, that the events C_i (the car is at i) are the *causes* that produce the *effects* H_j (the host opens door j). The probabilities of the *effects given the causes* we call the *productive* probabilities; these are the conditional probabilities $\mathbf{P}(H_j|C_i)$. What we have called q is the productivity probability $\mathbf{P}(H_3|C_1)$. We also let

$$\mathbf{P}(H_j) = \textit{a priori} \text{ probability that the host opens door } j,$$

a priori meaning without knowledge of the location of the car. Finally, we wish to know the probabilities of the *causes given the effects*—the *a posteriori* probabilities. These are the conditional probabilities $\mathbf{P}(C_i|H_j)$. What we have called P is the *a posteriori* probability $\mathbf{P}(C_2|H_3)$.

6. BAYES'S FORMULA. Bayes's formula is the fundamental equation relating the *a posteriori* to the productive probabilities:

$$\mathbf{P}(H_3)\mathbf{P}(C_i|H_3) = \mathbf{P}(C_i)\mathbf{P}(H_3|C_i). \tag{2}$$

(Technically, it is this equation solved for $\mathbf{P}(C_i|H_3)$.) It is an immediate consequence of the law of compound probability, according to which each side is equal to $\mathbf{P}(H_3 \cap C_i)$. In its general setting, the probabilities $\mathbf{P}(C_i)$ may be unequal for some choices of i; but $\mathbf{P}(H_3)$ is still independent of i. (We could write H instead of H_3). Therefore

$$\mathbf{P}(C_i|H_3) \sim \mathbf{P}(C_i)\mathbf{P}(H_3|C_i). \tag{3}$$

When all the $\mathbf{P}(C_i)$ *are* equal,

$$\mathbf{P}(C_i|H_3) \sim \mathbf{P}(H_3|C_i) \qquad (\mathbf{P}(C_i) = \text{const.}) \tag{4}$$

When the a priori probabilities are all equal, the a posteriori probabilities are proportional to the productive probabilities. The proposition seems intuitively clear (even when the hypothesis is not explicitly acknowledged). It underlies all the examples in Section 4. (To hammer this home, assign unequal *a priori* probabilities and rework an example using (3) instead of (4).)

In the example with $q = 1/2$, $\mathbf{P}(H_3|C_2) = 1$ and $\mathbf{P}(H_3|C_1) = q = 1/2$. So the host is twice as likely to open #3 when the car is at #2 as when it is at #1. By (4) when the host *does* open #3, the car is twice as likely to be at #2 as at #1; therefore $P = \mathbf{P}(C_2|H_3) = 2/3$. In general, for any q,

$$\mathbf{P}(C_2|H_3) : \mathbf{P}(C_1|H_3) = 1 : q,$$

and we get $P = 1/(1 + q)$.

We gave an example in Section 4 to illustrate that the solution of Game I is consistent with that of Game II. Here is a general proof:

$$\begin{aligned}
&\mathbf{P}(\text{You win Game II if you switch})\\
&\quad = \mathbf{P}(H_3 \cap C_2) + \mathbf{P}(H_2 \cap C_3)\\
&\quad = \mathbf{P}(C_2)\mathbf{P}(H_3|C_2) + \mathbf{P}(C_3)\mathbf{P}(H_2|C_3)\\
&\quad = \frac{1}{3} \times 1 + \frac{1}{3} \times 1 = \frac{2}{3}.
\end{aligned}$$

7. THE PARADOX OF THE SECOND ACE. My interest in problems of conditional probability was sparked, many years ago, by a passage in *Mathematical Recreations and Essays*, by W. W. Rouse Ball (now Ball and Coxeter [**1**, p. 44]). When a bridge hand is dealt from a deck of cards (13 cards from 52), the probability that it contains at least two aces turns out to be .26. *Question*: What is the probability that it contains at least two aces given that it contains (a) an ace, (b) the ace of hearts? We expect the answers to be the same and of course greater than .26. We are half right: they turn out to be (a) .37, (b) .56. I was able to wade through the binomial coefficients, but I still wondered *why* (b) should be greater than (a).

Here is a way to see why without computation. In terms of the complementary events, we wish to show that the probability of exactly one ace, given that the hand contains an ace, is greater than the probability of exactly one ace given that the hand contains the ace of hearts. This means we want

$$\frac{\mathbf{N}(!A)}{\mathbf{N}(A)} > \frac{\mathbf{N}(!A_H)}{\mathbf{N}(A_H)},$$

where $\mathbf{N}(!A)$ is the number of hands containing exactly one ace, $\mathbf{N}(!A_H)$ is the number of such hands whose unique ace is the ace of hearts, and $\mathbf{N}(A)$ and $\mathbf{N}(A_H)$ are the numbers of hands containing an ace or the ace of hearts, respectively. A slightly more convenient form is

$$\frac{\mathbf{N}(!A)}{\mathbf{N}(!A_H)} > \frac{\mathbf{N}(A)}{\mathbf{N}(A_H)}. \tag{5}$$

Since a unique ace in a hand must be one of the four specific aces, the numerator of the first fraction is exactly four times the denominator, and the fraction is equal to four. But in the second fraction the numerator is *less* than four times the denominator, because of overlaps—e.g., a hand containing the aces of both hearts and spades should be counted only once. This establishes (5).

8. EXAMPLES. The situation may be clarified further by considering a deck of four cards, two aces and two jacks, from which you are dealt a hand of two cards. There are six possible hands, one of them consisting of the two aces, so the probability you have both aces is $1/6$. If it is given that the hand contains an ace we have eliminated the two jacks, and the probability for both aces goes up to $1/5$. But if it is given that you have the ace of hearts, then your other card is either the ace of spades or one of the jacks, and the probability that you are holding both aces is now $1/3$.

Carrying this to the extreme, consider a two-card hand from a deck of *three* cards, two aces and a jack. There are three possible hands, and the probability that you have the two aces is $1/3$. If you state that the hand contains an ace, I smirk. But if we are given that the hand contains the ace of hearts, the probability for both aces goes up to $1/2$. At this point (if not long since) your friend enters the picture with a "proof" that the probability of both aces is $1/2$, with or without any condition: "You have an ace. Either it is the ace of hearts or the ace of spades. If it is the ace of hearts, then as we have just proved, the probability of both aces is $1/2$. If it is the ace of spades, then, similarly, the probability for both aces is $1/2$. So in either case it is $1/2$. So it is $1/2$." It is easier to detect the flaw in this reasoning than to get your friend to understand it. A suggested response (guaranteed not to help) is printed upside down at the end of the article.

9. OTHER PROBLEMS. While preparing this article I looked through a number of books for related material but found very little other than the classic gold and silver coins distributed in three two-drawer boxes (Bertrand's box paradox), and the family with two children of whom one is a girl. I felt that the car-and-goats problem must surely have appeared somewhere. Eventually I was steered to a 1959 column of Martin Gardner [2], who presents the problem in terms of three prisoners, one of whom is to be paroled. It is noteworthy that he states explicitly, as part of the hypothesis, that the warden is to flip a coin when he has a choice between two prisoners to name (corresponding to the host's picking Door 3 with probability $q = 1/2$). Gardner mentions in his column that the problem "is now making the rounds"; but he told me recently he has no recollection of how he came to hear about it.

I deliberately misquoted Ball's problem when I asked for the probability "given that" your hand contains an ace (or the ace of hearts). Ball says you *assert* that the hand contains an ace. In such a case I would want to know how you decide what statement to assert. My present rule is that you are to state whether your hand

contains an ace. Instead, suppose the rule in the four-card problem is that you are to pick a random card from your hand and tell whether it is an ace or a jack. Now when you pick an ace, the probability for both aces is higher than before, since if you had a jack you could have picked it. In fact the productive probabilities for picking an ace are in the proportions $2:1:1:1:1$, and the *a posteriori* probability that you have both aces, originally $1/5$, is now $1/3$.

ACKNOWLEDGMENTS. I wish to thank my colleague John Dollard for his insightful suggestions. I also got helpful comments from colleagues Stephen McAdam and Michael Starbird.

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1. W. W. Rouse Ball and H. S. M. Coxeter, *Mathematical Recreations and Essays*, 13th edition, Dover, New York, 1987 (Ball, first edition, 1892).
2. Martin Gardner, Mathematical Games, *Scientific American*, 201 (1959), October 180–182, November 188.
3. Leonard Gillman, The car and goats fiasco, *Focus* (the MAA newsletter), 11 (1991), June, 8.
4. Marilyn vos Savant, "Ask Marilyn," *Parade*, (a), September, 9 1990; (b), December 2 1990; (c), February 17 1991.

Suggested response:

Annoy your friend by asking, "What is your definition of 'it'?"

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> Mathematics not only demands straight thinking, it grants the student the satisfaction of knowing when he is thinking straight.
>
> —*D. Jackson*

A Continuous, Nowhere Differentiable Function

Mark Lynch

The examples of continuous, nowhere differentiable functions given in most analysis and topology texts involve the uniform limit of a series of functions in the former and the Baire category theorem in the latter. Below we give a simple example of such a function which uses elementary topological concepts of the real plane normally covered in the first semester of undergraduate analysis and topology courses. In addition, the example will show that these functions are dense in the set of continuous real-valued functions on a compact interval without appeal to the Baire category theorem. Two basic facts are needed to understand it:

(1) The nested intersection of non-empty compact sets is non-empty and compact;
(2) a function f: $[a, b] \to R$ is continuous if and only if its graph is compact.

THE EXAMPLE. Let Π: $R \times R \to R$ be the first coordinate projection and for $A \subset R \times R$ and $x \in R$, let $A[x] = \{y|(x, y) \in A\}$. Define a nested sequence of bands $C_n \supset C_{n+1}$ in $R \times R$ with the following properties:

(a) $\Pi(C_n) = [0, 1]$, for all $n \in N$;
(b) $\operatorname{diam}(C_n[x]) < 1/n$, for each $x \in [0, 1]$ and $n \in N$ and;
(c) for each $x \in [0, 1]$, there exists $y \in [0, 1]$ with $0 < |x - y| < 1/n$ such that if $p \in C_n[x]$ *and* $q \in C_n[y]$, *then* $|(p - q)/(x - y)| > n$.

The C_n will be defined as the closures of band neighborhoods of polygonal arcs defined on $[0, 1]$ (see diagram 2 for two stages of the construction). However, before we construct them, it may be instructive to first verify property (c) above for closed band neighborhoods of straight line segments. Let n be a given positive integer and $f(x) = mx + b$ with $m > n$.

Claim. For any $\delta > 0$, there exists an ε-neighborhood $N_\varepsilon(f)$ of the graph of f such that for any $x \in [0, 1]$, there exists $y \in [0, 1]$ with $|x - y| = \delta$ such that if $p \in \overline{N_\varepsilon(f)}[x]$ and $q \in \overline{N_\varepsilon(f)}[y]$, then $|(p - q)/(x - y)| > n$.

Proof: See diagram 1 below.

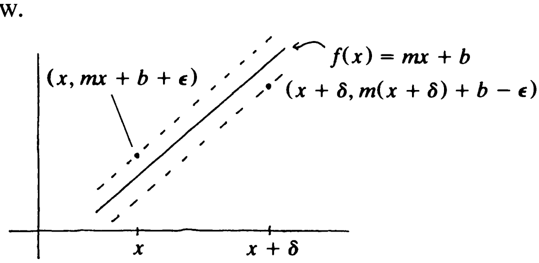

DIAGRAM 1

It is easy to see that we can choose ε small enough so that

$$|\{[m(x+\delta)+b-\varepsilon]-[mx+b+\varepsilon]\}/\{(x+\delta)-x\}|=|m-2\varepsilon/\delta|>n$$

since $m > n$. Hence, if $p \in \overline{N_\varepsilon(f)}[x]$ and $q \in \overline{N_\varepsilon(f)}[x+\delta]$, then

$$|(p-q)/(x-(x+\delta))|>n.$$

Take $y = x + \delta$.

To construct the C_n satisfying $(a)-(c)$, assume C_1 through C_{n-1} have been defined. Let P be a polygonal arc contained in the interior of C_{n-1} each of whose segments, say $P_1, P_2, \ldots, P_k$, has slope in absolute value exceeding n. For each $i = 1, \ldots, k$, let $0 < \delta_i < \min\{(\text{length of } \Pi(P_i))/2, 1/n\}$. Apply the above claim to the δ_i and the segments P_i defined on $\Pi(P_i)$ to obtain the desired ε_i-neighborhood of P_i (since $\delta_i <$ (length of $\Pi(P_i))/2$, y can always be chosen in $\Pi(P_i)$ for each $x \in \Pi(P_i)$). Let $\varepsilon = \min\{\varepsilon_i | i = 1, \ldots, k\}$. Then, $\overline{N_\varepsilon(P)}$ is a closed neighborhood of P satisfying condition (c). Clearly, ε can be chosen smaller, if necessary, so that $\overline{N_\varepsilon(P)} \subset C_{n-1}$ and satisfies (b). Take $C_n = \overline{N_\varepsilon(P)}$.

Now that the C_n have been defined, we define our continuous, nowhere differentiable function as follows. Let $C = \bigcap C_n$. By (b), $\operatorname{diam}(C[x]) = 0$ for each $x \in [0,1]$ so that C is the graph of a function $f\colon [0,1] \to R$ (this is the "vertical line test"). Since C is compact (by (1)), f is continuous (by (2)).

Claim. f is nowhere differentiable.

Let $x \in [0,1]$ and $\delta > 0$. Choose n so that $1/n < \delta$. By (c), there exists $y \in [0,1]$ with $0 < |x-y| < 1/n$ such that if $p \in C_n[x]$ and $q \in C_n[y]$, then $|(p-q)/(x-y)| > n$. Since $f(x) \in C_n[x]$ and $f(y) \in C_n[y]$, we have $|(f(x)-f(y))/(x-y)| > n$. Hence, f is not differentiable at x and this proves the claim.

Remark. This construction can be used to show that these functions are dense in the set of all continuous real-valued functions defined on $[0,1]$. Let $g\colon [0,1] \to R$ be continuous and let $\delta > 0$. Let P be a polygonal arc within $\delta/2$ of g. Construct $C_n \subset N_{\delta/2}(P)$ satisfying (a) through (c) above (note that $N_{\delta/2}(P)$ is a band neighborhood of P). Then, $\bigcap C_n$ defines a continuous, nowhere differentiable function within $\delta/2$ of P and hence, within δ of g.

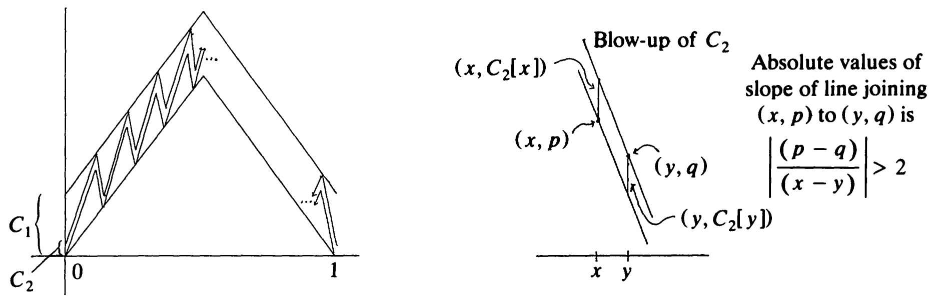

DIAGRAM 2

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Birthday Problem with Unlike Probabilities

Kumar Joag-Dev and Frank Proschan

1. INTRODUCTION. The Birthday Problem as described in Feller ([**1**], p. 33) has become an important example in a course on elementary probability. The problem is to find the probability that among n students in a class, no two or more students share the same birthday. This computation is to be done under the assumption that individuals' birthdays are independent and that for every individual, all 365 days of the year are equally likely as possible birthdays. The more inquiring student might ask "What happens when different dates of the year have different probabilities of being a birthday?" We know that the probability that no two or more students have the same birthday (called *coincidence*) is smaller in this case than in the standard case of all days being equally likely to be a birthday. Actually, the teacher may take this opportunity to teach a more detailed result: as the probabilities differ more and more from $1/365$, the required probability decreases. More precisely, the desired probability is a Schur-concave function of the 365 probabilities.

To explain this notion we choose a particular set of definitions of majorization and Schur concave and Schur convex functions. A vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$ is said to majorize a vector $\mathbf{y} = (y_1, y_2, \dots, y_n)$, or $\mathbf{x} \overset{m}{\geq} \mathbf{y}$, if $\mathbf{y}$ can be derived from $\mathbf{x}$ by a finite sequence of *averagings*. An *averaging* of $\mathbf{x}$ yields $(x_1, x_2, \dots, y_i, \dots, y_j, \dots, x_n)$, where $y_i + y_j = x_i + x_j$, while $|y_i - y_j| \leq |x_i - x_j|$. A function $g(\mathbf{x})$ is *Schur-concave* if $\mathbf{x} \overset{m}{\geq} \mathbf{y}$ implies $g(\mathbf{x}) \leq g(\mathbf{y})$. *Schur-convex* function is defined in a similar fashion. See [**2**].

Note that the convenience of the above definition is that we have to deal with only two variables at a time. Other definitions appear later.

The use and application of Schur-concavity has increased steadily since the publication of the excellent comprehensive book by Marshall and Olkin [**2**]. Unfortunately, it is still considered a research topic and too advanced to be used in elementary courses. Actually, the basic notions of majorization and Schur functions are quite elementary when properly introduced. In this note, we explain these notions and their application to the Birthday Problem with unlike probabilities; we believe this explanation can be used successfully in teaching elementary probability.

Suppose we start with the simpler model of just two periods: the first half of the year (January 1 through June 30) and the second half (July 1 through December 31). The probability of being born in the first half is p and in the second half is $1 - p$. With just two students in the class, the probability that they are born in different half-years is simply $2p(1 - p)$. Note that this function of p is symmetric about $1/2$. Furthermore, as p and $1 - p$ move further apart, this probability decreases.

2. A MORE GENERAL RESULT. Next consider the Birthday Problem, where there are 365 days. Again we are interested in the event that no two or more students have the same birthday, in a class of n students. We assume now that the probability of being born on day i is p_i, $i = 1, 2, \dots, 365$, where $\Sigma_{i=1}^{365} p_i = 1$. For simplicity, consider the simple case where $n = 3$. The probability that all three birthdays are different is

$$f(\mathbf{p}) = \sum_{i \neq j \neq k} p_i p_j p_k, \tag{1}$$

where $\mathbf{p} = (p_1, p_2, \dots, p_{365})$ and summation is over all possible distinct choices of ordered triplets out of 365 days.

We wish to study the effect on $f(\mathbf{p})$ of increasing the spread or variability among the $p_1, p_2, \dots, p_{365}$. Specifically, suppose we start with $0 < p_1 < p_2 < 1$ and then spread the p_1 and p_2 further apart while keeping this sum unchanged; the 363 remaining p's are kept fixed. How does $f(\mathbf{p})$ change?

Note that the terms in (1) that contain neither p_1 nor p_2 as factors are unaffected. The terms that contain p_1 but *not* p_2 are of the form

$$p_1 \sum_{i \neq j \neq 2} p_i p_j. \tag{2}$$

Similarly, the terms that contain p_2 but not p_1 are of the form

$$p_2 \sum_{i \neq j \neq 1} p_i p_j. \tag{3}$$

Adding (2) and (3) we get

$$(p_1 + p_2) \sum_{i \neq j \neq 1 \text{ or } 2} p_i p_j. \tag{4}$$

Since we are keeping $p_1 + p_2$ constant, (4) is unchanged. Thus the only change occurs in the remaining terms $p_1 p_2 \Sigma_{i \neq 1,2} p_i$. But we have seen above that $p_1 p_2$ *decreases* as p_1 and p_2 move apart while their sum $p_1 + p_2$ remains fixed. Also $\Sigma_{i \neq j \neq 1 \text{ or } 2} p_i p_j$ is unaffected by changes in p_1 or p_2. Hence, $f(\mathbf{p})$ must decrease as p_1, p_2 move apart while $p_1 + p_2$ remains fixed.

This proves that for the case of $n = 3$, the probability $f(\mathbf{p})$ of no coincidence in birthdays decreases as the spread between a pair of p values increases. The proof for general n is similar.

Note that using different pairs of p values, we can repeatedly spread the distance between the two elements of a pair keeping their sum fixed and achieve lower and lower values of $f(\mathbf{p})$. It follows that among the set of $(p_1, p_2, \dots, p_{365})$, the highest probability of no coincidence is achieved by $(1/365, 1/365, \dots, 1/365)$ (having 0-spread), while $(1, 0, 0, \dots, 0)$ has minimum $f(\mathbf{p})$, (actually 0), and has the maximum spread.

3. OTHER DEFINITIONS. Majorization and Schur concavity could be viewed as a multivariate generalization of the following concepts on the real line. A number x is said to majorize y if $|x| \geq |y|$. Clearly, x and $-x$ become equivalent. A nonnegative function h which has the property that x majorizes y implies $h(x) \leq h(y)$, is called a *symmetric unimodal* function. Note that the majorization on the real line can be expressed as: x majorizes y if y is in the convex set

(interval in this case) spanned by the equivalent points x and $-x$. The generalization to the n-dimensional case is achieved by making x and its *permutants*, the points obtained by permuting its n coordinates, equivalent. If a vector $\mathbf{y}$ is in the convex hull of x and its permutants, then $\mathbf{x} \overset{m}{\geq} \mathbf{y}$. Consider the set of vectors such that the sum of their co-ordinates is a fixed number s, say. Then it is clear that every vector in this set majorizes the vector $(s/n, s/n, \ldots, s/n)$.

An alternate definition of majorization frequently used is given as follows:

Let $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ denote the decreasing rearrangement of $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Then $\mathbf{x}$ majorizes $\mathbf{y}$ if

$$\sum_1^k x_{[i]} \geq \sum_1^k y_{[i]} \quad \text{for } k = 1, 2, \ldots, n-1,$$

and

$$\sum_1^n x_{[i]} = \sum_1^n y_{[i]}.$$

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Between twenty-five and thirty-five you're too young to do anything well; after thirty-five you're too old.

—*Fritz Kreisler*

Two Relatives of Picard's Theorem on Entire Functions

Robert M. Gethner

1. INTRODUCTION. According to Picard's 'great theorem', a transcendental (i.e., non-polynomial) entire function takes on every complex value, with one possible exception, infinitely many times in the complex plane. I will present here two theorems, related to Picard's theorem, whose proofs use only the techniques of a first course in complex analysis. They are weak versions of known results, weak enough (I hope) to be widely accessible, but still strong enough (I hope) to be interesting.

2. POWER SERIES WITH GAPS. From properties of the coefficients of a power series it is possible to deduce properties of the function represented by the series. For example, a power series is said to have 'gaps' if most of its coefficients vanish, and the existence of gaps implies information about the values taken on by the associated function. The following theorem illustrates this.

Theorem 1. *Let f be a transcendental entire function whose Maclaurin series $\sum_{k=1}^{\infty} a_k z^{n_k}$ satisfies the gap condition,*

$$n_k - n_{k-1} \geqslant k^2 \qquad \text{for all} \quad k \geqslant 2.$$

Then f takes on every complex value (with no exception) infinitely many times.

This is a special case of a theorem of Biernacki (which he proved by methods other than those used here—see [**9**, Theorem (35, 2), p. 164]).

The proof of Theorem 1 depends on a lemma which uses the rudiments of the theory of the 'maximum term' and 'central index' of a power series (see [**12**, part four, chapter 1]), presented here in a slightly non-standard way. Let $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ be a transcendental entire function. Then $|a_k| r^{n_k} \to 0$ as $k \to \infty$ for each $r \geqslant 0$. So $\operatorname{Max}_k\{|a_k| r^{n_k}\}$ exists; call it $m(r)$. In addition, for each positive integer k, let

$$I_k = \{r : |a_k| r^{n_k} = m(r)\}.$$

Then $\bigcup_{k=1}^{\infty} I_k = \{r : r \geqslant 0\}$. Also, each I_k is empty, a point, or a bounded, closed interval: when $k > 1$ this is because

$$I_k = \bigcap_{j : j<k} \{r : |a_k| r^{n_k} \geqslant |a_j| r^{n_j}\} \cap \bigcap_{j : j>k} \{r : |a_k| r^{n_k} \geqslant |a_j| r^{n_j}\},$$

and the intersection on the right is either $\{0\}$ or a closed, bounded interval, while the intersection on the left is either empty or a closed ray; a similar argument works when $k = 1$. Furthermore, the interiors of the sets I_k are pairwise disjoint (if there were an r in the interiors of both I_j and I_k, with $j \neq k$, then we would have simultaneously $|a_k| r^{n_k} > |a_j| r^{n_j}$ and $|a_j| r^{n_j} > |a_k| r^{n_k}$). Finally, if each of I_j

and I_k is non-empty and $j > k$, then I_j cannot lie to the left of I_k: for if $r \in I_j$ and $s \in I_k$, then $|a_j|r^{n_j} \geqslant |a_k|r^{n_k}$ and $|a_k|s^{n_k} \geqslant |a_j|s^{n_j}$, so that $r \geqslant |a_k/a_j|^{1/(n_j-n_k)} \geqslant s$.

Here is the lemma. It says, in effect, that, if I_k is long enough, then there is an r (near the middle of I_k) such that the maximum term $|a_k|r^{n_k}$ is large compared to the sum of the other terms of the series for f.

Lemma 1. *Let f satisfy the hypotheses of Theorem* 1. *Let k be an integer greater than one such that $I_k = [c, d]$, where*

$$\log\frac{d}{c} > \frac{2\log(3k)}{k^2}. \tag{1}$$

Then there exists an r in I_k such that $|f(z) - a_k z^{n_k}| < |a_k z^{n_k}|$ for each z such that $|z| = r$.

Here is the proof of Theorem 1; I will prove the lemma in a moment. We must show that, for each complex w, the equation $f(z) = w$ has infinitely many solutions z. We may assume without loss of generality that $w = 0$, since $f - w$ satisfies the hypotheses of the theorem.

For each k such that I_k is non-empty, set $I_k = [c_k, d_k]$ (where $c_k = d_k$ if I_k reduces to a point). Now if (1) holds for a given k, then, by Lemma 1 and Rouché's theorem, f has at least n_k zeros in $D(0; d_k)$. (Here and throughout the paper, $D(w; r)$ represents the open disk having center w and radius r.) It therefore suffices to prove that there are infinitely many values of k satisfying (1).

Suppose there are only finitely many such k. Then, in essence, the intervals I_k are too short to fill up $\{r: r \geqslant 0\}$. Here are the details. There exists K such that $c_K > 0$ and

$$\sum_{\substack{k=K \\ I_k \text{ non-empty}}}^{\infty} \log\frac{d_k}{c_k} \leqslant \sum_{k=K}^{\infty} \frac{2\log(3k)}{k^2} < \infty.$$

But this is impossible, because

$$\sum_{\substack{k=K \\ I_k \text{ non-empty}}}^{\infty} \log\frac{d_k}{c_k} = \sum_{\substack{k=K \\ I_k \text{ non-empty}}}^{\infty} \int_{c_k}^{d_k} \frac{dt}{t} = \int_{c_K}^{\infty} \frac{dt}{t} = \infty.$$

So there are infinitely many k satisfying (1), and Theorem 1 is established.

Proof of Lemma 1: Choose r in (c, d). For all z such that $|z| = r$,

$$|f(z) - a_k z^{n_k}| \leqslant \sum_{j=1}^{k-1} |a_j|r^{n_j} + \sum_{j=k+1}^{\infty} |a_j|r^{n_j}. \tag{2}$$

We need upper bounds on the two sums; first let's consider the second one. Pick $j > k$. The definition of I_k shows that $|a_j|d^{n_j} \leqslant |a_k|d^{n_k}$ for all j. Also, the gap condition gives

$$n_j - n_k = \sum_{t=k+1}^{j} (n_t - n_{t-1}) \geqslant \sum_{t=k+1}^{j} t^2 > \sum_{t=k+1}^{j} k^2 = k^2(j-k).$$

Therefore, since $r < d$,

$$\frac{|a_j|r^{n_j}}{|a_k|r^{n_k}} = \frac{|a_j|}{|a_k|} r^{n_j-n_k} \leqslant \left(\frac{r}{d}\right)^{n_j-n_k} < \left\{\left(\frac{r}{d}\right)^{k^2}\right\}^{j-k}$$

Thus, by the formula for the sum of a geometric series,

$$\frac{\sum_{j=k+1}^{\infty}|a_j|r^{n_j}}{|a_k|r^{n_k}} < \sum_{j=k+1}^{\infty}\left\{\left(\frac{r}{d}\right)^{k^2}\right\}^{j-k} = \frac{(r/d)^{k^2}}{1-(r/d)^{k^2}}.$$

Next we need an upper bound on the first sum in (2). For each j such that $1 \leqslant j \leqslant k-1$, we have $|a_j|c^{n_j} \leqslant |a_k|c^{n_k}$ and

$$n_k - n_j \geqslant n_k - n_{k-1} \geqslant k^2.$$

So, since $r > c$,

$$\frac{\sum_{j=1}^{k-1}|a_j|r^{n_j}}{|a_k|r^{n_k}} = \sum_{j=1}^{k-1}\frac{|a_j/a_k|}{r^{n_k-n_j}} < \sum_{j=1}^{k-1}\left(\frac{c}{r}\right)^{k^2} < k\left(\frac{c}{r}\right)^{k^2}.$$

It therefore follows from (2) that, if $|z| = \sqrt{cd}$, then

$$\frac{|f(z)-a_kz^{n_k}|}{|a_kz^{n_k}|} < \frac{\left(\sqrt{c/d}\right)^{k^2}}{1-\left(\sqrt{c/d}\right)^{k^2}} + k\left(\sqrt{c/d}\right)^{k^2}.$$

But $k(\sqrt{c/d})^{k^2} < 1/3$ by (1), so the right-hand side of the above inequality is less than $(1/3)/(1-1/3)+1/3$, which is less than one. This establishes Lemma 1.

The techniques used in proving Theorem 1 are common in the study of power series—see, for example, [**4**], [**5**], and [**7**].

3. LINES OF JULIA. There is an interesting refinement of Picard's theorem. Denote by $S(\varphi,\varepsilon)$ the sector $\{z: |\arg z - \varphi| < \varepsilon\}$, and by $R(\varphi)$ the ray $\{z: \arg z = \varphi\}$. Then $R(\varphi)$ is a *line of Julia* of f if*, for each $\varepsilon > 0$, f takes on every complex value, with one possible exception, infinitely many times in $S(\varphi,\varepsilon)$. The refinement is that every transcendental entire function has at least one line of Julia.

Here I will prove a weak version of Julia's theorem (which is also a strong version of the Casorati-Weierstrass theorem). Call the ray $R(\varphi)$ a *weak line of Julia* of f if, for each $\varepsilon > 0$, and each $r > 0$, the image under f of $S(\varphi,\varepsilon) \cap \{z: |z| > r\}$ is dense in the plane.

Theorem 2. *Every transcendental entire function has at least one weak line of Julia.*

The proof is similar to that of the existence of lines of Julia given by Cartwright [**3**, Theorem 63, p. 102]. The main difference is that the role played in [**3**] by Schottky's Theorem is played here by a weaker lemma with a simpler proof.

Lemma 2. *Let F be a function analytic in $D(0;1)$. If w is a complex number, and δ a positive number, such that*

$$|F(u) - w| \geqslant \delta \qquad \textit{for all } u \textit{ in } D(0;1),$$

then

$$|F(u)| < 5M^2/\delta \qquad \textit{for all } u \textit{ in } D(0;1/5),$$

where $M = \mathrm{Max}\{|w|, |F(0)|, \delta\}$.

Before proving the lemma, I will show how Theorem 2 follows from it.

*There are other definitions; see [**3**, p. 100].

Proof of Theorem 2: We will see in a moment that there exist disks $\{D_n\}_{n=1}^{\infty} = \{D(r_n e^{i\lambda_n}; r_n/n)\}_{n=1}^{\infty}$ such that $r_n \to \infty$, such that $0 \leqslant \lambda_n < 2\pi$, and such that, for each union U of infinitely many of the D_n, $f(U)$ is dense in the complex plane. The theorem follows from the existence of these disks because $\{\lambda_n\}_{n=1}^{\infty}$ has at least one accumulation point λ in $[0, 2\pi]$, and every sector $S(\lambda, \varepsilon)$ contains infinitely many of the D_n. The ray $R(\lambda)$ is therefore a weak line of Julia.

To construct the disks, choose a sequence $\{w_k\}_{k=1}^{\infty}$ which accumulates at all points of the complex plane, and pick a positive integer n. According to the Casorati-Weierstrass theorem, $f(\{z: |z| > r\})$ is dense in the complex plane for each $r > 0$. So there exist complex numbers z_0, of arbitrarily large modulus, such that $|f(z_0)| < 1$. Let z_0 be such a number, let $r = |z_0|$, and suppose that, for each real $\theta, f(D(re^{i\theta}; r/n))$ fails to intersect at least one of the disks $D(w_1; 1/n), \ldots, D(w_n; 1/n)$. This supposition leads to an upper bound on $|f|$ in $D(0; r)$ as follows. Let $f(D(z_0; r/n))$ omit $D(w_j; 1/n)$, say. Then Lemma 2, with $\delta = 1/n$, $w = w_j$, and $F(u) = f(z_0 + ru/n)$, gives

$$|f(z)| < 5n\left[\operatorname{Max}\left\{|w_j|, |f(z_0)|, \frac{1}{n}\right\}\right]^2 \leqslant 5nA^2 \qquad \textit{for all } z \textit{ in } D\left(z_0; \frac{r}{5n}\right),$$

where $A = \operatorname{Max}\{|w_1|, \ldots, |w_n|, 1\}$. Next let $z_1 = z_0 e^{i\pi/(15n)}$. Then $z_1 \in D(z_0; r/5n)$ because

$$|z_1 - z_0| = 2r\sin\left(\frac{\pi}{30n}\right) < (2r)\left(\frac{\pi}{30n}\right) < \frac{r}{5n}.$$

Therefore, by applying Lemma 2 (possibly with a different w_j) to the function $F(u) = f(z_1 + ru/n)$, we find that, for all z in $D(z_1; r/5n)$,

$$|f(z)| < 5n\left[\operatorname{Max}\left\{|w_j|, |f(z_1)|, \frac{1}{n}\right\}\right]^2 \leqslant 5n(5nA^2)^2 = (5n)^3 A^4.$$

This process can be repeated (with $z_2 = z_1 e^{i\pi/(15n)}$, etc.) to obtain $30n$ overlapping disks whose union contains the circle $\{|z| = r\}$, and such that, in each of these disks, $|f| < (5n)^{p-1}A^p$, where $p = 2^{30n}$. (The important thing is that this bound is independent of z_0). By the maximum modulus principle, $|f| < (5n)^{p-1}A^p$ in $D(0; r)$.

It follows that there is a disk $D_n = D(r_n e^{i\lambda_n}; r_n/n)$, with $r_n > n$, whose image intersects all the disks $D(w_1; 1/n), \ldots, D(w_n; 1/n)$: otherwise the argument above would be valid for arbitrarily large values of z_0 and f would be bounded, hence, by Liouville's theorem, constant. This would contradict the hypothesis that f is transcendental.

The disks $\{D_n\}_{n=1}^{\infty}$ so constructed have the required properties. In particular, $f(U)$ is dense in the plane for each infinite union U of the disks. For let U be such a union, let w be complex, and let $\{v_k\}$ be a subsequence of $\{w_j\}_{j=1}^{\infty}$ such that $v_k \to w$. A sequence of points α_k, and a sequence of disks D_{n_k}, can be chosen such that $\alpha_k \in D_{n_k} \subset U$ and $|f(\alpha_k) - v_k| \to 0$. Then $f(\alpha_k) \to w$. Since w was arbitrary, $f(U)$ is dense in the plane. This completes the proof of the theorem.

Proof of Lemma 2: I claim that, if g is a function analytic in $D(0; 1)$ such that $|g(u)| \geqslant 1$ for all u in $D(0; 1)$, then $|g(u)| < |g(0)|^2$ for all u in $D(0; 1/5)$. (It may

seem strange that a lower bound on $|g(u)|$ should imply an upper bound on the same quantity, but consider the special case $g(u) = a + bu$, where a and b are complex constants. Here $g(D(0;1)) = D(g(0); |g(u) - g(0)|)$ for each u of modulus 1, and the image disk will intersect $D(0;1)$ unless $|g(u) - g(0)| \leqslant |g(0)| - 1$. Thus, in this case, the hypothesis of the claim implies that $|g(u)| \leqslant 2|g(0)| - 1$ for all u in $D(0;1)$.) We can deduce Lemma 2 by applying the claim to $[F(u) - w]/\delta$: this gives, for all u in $D(0;1/5)$,

$$|F(u) - w| < |F(0) - w|^2/\delta \leqslant [|F(0)| + |w|]^2/\delta \leqslant 4M^2/\delta,$$

so that, since $M \geqslant \delta$,

$$|F(u)| < 4M^2/\delta + |w| \leqslant 4M^2/\delta + M \leqslant 5M^2/\delta.$$

Attempted proof of the claim: Let $h = 1/g$. Then h is analytic in $D(0;1)$, and $0 < |h| \leqslant 1$ there. To bound $|g|$ above, we will try to bound $|h|$ below. Now h, being small, cannot change very fast, so that $|h(u)|$ is not much smaller than $|h(0)|$. More precisely, for u in $D(0;1)$, the fundamental theorem of calculus implies that

$$\begin{aligned}|h(u)| &= |h(0) + h(u) - h(0)| \geqslant |h(0)| - |h(u) - h(0)| \\ &\geqslant |h(0)| - \int_0^u |h'(v)|\,|dv|,\end{aligned}$$

where the integral is taken over the line segment joining 0 and u. But for all v in $D(0;1/5)$, Cauchy's Inequality gives

$$|h'(v)| \leqslant \left(\frac{4}{5}\right)^{-1} \operatorname{Max}\{|h(z)| : z \in D(v;4/5)\} \leqslant 5/4.$$

So for each u in $D(0;1/5)$,

$$|h(u)| \geqslant |h(0)| - 5|u|/4 > |h(0)| - 1/4.$$

Here we run into a problem: this inequality is only helpful if $|h(0)| > 1/4$, and we have no positive lower bound on $|h(0)|$.

Proof of the claim: Let $h = 1/g$ again. For each $s > 0$, there is a branch H of $[h(u)]^s$ analytic in $D(0;1)$. (This branch has the form $\exp\{sL(z)\}$, where $L(z)$ is an analytic branch of $\log h(z)$ in $D(0;1)$. The latter branch exists because h is non-zero in $D(0;1)$: see [**2**, p. 93].) Then $|H(u)| \leqslant 1$, so the calculations in the attempted proof remain valid when h is replaced by H. Thus

$$|h(u)|^s > |h(0)|^s - 1/4 \qquad \text{whenever } |u| < 1/5. \tag{3}$$

The idea is to make the right-hand side positive by choosing a small s when $|h(0)|$ is small; this can be done by taking any s in the open interval $(0, -\log 4/\log|h(0)|)$. When s is the midpoint of the interval, then, because $x^{1/\log x} = e$ for $x > 0$, (3) becomes

$$|h(u)| > (|h(0)|^s - 1/4)^{1/s} = (e^{-\log 2} - 1/4)^{-\log|h(0)|/(\log 2)} = |h(0)|^2.$$

This proves the claim, and therefore establishes Lemma 2. (This choice of s happens to be the one that maximizes the right-hand side of (3), but any s in $(0, -\log 4/\log|h(0)|)$ would have led to a lemma similar to Lemma 2, and therefore to a proof of Theorem 2.)

4. EXAMPLES AND EXERCISES. The examples are on lines of Julia, and the exercises are on weak lines of Julia. (It would be interesting to know whether the two ideas are equivalent: is every weak line of Julia a line of Julia?)

There are functions having only one line of Julia. For example, there exists [**11**, problems 158–160, p. 135] a transcendental entire function g bounded outside the half-strip $\{x + iy: x > 0 \text{ and } -\pi < y < \pi\}$. The only line of Julia of this g is $R(0)$. Given any finite subset T of $[0, 2\pi]$, a function can be constructed whose lines of Julia are the rays $R(\varphi)$ such that $\varphi \in T$: just add together several rotations of g. More generally [**10**, Theorem Ib, p. 431 or **1**, Theorem 1, p. 61], every non-empty, closed subset of $[0, 2\pi]$ forms the set of lines of Julia of some entire function.

Certain types of behavior of an entire function f, however, imply specific information about where the lines of Julia lie. I will give two examples.

First suppose that $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ is a transcendental entire function. If $n_k = pk$ for all k, where p is a positive integer, then $f(ze^{2\pi i/p}) = f(z)$ for all z, so f has at least one line of Julia in each sector $S(\varphi, \varepsilon)$ such that $\varepsilon > \pi/p$. Thus the lines of Julia become more thickly distributed in all directions as p gets larger. This situation suggests that, if f is a function for which n_k grows quickly enough, then *every* ray $R(\varphi)$ might be a line of Julia.

EXERCISE 1. Prove that, if f is a transcendental entire function such that $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$, where $n_k = Q^k$ and Q is an integer greater than one, then every ray $R(\varphi)$ is a weak line of Julia.

Hayman [**6**] has shown that, if $n_k/k \to \infty$, then f takes on every complex value infinitely many times in every $S(\varphi, \varepsilon)$.

The second example concerns the rate of growth of a function. The *Mittag-Leffler functions* [**8**, pp. 127–128 or **3**, pp. 50–52] are defined, for α in $(0, 2)$, by $E_\alpha(z) = \sum_{n=0}^{\infty} z^n/\Gamma(1 + \alpha n)$, where Γ represents Euler's gamma function [**2**, p. 215]. Now $E_\alpha(z)$ is bounded for z outside $S(-\alpha\pi/2, \alpha\pi/2)$. Also if $z^{1/\alpha}$ is the principal branch of the power, then $E_\alpha(z) - \exp(z^{1/\alpha})$ is bounded for z inside that sector, so that, for each Ψ in $(0, \alpha\pi/2)$, $|E_\alpha(re^{i\theta})| \to \infty$ as $r \to \infty$, uniformly for θ in $(-\Psi, \Psi)$. These facts, along with the Schwarz Reflection Principle, imply that E_α has exactly two lines of Julia: $R(\pm\alpha\pi/2)$. The more quickly E_α grows on $R(0)$ (i.e., the smaller an α we choose), the closer these lines come to $R(0)$. This suggests the following result, which is a very weak form of a result of Pólya [**10**, Theorem Vb, p. 440].

EXERCISE 2. Show that, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function such that $a_n \geqslant 0$ for all n, and such that, for each $s > 0$, $f(x)/\exp(x^s) \to \infty$ as $x \to \infty$ through positive values, then $R(0)$ is a weak line of Julia of f.

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The shortest path between two truths in the real domain passes through the complex domain.

—*J. Hadamard*

An Unorthodox "Test"

Abe Shenitzer

Students are often taught, and tested for, technical skills. We miss the mark in not teaching the intellectual context and critical analysis of the material we present. What I mean is illustrated by the "test questions" below. These "test questions" reflect the kind of intellectual bias which I consider absolutely vital in the teaching of mathematics. The reader is encouraged to make up his or her own version of my "test."

1. What is the difference between intrinsic and extrinsic properties of mathematical objects?
2. What are some basic differences between Taylor series and Fourier series?
3. Can you describe the difference between Greek axiomatics and modern axiomatics? Between platonists and formalists?
4. What is the connection between equivalence relations and the intuitive notion of classification? How are equivalence relations used in mathematical constructions?
5. In 1952 Harvard University bestowed on Kurt Gödel an honorary doctorate "for the greatest intellectual discovery of the twentieth century." What was this discovery?
6. Have we "vanquished" the actual infinite? Have we eliminated paradoxes from mathematics?
7. What are some important independence results?
8. One way of solving a problem is to prove that it is unsolvable. Do you know some unsolvable problems and the theories that proved their unsolvability?
9. In what sense did Galois' work revolutionize algebra?
10. Are the three "Fundamental Theorems" of arithmetic, algebra, and the calculus really fundamental? If so, can you give some reasons?
11. Can you trace the crucial stages in the rigorization of analysis?
12. What were some of the crucial intellectual changes in mathematics between 1830–1930? What were some of their cumulative effects?
13. If you were asked to glorify the calculus, what would you say?
14. (a) "Curvature is an important characteristic of curves and surfaces." Can you substantiate this claim? (b) Can you give an indication of the importance of curvature in physics?
15. What is a Riemannian geometry? What is a Kleinian geometry?
16. What is topology? What are: combinatorial topology? algebraic topology? general topology? How did they arise?

What follows are brief sample discussions of the issues involved in the first 12 "test questions." *While none of these sample discussions is in any sense definitive, all contain significant remarks bearing on significant questions.*

(The bibliographical items not referred to in the sample discussions may help the reader to answer question 13–16.)

* * *

1. What is the difference between intrinsic and extrinsic properties of mathematical objects?

The difference between intrinsic and extrinsic properties of mathematical objects is far easier to pin down in algebra than in geometry. For example, the property of being commutative is, obviously, an intrinsic property of a group. On the other hand, the property of being normal is extrinsic, for it depends on the way a group is embedded in another group. Next, geometric examples.

(a) A surface can be thought of as rigid, or as flexible without being stretchable. In the latter case, the shape of the surface can be changed rather radically without affecting the intrinsic distances between points, that is distances measured on the surface. We could call such deformations of a surface intrinsic isometries. A less fancy term is "bending." An inhabitant of such a surface, call it S, unaware of the ambient space, would be unaware of bending transformations applied to his habitat. It is natural to ask what he could find out about the geometry of S. The things he could determine (known collectively as the intrinsic geometry of S) include, not surprisingly, the geodesics on S, angles between curves, area, and, surprisingly, the Gaussian curvature of S. (This last insight surprised Gauss, its very discoverer, who called it *Theorema Egregium*.) Another bending invariant is the geodesic curvature of a curve on S. See **[16]**, vol. 2, pp. 91–108, and **[18]**.

It is clear that bending will affect the disposition of S with respect to a fixed, preassigned coordinate system. This means that all positional data are extrinsic. See **[16]**, vol. 2, pp. 102–103, for a discussion of the connection between the intrinsic geometry of a surface and its form in space.

(b) A knotted cloverleaf and a circle are homeomorphic. But there is no homeomorphism of 3-space that maps a cloverleaf onto a circle. This means that the knottedness of the cloverleaf is an extrinsic property of this figure, a function of its embedding in 3-space.

It is interesting that, while there is no homeomorphism of 3-space that maps a cloverleaf onto a circle, there *is* a homeomorphism of 4-space that does just that. See **[10]**, pp. 593–594.

2. What are some basic differences between Taylor series and Fourier series?

Any "reasonably tame," say smooth, function on an interval that takes on the same values at its endpoints can be represented by a Fourier series, and, moreover, the terms of a Fourier series describe simple harmonic motions. Put differently, any "reasonably tame" function on an interval can be thought of as a (generally infinite) linear combination of harmonic motions. On the other hand, not even infinite differentiability guarantees representability of a function on an interval by a Taylor series (see **[1]**, section 24), and the terms of such a series have no physical significance comparable to that of the terms of a Fourier series.

3. Can you describe the difference between Greek axiomatics and modern axiomatics? Between platonists and formalists?

The idea of an axiomatic system is one of the greatest intellectual contributions of the Greeks. But great ideas evolve, and so did the idea of an axiomatic system.

Lectures 7 and 35 in [**6**] describe Greek "material axiomatics" and modern "formal axiomatics," respectively.

In material axiomatics meanings are attached to the basic terms, and the axioms "concerning the basic terms ... are felt to be acceptable to the reader as true on the basis of the properties suggested by the initial explanations" of the basic terms. In formal axiomatics some "technical terms ... are deliberately chosen as undefined terms," and the axioms about the technical terms "are deliberately chosen as unproved statements."

The transition from material axiomatics to formal axiomatics is the transition from Euclid's view of axioms and postulates as obvious truths that form the foundation of the unique system of geometric insights (now referred to as Euclidean geometry) to the view of axioms as initial propositions for which one claims neither truth nor meaning. This transition can be described as a transition from platonism to formalism.

The platonist view of Euclidean geometry espoused by Kant is of special interest. Kant claimed that "Euclidean geometry is a science which determines the properties of space synthetically and yet a priori," that is, without relying on experience.

The retreat from this position began in the nineteenth century with the emergence of hyperbolic geometry, which, while in one respect the logical opposite of Euclidean geometry, turned out to be logically its legitimate equal. The effect of this was that axioms in general, and the axioms of Eucliden geometry in particular, ceased to be "obvious truths" and gradually came to be viewed as variously motivated initial assumptions. By now

> there is no longer any question of defending the ancient and long-recurring rationalist hope that geometry gives us knowledge which is both synthetic and a priori. There is no one who can on this be quoted more aptly than Einstein. For he was the first to give a physical application to a [Riemannian] geometry: "As far as the laws of mathematics refer to reality they are not certain; and as far as they are certain, they do not refer to reality." [**8**], p. 427.

One should add that a strict formalist position can only be sustained by a computer, and that human mental activity often involves the risk of an "affair" with platonism. Here is a relevant "scenario."

You swear on a stack of bibles that you are a dyed-in-the-wool formalist. But when you realize that a mathematical "game" you—or someone else—has devised leads to an insight into the working of the physical world, or to the construction of some extraordinary—benevolent or evil—artifact, then you may find it hard to resist the feeling that the "game" is part of the reality it influences or sheds light on. But then you will have become a born-again platonist. And if you think that all you have to do in order to avoid the snares of platonism is stick to diophantine equations, then I must warn you that "Since equations in integers are encountered in physics, the solution of these equations is of more than theoretical interest." (A. O. Gelfond, *The Solution of Equations in Integers*.)

Speaking of the "dangers" of platonism, it is well to remember the physicist who arrives at the conclusion that the only changeless reality is the groups of symmetries and the probabilities he is led to when trying to penetrate the deceptive "exterior" of the "real" world.

Platonists and formalists would probably agree that axioms do not arise in a void, and that when we develop the consequences of a set of axioms we expect to be able to prove certain anticipated results. For example, in a rigorous course in

the calculus we expect to be able to prove the intermediate value theorem. If we can prove the anticipated results, then we become confident that we have chosen our axioms wisely and expect to reach the "payoff stage," the stage when we prove unanticipated, and possibly widely applicable, results.

4. What is the connection between equivalence relations and the intuitive notion of classification? How are equivalence relations used in mathematical constructions?

Classification is a common activity in mathematics as well as outside mathematics. You can classify nails by size, quadratic curves under isometries or affinities of the plane, algebraic curves of a certain degree under birational transformations of a suitable plane, abelian groups of n elements from the viewpoint of isomorphism, and so on.

When one does any of these things, one is guided by a criterion of sameness for the objects in some set S whose elements one is trying to classify, that is, one has a definition of an equivalence relation on S. Now one is faced with the following standard problem: Given an equivalence relation on a set S, to determine a set of functions on S—a "complete set of invariants"—whose values at an element of S enable one to assign that element unambiguously to a particular equivalence class. Such a function is called an invariant because it is constant on each equivalence class. A class representative is called a canonical form.

Having stated the standard problem associated with classification, and thus with equivalence relations, we give examples of the kinds of answers the problem leads to.

Example 1. Let S be the set of real plane quadratic curves $\Omega = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$. Put two such curves in the same equivalence class iff they "differ" by an isometry. A complete set of invariants for this equivalence relation on S consists of the functions $F_1(\Omega) = A + C$, $F_2(\Omega) = \begin{vmatrix} A & B \\ B & C \end{vmatrix}$, and $F_3(\Omega) = \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix}$. In other words, two plane quadratic curves Ω_a and Ω_b differ only in position iff $F_i(\Omega_a) = F_i(\Omega_b)$, $i = 1, 2, 3$. See **[16]**, vol. 1, pp. 238–239.

Example 2. The uniqueness of the decomposition of a finite abelian group into a direct sum of primary cyclic groups (see **[19]**, p. 59) implies that two such groups are isomorphic iff their respective decompositions into direct sums of primary cyclic groups have the same number of summands, and the summands of one decomposition can be so matched with the summands of the other decomposition that corresponding summands have the same order.

In the case of the class of abelian groups with, say, 200 elements, we have six classes of isomorphic groups characterized by the following six strings of prime powers:

$$2^3, 5^2; 2, 2^2, 5^2; 2, 2, 2, 5^2; 2^3, 5, 5; 2, 2^2, 5, 5; 2, 2, 2, 5, 5.$$

Here a complete set of invariants consists of a single function F; when applied to the representation of an abelian group with 200 elements as a direct sum of primary cyclic groups, F yields the orders of the summands. For example, $F(Z_8 \oplus Z_{25}) = (8, 25) = (2^3, 5^2)$.

Example 3. Consider the set W of compact, connected, orientable surfaces without boundary. Such a surface is homeomorphic to a sphere with handles. The number

of handles (the genus) characterizes the latter up to homeomorphism. Thus spheres with handles are the canonical forms of the countably many classes into which W is split by the equivalence relation of homeomorphism.

(Note that the Riemann surfaces of algebraic functions are compact, connected, orientable surfaces.)

Now we come to "doing" things with equivalence relations, that is, to their use in various constructions.

Equivalence relations are used repeatedly in the construction of the number system. See **[5]**, Chapters 1 and 2. Another important use is in quotient constructions: of a ring by an ideal; of a group by a normal subgroup; of a vector space by a subspace; and of a space by a discrete group acting on that space (for example, a cylinder is the quotient of the plane by a 1-generator group of translations, and a torus by a 2-generator group of translations).

5. In 1952 Harvard university bestowed on Kurt Gödel an honorary doctorate "for the greatest intellectual discovery of the twentieth century." What was this discovery?

The discovery of hyperbolic geometry eliminated from mathematics the notion of absolute truth.

A profound discovery made in 1931 by Kurt Gödel eliminated from mathematics the very possibility of certain forms of certainty. Roughly speaking, Gödel showed that any consistent axiomatic system that includes ordinary arithmetic contains undecidable propositions. Also—and this is of crucial importance—through an ingenious encoding, one of the unprovable propositions of such a system can be interpreted as asserting the consistency of the system. In other words, all we can do is *hope* that mathematics is consistent. See **[6]**, Lecture 38. Also, see **[17]**.

6. Have we "vanquished" the actual infinite? Have we eliminated paradoxes from mathematics?

The Greeks "feared" the actual infinite. This was in part due to Zeno's paradoxes.

Medieval speculations largely dispensed with the fear of the infinite in all areas of thought.

Awareness of the logical difficulties associated with the actual infinite reemerged at the end of the nineteenth century in connection with the growth of rigor, the interest in questions of the foundations of mathematics, and Cantor's set theory. In fact, the debate over the legitimacy of the use of actually infinite sets in mathematical reasoning has split the ranks of mathematicians. In particular, a constructivist (intuitionist) would not accept attempts to resolve, say, Zeno's *Dichotomy* which involve the use of the actual infinite.

Some of the well known paradoxes of relatively recent vintage are the Russell paradox and the related paradoxes of Hausdorff and Banach-Tarski. The Russell paradox is a genuine logical difficulty. The Hausdorff and Banach-Tarski paradoxes do not square with our everyday experience. We state all three and add brief comments.

(a) *The Russell paradox*. Consider the set R of sets that are not members of themselves:

$$R = \{S \mid (S \notin S)\}. \tag{1}$$

In other words,

$$S \in R \text{ iff } (S \notin S). \tag{2}$$

If we suppose that $R \in R$ then, by (2), $(R \notin R)$. But if $R \notin R$ then, again by (2), $R \in R$.

This shows the danger of specifying a set T by collecting all objects with a given property P:

$$T = \{X | X \text{ has the property } P\}.$$

One (standard) way of avoiding this difficulty is to form subsets of a *given* set W. Thus given a set W and a property P of sets one may form the set

$$T = \{X | X \in W \text{ and } X \text{ has the property } P\}.$$

See [**13**], pp. 55–56.

(b) *The Hausdorff Paradox.* Using the *axiom of choice* (= given a collection of nonempty sets, there exists a set of representatives of these sets) we can prove that it is possible to subdivide a sphere into disjoint sets A, B, C such that A, B, C and $B \cup C$ are all congruent to one another! See [**9**], p. 24.

(c) Also—this is the *Banach-Tarski theorem*—it can be shown that a ball can be divided into a finite number of disjoint parts which can be rearranged into two copies of the ball. See [**9**].

The last two paradoxes are certainly worrisome. On the other hand, the axiom of choice seems to be an indispensable tool of mathematics. Small wonder that the question of the legitimacy of the unrestricted use of this axiom has given rise to factions among mathematicians.

7. What are some important independence results?

The discovery that the parallel postulate is independent of the axioms of plane absolute geometry revolutionized mathematics. See [**6**], Lecture 27.

Here are two more-recent independence results, of more or less revolutionary nature, that pertain to the axiomatics of set theory:

(α) The axiom of choice is independent of the Zermelo-Fraenkel axioms for set theory. Given the importance of the axiom of choice in mathematical practice, it is nice to know that adding it to the axioms of set theory does no harm to their consistency.

(β) The continuum hypothesis (i.e. the nonexistence of a set with cardinality intermediate between $\aleph_0$ and $2^{\aleph_0}$) is independent of *ZFC* (= the Zermelo-Fraenkel axioms and the axiom of choice). See [**13**], pp. 362–368, and [**17**].

This discovery was made by Gödel (1938) and Cohen (1963).

The continuum hypothesis was formulated by Cantor, the creator of set theory. Cantor was a confirmed platonist. He spent many years attempting to show that there can be no power between that of the countable sets and the continuum. The Gödel-Cohen finding would not have pleased him. See [**4**], vol. 3, p. 57.

8. One way of solving a problem is to prove that it is unsolvable. Do you know some unsolvable problems and the theories that proved their unsolvability?

(a) The Greeks were aware that it is possible to double a cube and trisect an angle by using curves other than lines and circles, but it was only modern algebra that clarified the profound issues involved in these and many other construction problems. This is a telling demonstration that, at least in mathematics, the solution

of a "simple" problem may require the use of tools of such sophistication that the time span between the formulation of the problem and its solution is of the order of millennia.

(b) The unsolvability of the general quintic was first established (in 1799) by Ruffini. Ruffini's proof was incomplete. A complete proof was first given by Abel. Galois' "translation" of solvability of equations into solvability of their Galois groups "explained" the mystery of the unsolvability of the general quintic. The "reason" is that its Galois group is the group S_5, which is not solvable. See [21].

These problems led to the invention of fundamental concepts with far-reaching applications beyond the problems in question. This is a common phenomenon in the history of mathematics.

9. In what sense did Galois' work revolutionize algebra?

Galois did not invent modern algebra. Modern algebraic ideas are abundantly present in implicit rather than explicit form in Gauss' monumental *Disquisitiones* of 1801. But Galois must be given partial or full credit for the introduction—all in a concrete setting—of such key notions as field, finite group, normal subgroup, group of an equation, and algebraic extension of a field. He made his discoveries around 1830, but they did not significantly influence mathematics until about 1870. *When they did, they began to replace the theory of equations as the main concern of algebra by the study of structures such as groups and fields.* See [21], and Chapter 2 in [15].

10. Are the three "Fundamental Theorems" of arithmetic, algebra, and the calculus really fundamental? If so, can you give reasons?

Many of the preceeding issues involve theorems which would end up on everybody's "fundamental" list. Here we have in mind Gauss' Theorema Egregium (under 1(a)), the possibility of representing a "reasonably tame" function on an interval in a Fourier series (under 2), the Gödel incompleteness theorems (under 5), and various independence results (under 7). We add to this list three results known as The Fundamental Theorem of Arithmetic, The Fundamental Theorem of Algebra, and The Fundamental Theorem of the Calculus, and try to show that they are indeed fundamental.

The Fundamental Theorem of Arithmetic (FTAr) asserts that every integer $\neq 0, \pm 1$ is an essentially unique product of primes (unique up to order and multiplication by ± 1). In more picturesque terms, the primes, or the prime powers, are the essentially unique multiplicative building blocks for the integers.

The FTAr is very old. Its essence seems to have been known to Euclid (300 B.C.). The uniqueness aspect of the FTAr is both important and nonobvious; indeed, it fails in the system of positive even integers $2, 4, 6, \ldots$. Here the primes are $2, 6, 10, \ldots$ and the prime factorizations $36 = 6 \cdot 6 = 2 \cdot 18$ are genuinely different. The number and variety of applications of the FTAr are staggering. Chapter 2 of [11] shows how to use the FTAr to derive formulas for various arithmetic functions, to derive important properties of such key number-theoretic functions as the Euler ϕ-function and the Möbius function, to prove the divergence of the series $\Sigma 1/p$ of reciprocals of the positive primes, and to study the growth of the function $\pi(x)$, which gives the number of primes $\leq x$.

The Fundamental Theorem of Algebra asserts that every polynomial of positive degree with complex coefficients has a complex root. It follows readily that the only irreducible polynomials over $\mathbb{C}$ are linear.

An important partial generalization of the FTAl is that any field F has an algebraically complete extension G (that is, $q(x)$ in $F[x]$ splits into linear factors in a suitable extension $G[x]$ of $F[x]$).

The following comment points to the "indirect" importance of the FTAl.

> Viewed as an elementary proposition of the theory of functions of a complex variable the fundamental theorem of algebra is of little interest. And yet mathematicians as great as Euler, Lagrange, Laplace, and Gauss worked on it, and Gauss gave four different proofs of it. What was interesting about this theorem? We can now answer this question insofar as it pertains to the algebraic proofs of the fundamental theorem. As it turned out, there was an intimate connection between such proofs and the general theory of equations. The connection between algebraic proofs of the theorem and the theory of symmetric and similar functions[1] of the roots of an equation became apparent already in the proofs of Euler and Lagrange. The study of the latter functions is an essential part of Galois theory. To give a proof not based on the existence of a splitting field Gauss used his "principle of continuation of identities" and may be said to have constructed a field in which a given polynomial had a quadratic factor. Gauss' method of construction of this field was subsequently developed by Kronecker and became one of the most powerful tools of algebra. In this way the fundamental theorem of algebra stimulated the creation of new algebraic methods. **[15]**, p. 51 (in Russian).

The Fundamental Theorem of the Calculus is stated in different versions. One of these versions is $\int_a^b f'(x)\,dx = f(b) - f(a)$. The evaluation of definite integrals of functions having an antiderivative is still based on the FTC in the form stated above.

Using the language of forms and their derivatives (see **[2]**, 376 ff) we can rewrite $f(b) - f(a) = \int_a^b f'(x)\,dx$ as $\int_{\partial M} f = \int_M df$. Here $df = f'(x)\,dx$, $M = [a, b]$, $\partial M = \{a, b\}$, f is a 0-dimensional form whose integral $\int_{\partial M} f$ over the boundary $\partial M = \{a, b\}$ of the 1-dimensional manifold $M = [a, b]$ is $\int_{\partial M} f = f(b) - f(a)$, and df is a 1-form whose integral $\int_M df$ over M is $\int_{\partial M} f$. This shows that the above version of the FTC is the simplest case of the remarkable theorem known as *the generalized Stokes theorem*. This theorem asserts that if M is an oriented manifold in $\mathbb{R}^k$ with boundary ∂M, and ω is an $r - 1$ form, $r = 1, 2, \ldots, k$, then, under suitable conditions,

$$\int_{\partial M} \omega = \int_M d\omega.$$

Green's theorem, Stokes' theorem and Gauss' theorem (the divergence theorem) (see **[2]**, p. 406) are all special cases of this fundamental result of calculus on manifolds. All of these theorems have many applications in mathematics and in mathematical physics.

[1]Functions $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ of the roots of an equation of degree n are called similar if they belong to the same subgroup H of the group S_n of permutations of the roots of this equation, that is, they are unchanged under the permutations in H and are changed by all other permutations in S_n.

11. Can you trace the crucial stages in the rigorization of analysis?

In 1734 George Berkeley published *The Analyst*, in which he attacked the foundations of the calculus as formulated by Newton and Leibniz. Newton's "ultimate ratios of evanescent quantities," and Leibniz's nebulous infinitesimals invited and merited the good bishop's devastating criticism. His summary indictment was as apt as it was brief:

> I say that in every other science men prove their conclusions by their principles and not their principles by their conclusions. [**14**], p. 6.

Berkeley's strictures could also have been applied to the following utterance of the great Euler dating to 1755:

> There is no doubt that every quantity can be diminished to such an extent that it vanishes completely and disappears. But an infinitely small quantity is nothing other than a vanishing quantity and therefore the thing itself [i.e. the quantity] = 0. [**14**], p. 10.

All this was bound to change.

First a qualified success. Lagrange tried to eliminate the gremlins in the foundations of the analysis of his time by a kind of detour. His was the debatable attempt to reduce the calculus to algebraic processes by working only with power-series representations of functions. See [**14**], pp. 20–23. Next a sketch of the "success story."

Roughly speaking, the rigorization of analysis (or rather, of the calculus) occurred in two stages. The first stage was due, in large measure, to Cauchy. As a result of his work

> the subject was transformed from a collection of powerful methods and useful results into a mathematical discipline based on clear definitions and rigorous proofs. [**7**], pp. 571–572.

The "clear definitions" were those of such key concepts as limit, convergence, continuity, derivative, and integral. But Cauchy's use of expressions such as "a variable approaches a fixed value," and his view of the real numbers as largely an intuitive datum, left room for yet another stage of rigorization.

This second stage was dominated by Weierstrass and Dedekind. It was marked by elimination of the remaining borrowings from geometry and physics, and (to use a term coined by Klein) by arithmetization, that is, the basing of analysis on a number system that is not an intuitive datum, but a structure in which the intuitive component is limited to the choice of a few axioms, namely those of Peano.

In 1960, A. Robinson extended the reals to the system of hyper-reals. While the analysis of Cauchy and Weierstrass revolves around the notion of limit, in the Robinson version of analysis the limit concept is swallowed up by the hyperreal number system with its infinitely small and infinitely large "numbers," and all looks thoroughly algebraic. It is safe to say that Robinson made the infinitesimals of the founders of the calculus respectable. See, for example, [**12**].

A word about the axiomatic buildup of the number system. Here the key step was the transition from the rationals to the reals. The two familiar variants of this transition are those of Dedekind and Cantor. The advantage of the Cantor procedure is that it can be used to complete any metric space, that is, to embed it in a metric space in which every Cauchy sequence has a limit. This closure property is a must in analysis.

We conclude this summary with a surprise. Wouldn't everyone applaud all steps of the rigorization process just sketched? Two important mathematicians didn't. They objected to the separation of number from magnitude. They were du Bois-Reymond and, surprisingly, Herman Hankel, "the man who ... created a purely formal theory of rational numbers, but turned against a formal theory of irrationals." See **[14]**, pp. 92–93.

The dominant position of number in modern mathematics does not have the support of all leading mathematicians. Thus the great French mathematician René Thom of catastrophe theory fame feels that, given the great intuitive appeal of geometry, it, rather than number, should serve as the foundation for the edifice of mathematics.

12. What were some of the crucial intellectual changes in mathematics between 1830–1930? What were some of their cumulative effects?

A century separates the discovery of hyperbolic geometry (ab. 1830) by Lobachevski, Bolyai, and Gauss from Gödel's discovery (1931). This period witnessed some of the most decisive developments that have reshaped our view of mathematics. Any list of such developments must include the following:

1. Elimination of the special role of Euclidean geometry.
2. Axiomatization of arithmetic.
3. Arithmetization of analysis.
4. Improvement of the logical basis of Euclidean geometry and insight into the logical consequences of the various groups of axioms comprising the system of axioms of Euclidean geometry. The birth of formal axiomatics.
5. Discovery of paradoxes in set theory and efforts aimed at their elimination. The reemergence of the ancient debate on the actual infinite.
6. Hilbert's program to prove the consistency of arithmetic and its termination by the discoveries of Gödel.

In addition to changing our view of mathematics, these developments have also changed our view of the nature of mathematical activity. In the words of H. Weyl, "'Mathematizing' may well be a creative activity of man, like language or music, of primary originality, whose historical decisions defy complete objective rationalization" [Obituary Notices of Fellows of the Royal Soc., 4, 1944, 547–553].

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> Teaching is not a lost art, but the regard for it is a lost tradition.
>
> —*Jacques Barzun*

Replication and Stacking in Ergodic Theory

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1. INTRODUCTION. One of the beautiful ideas in mathematics is construction by replication. For example, replication is the basic idea underlying the construction of the fractal sets discussed by Mandelbrot in **[13]**. In ergodic theory stacking constructions have been used to obtain a variety of important examples of point transformations on the unit interval. These stacking constructions can also be viewed as an application of the idea of replication. Our purpose is to present two examples of transformations constructed by stacking along with related concepts and results.

In the case of fractal sets such as the Cantor set, the construction is essentially a picture. The same is true for the stacking examples constructed below. The pictures are quite simple.

2. PRELIMINARIES. Our discussion will take place on the unit interval $X = [0,1)$ with $\mathscr{B}$ its family of Lebesgue measurable sets and m its Lebesgue measure. All sets and functions discussed will be assumed measurable. Given sets A and B, their symmetric difference is $A \Delta B = (A - B) \cup (B - A)$. We write $A = B$ if $m(A \Delta B) = 0$. Given functions f and g, we write $f = g$ if $m(\{x: f(x) \neq g(x)\}) = 0$.

Let T denote an invertible point transformation mapping X onto X. Given a set B and an integer i, let $T^i(B) = \{T^i(x): x \in B\}$ and $B^T = \bigcup_{i=-\infty}^{\infty} T^i(B)$. We refer to B^T as the set *swept out by B* when T is understood.

A transformation T is *measurable* if $B \in \mathscr{B}$ implies $T(B) \in \mathscr{B}$ and $T^{-1}(B) \in \mathscr{B}$. All transformations will be assumed measurable. Hence $B \in \mathscr{B}$ implies $T^i(B) \in \mathscr{B}$ for each integer i and therefore $B^T \in \mathscr{B}$. A transformation is *nonsingular* if $m(B) = 0$ if and only if $m(T(B)) = 0$. That is, T preserves sets of measure zero. A nonsingular transformation is *ergodic* if each set of positive measure sweeps out X. That is, T is ergodic if $m(B) > 0$ implies $m(B^T) = 1$. A set A is *T-invariant* if $TA = A$, in which case $A^T = A$. It follows that a transformation is ergodic if and only if invariant sets have measure zero or one. A transformation T is *measure preserving* if $m(T(B)) = m(B)$, $B \in \mathscr{B}$. The examples constructed below will be ergodic and measure preserving.

The positive integers will be denoted by N and the integers will be denoted by Z. A transformation T is a *σ-translation* if there exist disjoint intervals I_n, $n \in N$, and disjoint intervals J_n, $n \in N$, such that $X = \bigcup_{n=1}^{\infty} I_n = \bigcup_{n=1}^{\infty} J_n$, I_n and J_n have the same length, and T translates I_n onto J_n, $n \in N$. Since a translation preserves measure, we have $m(T(B \cap I_n)) = m(B \cap I_n)$, $B \in \mathscr{B}$, $n \in N$. There-

fore $B \in \mathscr{B}$ implies

$$m(T(B)) = m\left(T\left(\bigcup_{n=1}^{\infty}(B \cap I_n)\right)\right) = m\left(\bigcup_{n=1}^{\infty} T(B \cap I_n)\right)$$

$$= \sum_{n=1}^{\infty} m(T(B \cap I_n)) = \sum_{n=1}^{\infty} m(B \cap I_n) = m(B).$$

Thus all σ-translations are measure preserving. The examples constructed below will be σ-translations. Moreover, it is shown in [**1, 2**] that every ergodic measure preserving transformation on the unit interval can be realized as a σ-translation.

3. LADDERS. The ergodicity of the examples will follow from viewing the construction of the examples via ladders. A *ladder* L of *height* h and *width* w is an ordered set of h disjoint subintervals I_i contained in the unit interval $[0,1)$ such that all h intervals have length w and are left-closed and right-open. Thus $L = (I_i\colon 1 \leqslant i \leqslant h)$ and we can view a ladder as in Figure 1. We refer to I_i as the ith *rung*, $1 \leqslant i \leqslant h$.

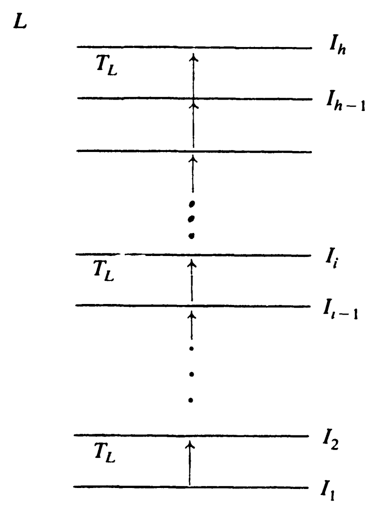

Figure 1.

The rung I_1 is the *base* of L and I_h is the *top* of L. Since all rungs in L are left-closed, right-open, and have the same length, we can define a map T_L that translates I_{i-1} onto I_i, $2 \leqslant i \leqslant h$. In Figure 1 I_i is directly above I_{i-1}, $2 \leqslant i \leqslant h$, so T_L simply maps a point to the point directly above, as indicated by the arrows. Let L^* denote the union of the rungs in L; hence T_L is defined on $L^* - I_h$ and T_L^{-1} is defined on $L^* - I_1$.

Given a transformation T, a ladder L is a *T-ladder* if $T = T_L$ on $L^* - I_h$. In this case iterates of T move a rung up and down the ladder; hence $I^T \supset L^*$ if I is a rung in L. In particular, if $L^* = [0,1)$, then each rung sweeps out the whole space. Thus a ladder is a natural picture for seeing sweeping out. This picture is due to J. von Neumann and S. Kakutani.

Suppose we start with a ladder L and the partially defined mapping T_L. If I_i is the ith rung in L, as in Figure 1, then $\bigcup_{j=-i+1}^{h-i} T_L^j I_i = L^*$. Thus we can say rungs in L sweep out L^* under iterates of T_L. Now we can extend T_L so that the bisected rungs of L also sweep out L^*. This is accomplished by cutting L in

Figure 1 in half by a vertical cut down the middle of L. We then obtain two ladders of height h and width $w/2$ each. Let L_1 be the left half and let L_2 be the right half, as in Figure 2. We assume the rungs in L_1 are right-open and the rungs in L_2 are left-closed. We now stack L_2 and L_1 to obtain a new ladder L_3 of height $2h$ and width $w/2$, as in Figure 3.

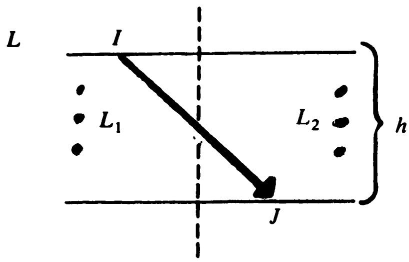

Figure 2.

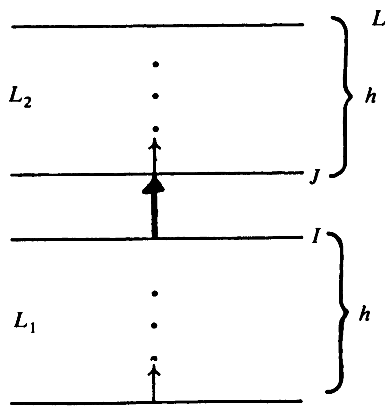

Figure 3.

Note that T_{L_3} extends T_L to map the left half I of the top of L onto the right half J of the base of L, as indicated by the heavy arrows in Figures 2 and 3. Thus the construction of L_3 extends T_L to I which is half of where T_L was not defined. The extension of T_L is measure preserving since I and J have the same length. Now L_3 consists of the bisected rungs of L and each rung in L_3 sweeps out $L_3^* = L^*$.

The preceding construction of cutting in half and stacking the right half above the left half can be repeated inductively. This is the construction in Example 1 below. Thus the construction consists of a sequence of ladders L_n, $n \geqslant 1$, where L_{n+1} is obtained by cutting L_n in half and stacking the right half above the left half.

4. REPLICAS. The Cantor set C can be written as a union of two disjoint sets C_1 and C_2, where each set C_i looks just like C, except scaled down by a factor of $1/3$. Thus the Cantor set consists of two replicas of itself. The transformations constructed below also admit so-called replicas of themselves, as defined below.

In general, let $(X_i, \mathscr{B}_i, m_i)$ be measure spaces with $m_i(X_i) = 1$, $i = 1, 2$. Let φ be an invertible mapping from X_1 to X_2 such that $m_1(B_1) = m_2(\varphi(B_1))$, $B_1 \in \mathscr{B}_1$, and $m_2(B_2) = m_1(\varphi^{-1}(B_2))$, $B_2 \in \mathscr{B}_2$. We refer to φ as an *isomorphism*. Transformations T_i on X_i, $i = 1, 2$, are *isomorphic* if there exists an isomorphism φ such that $T_1(x) = \varphi^{-1}(T_2(\varphi(x)))$ for $x \in X_1$. We refer to T_2 as a *copy of* T_1. This relationship is symmetric since T_1 is also a copy of T_2 via φ^{-1}.

Now given a measure preserving transformation T and $m(B) > 0$, we will define the so-called induced transformation T_A on $A \subset B$ with $m(B - A) = 0$. Let $\{x \in B\colon T^n(x) \notin B, n \geqslant 1\}$; hence $T^n(W) \cap W = \phi$, $n \geqslant 1$. Therefore $i > j$ implies $T^i(W) \cap T^j(W) = T^j(T^{i-j}(W) \cap W) = \phi$. Thus the iterates $T^i(W)$, $i \in Z$, are disjoint. Therefore $m(\bigcup_{i=-\infty}^{\infty} T^i W) = \sum_{i=-\infty}^{\infty} m(T^i W) = \sum_{i=-\infty}^{\infty} m(W)$. Since $m(X) = 1$, we conclude $m(W) = 0$; hence $m(W^T) = 0$. If $A = B - W^T$, then $x \in A$ implies $T^n(x) \in A$ for some $n \geqslant 1$. Thus each $x \in A$ returns to A under an

iterate of T. Let $n_A(x)$ be the smallest such n and define the *induced transformation* T_A on A by $T_A(x) = T^{n_A(x)}(x)$. Induced transformations are due to Kakutani **[9]**.

The induced transformation T_A is considered to act in the measure space $(A, \mathscr{B}_A, m_A)$ where $\mathscr{B}_A = \{E \in \mathscr{B}: E \subset A\}$ and $m_A(E) = m(E)/m(A)$, $E \in \mathscr{B}_A$. In general, T_A is not a copy of T for any $A \in \mathscr{B}$. However, if T is ergodic and measure preserving, then T_A will also be ergodic and measure preserving **[6, 9]**. As A varies, one obtains a large variety of ergodic measure preserving transformations T_A **[15]**.

Let $J = [a, b)$ with measure m_J and let λ be the order-preserving linear isomorphism of J onto $[0, 1)$ given by $\lambda(x) = (x - a)/(b - a)$, $x \in J$. If T is a transformation that is isomorphic to T_J by λ, then we refer to T_J as a *replica of* T. Thus T_J is a replica of T if T_J acting on J looks just like T acting on $[0, 1)$.

The binary intervals are $[(k - 1)/2^n, k/2^n)$, $1 \leqslant k \leqslant 2^n$, $n \in N$. The transformation T in Example 1 has the property that T_J is a replica of T for every binary interval J. For T in Example 2 there exist certain intervals J of arbitrarily small length such that T_J is a replica of T.

5. THE VON NEUMANN-KAKUTANI TRANSFORMATION. The first example is due to J. von Neumann and S. Kakutani (1940, unpublished). It is the basic example of an ergodic measure preserving transformation constructed by cutting and stacking. The transformation is a σ-translation that is constructed inductively. At the nth stage of the construction an interval I_n is mapped linearly onto an interval J_n of the same length, $n \geqslant 1$.

Example 1 (von Neumann-Kakutani transformation). The first ladder L_1 is constructed to guarantee that the two binary intervals of length $1/2$ sweep out. Cut $[0, 1)$ in half and define $L_1 = ([0, 1/2), [1/2, 1))$ as in Figure 4.

Now L_2 is formed to guarantee that the four binary intervals of length $1/4$ sweep out. Cut L_1 in half and stack the right half above the left half to form L_2 as in Figure 5. In general, denote $T_n = T_{L_n}$, $n \geqslant 1$. T_2 extends T_1 by mapping $[1/2, 3/4)$ onto $[1/4, 1/2)$, which is indicated by the heavy arrow in Figures 4 and 5.

The induction step starts with a ladder L_n of height 2^n whose rungs are the binary intervals of length 2^{-n}, but not in the usual order. The base of L_n is $[0, 2^{-n})$ and the top of L_n is $[1 - 2^{-n}, 1)$, as in Figure 6.

Thus L_n guarantees the binary intervals of length 2^{-n} sweep out. Now L_{n+1} is formed to guarantee that the binary intervals of length 2^{-n-1} sweep out. Cut L_n in half and stack the right half above the left half to obtain L_{n+1}, as in Figure 7. If $I_n = [1 - 2^{-n}, 1 - 2^{-n-1})$ and $J_n = [2^{-n-1}, 2^{-n})$, then T_{n+1} extends T_n by mapping I_n onto J_n which is indicated by the heavy arrow in Figures 6 and 7. Thus $T_{n+1}(I_n) = J_n$, $n \geqslant 1$, by induction.

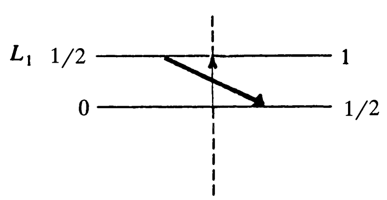

Figure 4.

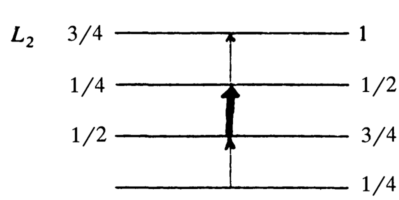

Figure 5.

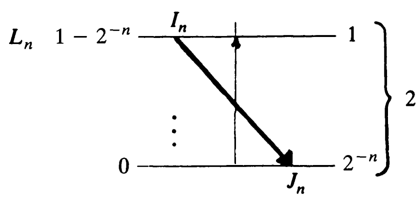

Figure 6.

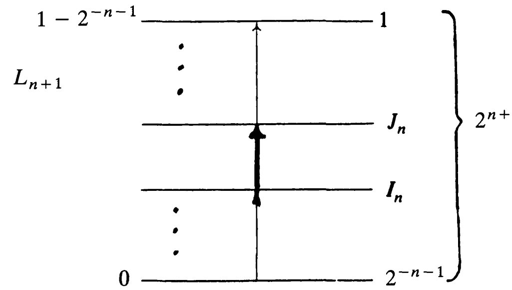

Figure 7.

If $I_0 = [0, 1/2)$ and $J_0 = [1/2, 1)$ then $[0,1) = \bigcup_{n=0}^{\infty} I_n = \bigcup_{n=0}^{\infty} J_n$ and $T_{n+1}(I_n) = J_n$, $n \geqslant 0$. If $x \in [0,1)$, then $x \in I_n$ for some $n \geqslant 0$ and we define $T(x) = T_{n+1}(x)$. Since T_k extends T_n for $k > n$, we have $T_k(x) = T_{n+1}(x)$, $k > n$, $x \in I_n$. Therefore we can write $T(x) = \lim_{n\to\infty} T_n(x)$, $x \in [0,1)$. The transformation T extends T_n, $n \geqslant 1$, hence L_n is a T-ladder, $n \geqslant 1$, and $T(I_n) = J_n$, $n \geqslant 0$. Thus T is a σ-translation. The graph of T is shown in Figure 8.

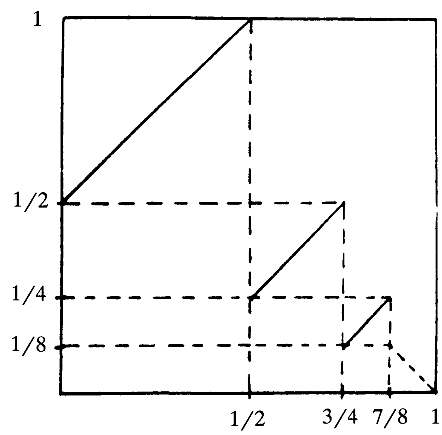

Figure 8.

Theorem 1. *The von Neumann-Kakutani transformation T is measure preserving and ergodic. If I is a binary interval, then T_I is a replica of T.*

Proof: Since T is a σ-translation, T is measure preserving. Before verifying ergodicity for T in the general case, first note that $I^T = [0,1)$ if I is a rung in L_n, $n \geqslant 1$. Since L_n consists of the 2^n binary intervals of length 2^{-n}, $n \geqslant 1$, we have $I^T = [0,1)$ if I is a binary interval. Since every interval contains binary intervals, we have $I^T = [0,1)$ if I is any interval.

In general, let $m(B) > 0$ and choose a point $x \in \mathscr{B}$ such that the Lebesgue density of B at x is 1. This means that given $\varepsilon > 0$ there exists $\delta > 0$ such that if I is any interval with $x \in I$ and $m(I) < \delta$, then $m(B \cap I) > (1 - \varepsilon)m(I)$. Choose n so large that $2^{-n} < \delta$ and let $h = 2^n$. There exists a binary interval I in L_n with $x \in I$. Suppose I is the rth rung in L_n. Since T is a translation on the rungs of L_n, we have $m(T^i(B \cap I)) = m(B \cap I) > (1 - \varepsilon)m(I)$, as long as $T^i(I)$ is a rung in

L_n. Therefore,

$$m(B^T) \geqslant m\big((B \cap I)^T\big) \geqslant \sum_{i=-r+1}^{h-r} m(T^i(B \cap I))$$
$$> \sum_{i=-r+1}^{h-r} (1-\varepsilon)m(I)$$
$$= (1-\varepsilon)hm(I) = (1-\varepsilon), \qquad \varepsilon > 0.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $m(B^T) = 1$. Hence T is ergodic.

We have now seen that the ladder construction helps to verify ergodicity. The ladder construction will next be used to find replicas of T. Let $I = [0, 1/2)$. It will first be shown that T_I is a replica of T. It is helpful to regard I as having a color, say blue. In Figure 5 the blue rungs are $[0, 1/4)$ and $[1/4, 1/2)$. Assume every other rung is blue in Figure 6. Hence the construction implies every other rung is blue in Figure 7. Thus, by induction, we conclude every other rung is blue in L_n, $n \geqslant 1$. Let L'_n denote the blue rungs in L_{n+1}, $n \geqslant 1$. Thus L'_n is L_{n+1} restricted to I. There are 2^n rungs in L'_n, $n \geqslant 1$. Thus L'_n looks like L_n, $n \geqslant 1$.

Suppose x is in a blue rung in L_{n+1} that is not the top blue rung. Therefore $T_I(x)$ is the point in the first blue rung above x. Since every other rung is blue, we have $T_I(x) = T^2(x) = T_{L'_n}(x)$. Thus the construction of T_I on I can be viewed in terms of the ladders L'_n, $n \geqslant 1$.

Let λ be the mapping from I to $[0,1)$ defined by $\lambda(x) = 2x$, $x \in I$. Hence λ is an isomorphism from $(I, \mathscr{B}_I, m_I)$ to $([0,1), \mathscr{B}, m)$. Now L'_n has 2^n rungs and $T_{L'_n}$ looks just like T_{L_n}. It will follow that T_I is isomorphic to T by λ if λ maps L'_n to L_n, $n \geqslant 1$.

In Figure 5 we see that λ maps L'_1 to L_1 in Figure 4. Suppose λ maps L'_n in Figure 7 to L_n in Figure 6. Since λ maps left and right halves of rungs in L'_n to left and right halves of rungs in L_n, respectively, it follows that λ maps L'_{n+1} to L_{n+1}. Thus, by induction, λ maps L'_n to L_n, $n \geqslant 1$. Therefore, T_I is isomorphic to T by λ. Since λ is linear, T_I is a replica of T.

In general, if I is a rung in L_k, then the construction of T_I can be viewed as the ladders L_n restricted to I, $n \geqslant k$. The ladder L_{k+n} restricted to I looks just like L_n under the isomorphism $\lambda(x) = 2^k(x-u)$, $x \in I = [u, u + 2^{-k})$. Thus T_I is a replica of T for all binary intervals I. Q.E.D.

It is not obvious that for any set B of positive measure there exists $A \subset B$ such that T_A is a copy of T. This result follows from the general theory in **[15]**.

6. MIXING. To motivate Example 2, we will discuss mixing properties of transformations. Mixing can be viewed as a form of asymptotic independence, where sets A and B are independent if $m(A \cap B) = m(A)m(B)$. If a set A is independent of all sets, then A is independent of A; hence $m(A) = m(A)^2$. Thus a set A is independent of all sets if and only if $m(A) = 0$ or 1. However, it is possible for a nontrivial sequence of sets to be asymptotically independent of all sets. A transformation T is *mixing* if

$$\lim_{n \to \infty} m(T^n(A) \cap B) = m(A)m(B), \qquad A, B \in \mathscr{B}. \tag{6.1}$$

It is easy to check that mixing implies ergodic and measure preserving but the converse is false. The transformation T in Example 1 has the property that if $A = [0, 1/2)$, then $T^n(A) = A$ for even n, as seen in Figure 4. Thus T is not mixing.

Intuitively, mixing implies that the iterates $T^n(A)$ spread out and approach a uniform distribution where the amount in a set B is proportional to the measure of B. Mixing transformations are easily constructed as shifts in sequence spaces [**8**, **17**].

Mixing is also called two-fold mixing. A transformation T is *three-fold mixing* if

$$\lim_{n,k\to\infty} m(T^k(T^n(A)\cap B)\cap C) = m(A)m(B)m(C), \qquad A,B,C\in\mathscr{B}. \quad (6.2)$$

A long-open problem is whether two-fold mixing implies three-fold mixing. This result has been proved for the class of rank one mixing transformations by S. Kalikow [**11**].

A transformation may not be mixing but can be mixing "on the average", which is Césaro-mixing. A transformation T is *Césaro-mixing* if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} m(T^i(A)\cap B) = m(A)m(B), \qquad A,B\in\mathscr{B}. \qquad (6.3)$$

It can be shown that a transformation T is Césaro-mixing if and only if T is ergodic and measure preserving [**6**, **8**, **17**].

Césaro-mixing can be verified directly for T in Example 1. Since T maps the top of L_n onto the base of L_n, for each rung I in L_n the iterates T^iI, $i\geqslant 1$, cycle through the rungs in L_n. It follows that (6.3) holds if A and B are rungs in L_n, $n\geqslant 1$. By finite additivity, (6.3) holds if A and B are unions of rungs in L_n, $n\geqslant 1$. In general, given sets A and B of positive measure and $\varepsilon>0$, we can choose a positive integer n sufficiently large and sets C and D that are finite unions of rungs in L_n such that $m(A\Delta C)<\varepsilon$ and $m(B\Delta D)<\varepsilon$. Since T is measure preserving, we have $m(T^i(A)\,\Delta T^i(C))<\varepsilon$ and $m(T^i(B)\,\Delta T^i(D))<\varepsilon$, $i\geqslant 1$. Since (6.3) holds with $A=C$ and $B=D$ and $\varepsilon>0$ is arbitrary, it follows that (6.3) holds for A and B.

In general, let T be measure preserving and let U_T be the unitary operator defined on $L^2(m)$ by $U_Tf(x)=f(T(x))$ for $f\in L^2(m)$. A complex number c is an *eigenvalue for* T if there exists a corresponding eigenfunction f such that $U_Tf=cf$. Constant functions are eigenfunctions with $c=1$. It can be shown that T is ergodic if and only if constant functions are the only eigenfunctions for $c=1$ [**8**]. A transformation T has *continuous spectrum* if $c=1$ is the only eigenvalue for T and constant functions are the only eigenfunctions. The mixing condition corresponding to continuous spectrum is *weakly mixing*. A transformation T is *weakly mixing* if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|m(T^i(A)\cap B)-m(A)m(B)|=0, \qquad A,B\in\mathscr{B}. \qquad (6.4)$$

Weakly mixing is difficult to verify directly. The following result of Koopman and von Neumann [**8**, **12**] is generally used to verify weakly mixing. This is the case in Example 2 below.

Theorem. *An ergodic measure preserving transformation T is weakly mixing if and only if T has continuous spectrum.*

It is clear that mixing implies weakly mixing and weakly mixing implies Césaro-mixing. In Example 1 T is not weakly mixing since for $A=[0,1/2)$, $T^n(A)=A$, n even, and $T^n(A)=A^c$, n odd. Thus (6.4) is not satisfied with $B=A$.

Weakly mixing can also be characterized as mixing on a sequence. A transformation T is *mixing on a sequence* $s = (k_n)$ if

$$\lim_{n\to\infty} m(T^{k_n}(A) \cap B) = m(A)m(B), \qquad A, B \in \mathscr{B}. \tag{6.5}$$

It can be shown that T is weakly mixing if and only if T is mixing on some sequence s [**7**]. Furthermore, the sequence s can be chosen to have density one [**17**]. A nice proof due to Kakutani is given in [**7**].

7. CHACON'S TRANSFORMATION. Although it is easy to construct mixing transformations, it is relatively difficult to construct a weakly mixing transformation that is not mixing. The first example was constructed by von Neumann and Kakutani (1940) using stacking but remained unpublished until [**10**]. A modified version of [**10**] due to Chacon [**6**, p. 86] is given in Example 2. A similar example is given in [**3**].

To get a feeling for Chacon's construction, consider a transformation T with a ladder L of height h, as in Figure 9. We cut L into three ladders of equal width and add an extra interval E above the top of the middle third, as in Figure 9.

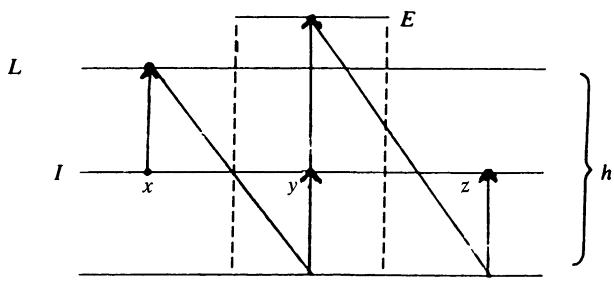

Figure 9.

The middle ladder with E will be stacked above the left ladder and the right ladder will then be stacked above the middle ladder. The resulting ladder will have height $3h + 1$. Consider a point x in the left third of a rung I, as in Figure 9. Then $T^h(x) = y$ will be a point in the middle third of I and $T^{h+1}(y) = z$ will be a point in the right third of I.

Suppose T admits an eigenvalue c with eigenfunction f. For simplicity, assume f is a constant k on I in Figure 9. Hence $U_T f = cf$ and $f \equiv k$ on I. Therefore $k = f(y) = f(T^h(x)) = c^h f(x) = c^h k$ and $k = f(z) = f(T^{h+1}(y)) = c^{h+1} f(y) = c^{h+1} k$. Thus $c^h k = c^{h+1} k$; hence $c = 1$. The reason for E in Figure 9 is now clear. The proof of continuous spectrum is a refinement of this simple case, as seen below.

Example 2 (Chacon's transformation). Let $a_0 = 0$ and

$$a_n = \sum_{i=1}^{n} \frac{2}{3^{i+1}}, \qquad n \geqslant 1.$$

Let $E_n = [2/3 + a_{n-1}, 2/3 + a_n)$, $n \geqslant 1$. The sum of the lengths $2/3^{n+1}$ of E_n, $n \geqslant 1$, is $1/3$; hence $\bigcup_{n=1}^{\infty} E_n = [2/3, 1)$. Let L_1 consist of a single rung $[0, 2/3)$.

This rung is cut into three equal subintervals and E_1 is placed above the middle third, as in Figure 10.

We map $[0, 2/9)$ onto $[2/9, 4/9)$, $[2/9, 4/9)$ onto E_1 and E_1 onto $[4/9, 2/3)$, as indicated by the heavy arrows in Figures 10 and 11. This results in the ladder L_2 in Figure 11.

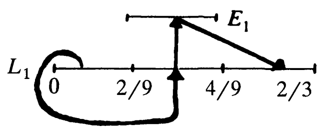

Figure 10.

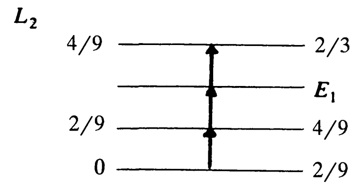

Figure 11.

For the induction step we start with a ladder L_n of height h_n and width $2/3^n$, as in Figure 12. The base of L_n, is $[0, 2/3^n)$ and the top of L_n is $[2/3 - 2/3^n, 2/3)$.

Cut L_n into three ladders of width $2/3^{n+1}$ each. Stack the middle ladder above the left ladder. Place E_n above the middle ladder and stack the right ladder above E_n. This results in the ladder L_{n+1} in Figure 13 of width $2/3^{n+1}$ and height $h_{n+1} = 3h_n + 1$. The stacking is equivalent to mapping the top of the left ladder onto the base of the middle ladder, the top of the middle ladder onto E_n, and E_n onto the base of the right ladder, as indicated by the heavy arrows in Figures 12 and 13.

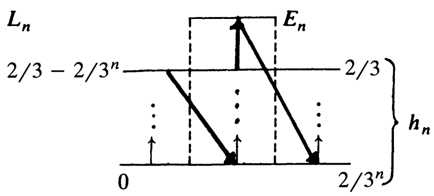

Figure 12.

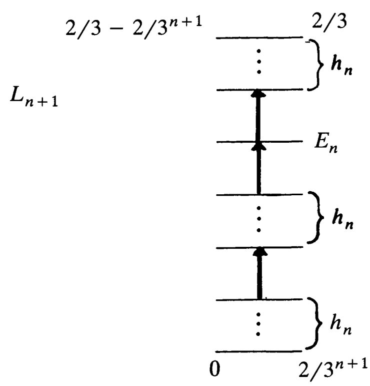

Figure 13.

If $x \in [0, 1)$, then $x \in [0, 2/3)$ or $x \in [2/3, 1)$. If $x \in [2/3, 1)$, then $x \in E_n$ for some n and $T_{n+1}(x)$ is defined. If $x \in [0, 2/3)$, then x is not on top of L_n for n sufficiently large and $T_n(x)$ is defined. Since T_k extends T_n for $k > n$, we can define $T(x) = \lim_{n \to \infty} T_n(x)$, $x \in [0, 1)$.

Theorem. *Chacon's transformation T is measure preserving, ergodic, not mixing, and has continuous spectrum. If $J = [1 - 1/3^k, 1)$, the T_J is a replica of T, $k \geqslant 1$.*

Proof: The transformation T is a σ-translation and is therefore measure preserving. To prove T is ergodic, we proceed exactly as in Example 1 with the refinement that n must be chosen sufficiently large so that $m(L_n^*) > 1 - \varepsilon$.

To see that T is not mixing, choose $A = [0, 2/9)$. The interval A appears as a rung in L_2 and A will be a union of rungs I in L_n for $n \geqslant 2$. Consider $L_n = L$ in Figure 9. If I_1 is the left third of I and I_2 is the middle third of I, then $T^h(I_1) = I_2$; hence $m(T^h(I) \cap I) \geqslant m(I)/3$. Since A is a union of rungs in L, it follows that $m(T^h(A) \cap (A) \geqslant m(A)/3$. Since $m(A) < 1/3$ and $h = h_n \to \infty$, it follows that T cannot be mixing.

To prove T has continuous spectrum, suppose there exist f and c such that $f(T(x)) = cf(x)$. Since T is measure preserving, $\|f\|_2 = \|f(T)\|_2 = |c|\,\|f\|_2$; hence $|c| = 1$. Therefore $|f(T(x))| = |f(x)|$; hence $|f|$ is invariant under T. Since T is ergodic, invariance implies $|f|$ is a constant which we can assume is 1. Thus $c = e^{ia}$ where $0 \leqslant a < 2\pi$ and $f(x) = e^{i\theta(x)}$, where $\theta(x)$ is measurable. By Lusin's Theorem there exists a closed set F of measure arbitrarily close to 1 such that θ is uniformly continuous on F. Therefore given $\eta > 0$ there exists $\delta > 0$ such that x, y in F and $|x - y| < \delta$ imply $|\theta(x) - \theta(y)| < \eta$.

Since $m(F) > 0$, we can choose a point $p \in F$ such that F has Lebesgue density one at p. Let $\varepsilon > 0$. We can choose n sufficiently large so that $2/3^n < \delta$ and there exists a rung I in L_n with $p \in I$ and $m(I \cap F) > (1 - \varepsilon)m(F)$. If ε is sufficiently small, then there must exist x, y, z in $I \cap F$, where x, y, z are as in Figure 9 with $L = L_n$. We, therefore, have

$$e^{i\theta(y)} = f(y) = e^{iha}\, e^{i\theta(x)} \tag{1}$$

$$e^{i\theta(z)} = f(z) = e^{i(h+1)a}\, e^{i\theta(y)}, \tag{2}$$

hence

$$\theta(y) = ha + \theta(x) \tag{3}$$

$$\theta(z) = (h+1)a + \theta(y). \tag{4}$$

Equalities (3) and (4) are mod 2π. Since $|x - y| < \delta$ and $|z - y| < \delta$, subtracting (3) from (4) yields $|a + \theta(y) - \theta(x)| = |\theta(z) - \theta(y)| < \eta$, hence $|a| \leqslant \eta + |\theta(y) - \theta(x)| < 2\eta$. Since $\eta > 0$ is arbitrary, we obtain $a = 0$; hence $c = 1$. Since T is ergodic, only constant functions can have eigenvalue 1. Thus T has continuous spectrum and is therefore weakly mixing. We note that since T is weakly mixing, there exists a sequence of density one on which T is mixing.

To find replicas of T, first consider the interval $J = [2/3, 1) = \bigcup_{n=1}^{\infty} E_n$. The stacking picture for T_J begins with E_1 playing the role of $[0, 2/3)$ in the stacking picture for T. Now E_1 in Figure 11 will be cut in three subintervals (Figure 12 with $n = 2$) and E_2 is placed above the middle third. Thus E_2 for T_J plays the role of E_1 for T. In general, E_{n+1} for T_J plays the role of E_n for T. It follows that T_J is a replica of T.

Fix k and let $J = \bigcup_{i=k}^{\infty} E_i = [\,1 - 1/3^k, 1)$. The stacking picture for T_J begins with E_k playing the role of $[0, 2/3)$ for T. In general, E_{n+k} for T_J plays the role of E_n for T. It follows that T_J is a replica of T for each k. Q.E.D.

In general, one can see that if I is a rung in L_n and $A = I \cup \bigcup_{i=n}^{\infty} E_i$, then T_A is a copy of T. Furthermore, it follows from the general theory in **[15]** that for $m(B) > 0$ there exists $A \subset B$ such that T_A is a copy of T.

The construction of Chacon's transformation is relatively simple and yet the transformation has some remarkable properties [5], two of which we shall describe. Ergodicity states that the iterates of a set of positive measure sweep out the whole

space. A transformation is *prime* if the iterates of each non-trivial set generate the whole σ-algebra $\mathscr{B}$. In general, let $\mathscr{B}(T, B)$ denote the smallest complete σ-algebra containing all iterates $T^i(B)$, $i \in Z$. A transformation T is prime if $0 < m(B) < 1$ implies $\mathscr{B}(T, B) = \mathscr{B}$. The first transformation shown to be prime was a mixing transformation constructed by Ornstein using stacking **[14]**. It was later discovered that Chacon's transformation was also prime **[4, 5]**.

The original purpose of Ornstein's example **[14]** was to construct a mixing transformation with no roots. A transformation S is a kth root of T if $S^k = T$. S *commutes with* T if $ST = TS$. The *centralizer* $C(T)$ *of* T is the class of all transformations commuting with T. All iterates T^i, $i \in Z$, are in $C(T)$ and roots of T are in $C(T)$. It can be shown that if T is ergodic, then a root of T cannot be an iterate of T. A transformation has *trivial centralizer* if $C(T) = \{T^i : i \in Z\}$. Ornstein's mixing transformation **[14]** has trivial centralizer and, hence, no roots. It was later shown that Chacon's transformation also has trivial centralizer **[4]** and hence also has no roots.

The properties of having trivial centralizer and primeness are both implied by the deep property of having *minimal self-joinings*, which was introduced by Rudolph **[16]**. In **[5]** it was shown that Chacon's transformation has the latter property, and therefore Chacon's transformation can be used to construct the exotic examples in **[16]**.

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Improving the Cayley-Hamilton Equation for Low-Rank Transformations

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I. INTRODUCTION. According to the classical Cayley-Hamilton theorem (e.g., **[1]**), every matrix satisfies its characteristic equation. As perhaps first noticed by Lehti (**[3]**), the usual Cayley-Hamilton equation for a low rank linear transformation (or matrix) contains redundant factors, whose removal results in an equation similar to the original, but of lower degree. The purpose of this article is to present a simple derivation of this modified Cayley-Hamilton equation. Alternative approaches can be found in (**[3]**) and (**[4]**).

II. A PROOF OF THE MODIFIED CAYLEY-HAMILTON EQUATION. Let V be an n-dimensional real or complex linear space and let A: $V \to V$ be a linear transformation. The equation $\det(A - \lambda I) = 0$, where I denotes the identity transformation (or the unit matrix), is the *characteristic equation* of A. Expanding and multiplying by $(-1)^n$, we obtain

$$\lambda^n - \alpha_1\lambda^{n-1} + \cdots + (-1)^p\alpha_p\lambda^{n-p} + \cdots + (-1)^n\alpha_n = 0. \tag{1}$$

The coefficients α_1 and α_n are the familiar *trace* and *determinant*, respectively, of A. We shall refer to the coefficients α_k, $k = 1, 2, \dots, n$, as the *scalar invariants* of A, a convenient term probably coined by Lehti (e.g. [2]). Replacing λ by A in (1), we have the well-known *Cayley-Hamilton equation* of A:

$$A^n - \alpha_1 A^{n-1} + \cdots + (-1)^p\alpha_p A^{n-p} + \cdots + (-1)^n\alpha_n I = 0. \tag{2}$$

For arbitrary $x_1, x_2, \dots, x_n \in V$ the number α_k, $k = 1, 2, \dots, n$, satisfies the identity

$$\alpha_k\Delta(x_1, \dots, x_n) = \sum \Delta(u_1, \dots, u_n), \tag{3}$$

where Δ is a determinant-function in V, and the sum on the right contains $\binom{n}{k}$ terms, each having $u_i = Ax_i$ for k indices, and $u_i = x_i$ for the remaining $(n-k)$ indices. With $n = 3$ and $k = 2$ we have thus for instance $\alpha_2\Delta(x_1, x_2, x_3) = \Delta(Ax_1, Ax_2, x_3) + \Delta(Ax_1, x_2, Ax_3) + \Delta(x_1, Ax_2, Ax_3)$. The case $k = n$ ($\alpha_k = \alpha_n = \det(A)$) is well-known (**[1]**, p. 50; the cases $k = 1$ and $k = 2$ are also touched upon the **[1]**, p. 67 and p. 70). From this special case the general formula follows fairly easily, if we replace A by $A - \lambda I$ and use the multilinearity of the determinant-function. The identity (3) can also be considered as the definition of the α_k:*s* (**[2]**).

Let p be the rank of A ($p \leqslant n$), let $W \subset V$ denote the range $A(V)$ of A, and let $\tilde{A}$: $W \to W$ be the restriction of A to W. We have $p = \dim(W)$. The Cayley-Hamilton equation of $\tilde{A}$ has the form

$$\tilde{A}^p - \beta_1 \tilde{A}^{p-1} + \cdots + (-1)^p \beta_p \tilde{I} = 0, \tag{4}$$

where the β_k:*s* are the scalar invariants of $\tilde{A}$.

Lemma. *The* β_k:*s equal the corresponding* α_k:*s*:

$$\beta_k = \alpha_k, \qquad k = 1, 2, \ldots, p. \tag{5}$$

Proof: Let $\{a_1, \ldots, a_p\}$ be a basis of W, $\{a_1, \ldots, a_n\}$ a basis of V, and Δ a determinant-function in V. Choosing $x_i = a_i$ in (3), we have

$$\alpha_k \Delta(a_1, \ldots, a_n) = \sum \Delta(u_1, \ldots, u_n). \tag{6}$$

The terms $\Delta(u_1, \ldots, u_n)$ of (6) in which A operates on at least one, say a_m, of the vectors $a_{p+1}, \ldots, a_n$, vanish, because the vectors $u_1, \ldots, u_p$, Aa_m, all belonging to the p-dimensional space W, are linearly dependent. Consequently, the sum in (6) shrinks down to a sum

$$\sum \Delta(u_1, \ldots, u_p, a_{p+1}, \ldots, a_n)$$

of only $\binom{p}{k}$ terms, and equation (6) can be written in the form

$$\alpha_k \Delta_1(a_1, \ldots, a_p) = \sum \Delta_1(u_1, \ldots, u_p), \tag{7}$$

where $\Delta_1(x_1, \ldots, x_p) \equiv \Delta(x_1, \ldots, x_p, a_{p+1}, \ldots, a_n)$ is a determinant-function in W. Equation (7), the analogue in W to (3) in V, shows that the α_k:*s* coincide with the scalar invariants of the linear transformation $\tilde{A}$ in W, q.e.d.

Application of our lemma to (4) gives us the equation

$$\tilde{A}^p - \alpha_1 \tilde{A}^{p-1} + \cdots + (-1)^p \alpha_p \tilde{I} = 0. \tag{8}$$

We now note that

$$\tilde{A}A = A^2. \tag{9}$$

Indeed, $\tilde{A}(Ax) = A(Ax)$ for all $x \in V$, since $Ax \in W$. Multiplying (8) by A from the right and using (9), we obtain

$$A^{p+1} - \alpha_1 A^p + \cdots + (-1)^p \alpha_p A = 0. \tag{10}$$

If p is n or $n-1$, equation (10) does not represent anything new or interesting compared to (2). For $p < n-1$, however, the left-hand side of (10) is of lower degree than the left-hand side of (2). In this case (10) can accordingly be considered as an improvement of the original Cayley-Hamilton equation. For $k > p$, all terms in the right-hand sum of (3) contain linearly dependent argument-vectors. Hence $\alpha_{p+1} = \alpha_{p+2} = \cdots = \alpha_n = 0$ and (2) assumes, in fact, the form

$$A^n - \alpha_1 A^{n-1} + \cdots + (-1)^p \alpha_p A^{n-p} = 0. \tag{11}$$

Comparing (11) to (10), we observe that our result may be interpreted as follows: If $p < n-1$, the Cayley-Hamilton equation (10) of A contains $n-p-1$ redundant factors A.

III. EXAMPLE. For the matrix

$$M = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 3 & 2 & 1 & 0 \\ 4 & 4 & 4 & 4 \end{pmatrix}$$

n is four and p is two, as the third and fourth row are constructed as simple linear combinations of the first and second row. The characteristic polynomial of M is $\lambda^4 - 12\lambda^3 - 36\lambda^2$ and the Cayley-Hamilton equation accordingly $M^4 - 12M^3 - 36M^2 = 0$. Dropping the redundant factor M, we obtain the "improved Cayley-Hamilton equation"

$$M^3 - 12M^2 - 36M = 0. \tag{12}$$

IV. REMARK. The so-called minimum polynomial of A always provides the best possible improvement of the Cayley-Hamilton equation. Finding the minimum polynomial is, however, generally a rather laborious task compared to a straightforward computation of the characteristic polynomial.

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Bessel Functions and Kepler's Equation

Peter Colwell

1. INTRODUCTION. Toward the end of a first course in differential equations, we may often use the method of Frobenius to show that *Bessel's differential equation of integral order n*

$$x^2y'' + xy' + (x^2 - n^2)y = 0 \tag{1}$$

has a solution

$$y = J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+n}}{2^{2k+n}k!(k+n)!} \tag{2}$$

called the *Bessel function of the first kind of order n*.

This essay is a partial account of connections between (1), (2), Bessel, and Kepler's Equation, a transcendental equation of celestial mechanics with a rich and extensive history.

2. KEPLER'S EQUATION. After years of work, Johannes Kepler announced three laws of planetary motion early in the seventeenth century.

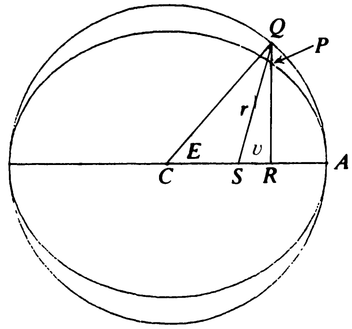
Kepler's three laws state that the planets move in elliptical orbits in a common plane with the sun at one focus, that for each planet the line connecting the sun with the planet sweeps out equal areas in equal times and that the ratio of the square of the period of revolution of each planet to the cube of the semimajor axis of its orbit is the same for all planets.

Kepler stated the first two laws in 1609 in the *Astronomia Nova* and the third in 1619 in *The Harmony of the World*. As we know now, these laws are only approximations, but for the six planets known at the time and to the limits of observation then they were essentially exact.

Kepler's Equation is a consequence of the first two laws only.

Suppose a planet moves in the counterclockwise direction in an elliptical orbit with the sun at one focus which has eccentricity $e, 0 < e < 1$, has semimajor axis a, and is traveled once in time T. In the figure, A denotes perihelion, C center of the orbit, and S the position of the sun. If, having passed through A, the planet after elapsed time t is at position P, we wish to express the polar coordinates of $P, (r, v)$, relative to S at time t.

The quantity v = angle PSA is called the *true anomaly* of the planet at time t. The circle centered at C with radius a is called the eccentric circle. If we draw the line P perpendicular to radius CA and mark R, its intersection with CA, and Q, its intersection with the eccentric circle, the quantity E = angle QCA is called the *eccentric anomaly* of the planet at time t.



The relation between r and v is

$$r = \frac{a(1-e^2)}{1+e\cos v}. \tag{3}$$

With trigonometry and algebra, we may derive

$$r = a(1 - e\cos E) \quad \text{and} \quad \tan\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\,\tan\frac{E}{2}. \tag{4}$$

Thus (r, v) may be obtained from E.

Kepler's Equation relates E to t by means of a quantity $M = 2\pi t/T$ called the *mean anomaly* of the planet at time t.

The relation between E and M (and so t) comes through Kepler's second law.

$$\text{Area } PSA = (t/T)(\text{Area enclosed in the orbit}) = (1/2)Ma^2\sqrt{1-e^2}$$

and

$$\text{Area } PSA = \text{Area } PSR + \text{Area } PRA = \frac{1}{2}a^2\sqrt{1-e^2}\,(E - e\sin E).$$

The result is Kepler's Equation (KE):

$$M = E - e\sin E.$$

If we know t and M, and if we can solve (KE) for E, then we can find (r, v) from (4). More details and background appear in [**10**].

3. LAGRANGE'S SOLUTION OF (KE). From the time of Kepler, many efforts had been made to solve (KE), at least approximately. In 1770, J. L. Lagrange [**7**] showed that under suitable conditions an equation of the form

$$w = a + t\phi(w) \tag{5}$$

would have a solution for w and for any function f it would be true that

$$f(w) = f(a) + \sum_{n=1}^{\infty} \frac{t^n}{n!} \frac{d^{n-1}}{da^{n-1}} \{f'(a)[\phi(a)]^n\}. \tag{6}$$

In the special case: $f(z) = z, \phi(z) = \sin z, a = M, t = e$ and $w = E$, (5) becomes (KE) and (6) reads

$$E = M + \sum_{n=1}^{\infty} \frac{e^n}{n!} \frac{d^{n-1}}{dM^{n-1}} (\sin^n M) = M + \sum_{n=1}^{\infty} a_n(M)e^n. \tag{7}$$

Lagrange applied his result to (KE) in [8], and later there were many efforts to find explicit formulas for the coefficients $\{a_n(M)\}$ in (7).

4. BESSEL AND (KE). F. W. Bessel, an eminent German astronomer of the early nineteenth century, was well-acquainted with Lagrange's solution of (KE). Where Lagrange used repeated differentiation, Bessel used integration and described a pretty solution of (KE) in the form

$$E = M + \sum_{n=1}^{\infty} b_n(e)\sin nM, \tag{8}$$

that is, as a Fourier sine series.

Here is what he did. If $E = g(M)$ is the solution of (KE), g has $M = 0$ and $M = \pi$ as fixed points, and if

$$g(M) - M = \sum_{n=1}^{\infty} b_n(e)\sin nM \tag{9}$$

on the interval $0 \leq M \leq \pi$, then

$$b_n(e) = \frac{2}{\pi}\int_0^{\pi}[g(M) - M]\sin nM\,dM = \frac{2}{n\pi}\int_0^{\pi}\cos nM\,dg(M).$$

Since $M = E - e\sin E = g(M) - e\sin(g(M))$,

$$b_n(e) = \frac{2}{n}\left\{\frac{1}{\pi}\int_0^{\pi}\cos(nE - ne\sin E)\,dE\right\}.$$

In modern notation Bessel's definition of $J_n(x)$ was

$$J_n(x) = \frac{1}{\pi}\int_0^{\infty}\cos(nE - x\sin E)\,dE \tag{10}$$

and Bessel's solution of (KE) was

$$E = M + \sum_{n=1}^{\infty}\left\{\frac{2}{n}J_n(ne)\right\}\sin nM. \tag{11}$$

It is not at all obvious that (10) and (2) describe the same function. In a landmark paper of 1824 [2] Bessel showed that the functions (10) indeed do satisfy the differential equation (1) and at $x = 0$ have the same value and derivative value as (2). In this paper Bessel also derived many of the standard identities for $J_n(x)$, but there is no explicit mention of (KE). The solution of (KE) leading to (11) was actually done in an 1818 letter to W. Olbers [1] and Bessel expressed his surprise that nobody had heretofore discovered it.

5. OTHER ANTECEDENTS TO BESSEL FUNCTIONS. Although it is simpler to say that Bessel invented Bessel functions in order to solve (KE), he didn't invent them and he didn't consider (KE) the most important of the problems of celestial mechanics which lead him to them.

Lagrange, [8], also tried to write his solution of (KE) in the form (8) and determined in series form the coefficients $b_1(e), b_2(e), b_3(e)$. While Lagrange seems to be the first to have encountered Bessel functions in form (2) in the

context of (KE), functions in form (2) had been encountered in special cases as early as 1703 and in considerable generality by Euler around 1764, [**13**, p. 356].

Even in the area of celestial mechanics Bessel's definition (10) finds some close antecedent in the work of S. D. Poisson [**9**], [**12**, p. 6]. And almost simultaneously with Bessel, F. Carlini [**3**], [**4**] derived a series expression for the true anomaly, v, in the form

$$v = M + \sum_{n=1}^{\infty} G_n(e)\sin nM$$

by starting with Lagrange's theorem. Carlini's expressions for $G_n(e)$ were not in the form of integrals, but in modern notation they satisfy the identities, [**11**, p. 59],

$$G_n(e) = \frac{2}{n}J_n(ne) + \sum_{m=1}^{\infty} \alpha^m\left[J_{n-m}(ne) + J_{n+m}(ne)\right], \qquad e = 2\alpha/(1+\alpha^2).$$

Carlini's work attracted very little attention until C. G. J. Jacobi in 1850 translated it into German, correcting parts of it and adding extensive commentary, [**5**]. In 1893, M. W. Kapteyn, [**6**], motivated by the literature of (KE) and celestial mechanics, studied the possibility of representing functions in the form

$$f(x) = \sum_{n=0}^{\infty} c_n(f)J_n(nx),$$

which are now called *Kapteyn series*, and of which the first is (11) solving (KE).

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Löwner's Inverse Coefficients Theorem for Starlike Functions

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DeBranges' proof of Bieberbach's conjecture resolved the most popular problem in geometric function theory for the class $\mathcal{S}$ ([1], [2], [3], [5]). $\mathcal{S}$ consists of all functions $f(z)$ holomorphic and one-to-one in the disk Δ: $|z| < 1$ with a series representation of the form $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ in Δ. DeBranges proved that $|a_n| \leq n$, $n = 2, 3, \ldots$, for all $f(z)$ in $\mathcal{S}$ and that equality holds only for the Koebe function $k(z) = z/(1+z)^2 = z - 2z^2 + 3z^3 + \cdots$ or its rotations $e^{i\alpha}k(e^{-i\alpha}z)$.

DeBranges' proof requires use of the parametric method due to K. Löwner ([4], [6]) with which Löwner was able to resolve completely the analog of Bieberbach's conjecture for inverses. Suppose $f(z)$ is in $\mathcal{S}$ and $F(w) = w + \beta_2 w^2 + \beta_3 w^3 + \cdots$ is its inverse. Löwner showed that for each n, $|\beta_n| \leqslant K_n$, where $K_n = (2n)!/n!(n+1)!$ and $K(w) = w + K_2 w^2 + K_3 w^3 + \cdots$ is the inverse of the Koebe function.

A function $f(z)$ in $\mathcal{S}$ is starlike if $f[\Delta]$, the image of Δ under $f(z)$, is starlike with respect to the origin, i.e., the segment $[0, f(z)]$ lies in $f[\Delta]$ for each z in Δ. The purpose of our note is to give a proof of Löwner's theorem for starlike functions which is elementary and does not require Löwner's technique.

Theorem. *If $s(z)$ is starlike and in $\mathcal{S}$, and its inverse is $S(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \cdots$, then $|\gamma_n| \leqslant (2n)!/n!(n+1)!$; equality holds for the inverse of the Koebe function.*

Proof: $s(z)$ is starlike if and only if $Re\{zs'(z)/s(z)\} > 0$ for z in Δ, ([1], [5]). This means there is a $P(z) = 1 + c_1 z + c_2 z^2 + \cdots$ with $Re\{P(z)\} > 0$ such that

$$\frac{zs'(z)}{s(z)} = \frac{1}{P(z)}.$$

Letting $w = s(z)$ and recalling that $s'(z)S'(w) = 1$, we may write

$$\frac{wS'(w)}{S(w)} = P(S(w)),$$

or

$$wS'(w) - S(w) = S(w)\{P(S(w)) - 1\}$$
$$= \sum_{j=1}^{\infty} c_k (S(w))^{j+1}.$$

Equating coefficients of like powers of w gives

$$(n-1)\gamma_n = \sum_{j=1}^{n-1} c_j\{S^{j+1}(w)\}_n,$$

where $\{S^{j+1}(w)\}_n$ is the coefficient of w^n for $S^{j+1}(w)$. From this relation it is easy to conclude the theorem is true for $n = 2$.

Furthermore,

$$\{S^{j+1}(w)\}_n = Q_j(\gamma_2, \gamma_3, \dots, \gamma_{n-1})$$

is a polynomial in $\gamma_2, \gamma_3, \dots, \gamma_{n-1}$ with *non-negative* coefficients, consequently

$$\left|\{S^{j+1}(w)\}_n\right| \leqslant Q_j(|\gamma_2|, |\gamma_3|, \dots, |\gamma_{n-1}|).$$

We now proceed inductively and assume the theorem to be true for $j \leq (n-1)$. Then $|\gamma_j| \leq K_j$ for such j and

$$\left|\{S^{j+1}(w)\}_n\right| \leq Q_j(K_2, K_3, \dots, K_{n-1}) = \{K^{j+1}(w)\}_n.$$

For each j, $|c_j| \leqslant 2$, ([1], [5]) and it follows that

$$(n-1)|\gamma_n| \leqslant \sum_{j=1}^{n-1} |c_j| \cdot \left|\{S^{j+1}(w)\}_n\right| \leqslant 2\sum_{j=1}^{n-1} \{K^{j+1}(w)\}_n.$$

To complete the proof we need only show that the last sum is $(n-1)K_n$.

$k(z)$ is itself starlike, it maps the disk onto the plane cut along $[\frac{1}{4}, +\infty)$, and satisfies the equation

$$\frac{zk'(z)}{k(z)} = \frac{1-z}{1+z}.$$

Proceeding as above we have

$$\frac{wK'(w)}{K(w)} = \frac{1+K(w)}{1-K(w)},$$

or

$$K'(w) - K(w) = 2\frac{K^2(w)}{1-K(w)} = 2\sum_{j=2}^{\infty} K^j(w).$$

Comparing coefficients of w^n gives $(n-1)K_n = 2\Sigma_{j=1}^{n-1}\{K^{j+1}(w)\}_n$. This completes the proof. ■

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Bôcher's Theorem

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INTRODUCTION. Bôcher's Theorem characterizes the behavior of positive harmonic functions in the neighborhood of an isolated singularity. Let n denote a positive integer greater than 1. Recall that a real-valued function u defined on an open set $\Omega \subset \mathbf{R}^n$ is said to be *harmonic* in Ω if u is twice continuously differentiable and

$$\Delta u \equiv 0$$

in Ω, where

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}.$$

Let B_n denote the open unit ball in $\mathbf{R}^n$. If $n = 2$, the function $\log(1/|x|)$ is positive and harmonic in $B_2 \setminus \{0\}$, while if $n > 2$, the function $|x|^{2-n}$ is positive and harmonic in $B_n \setminus \{0\}$. Bôcher's Theorem illustrates how important these particular functions are:

Bôcher's Theorem. *Suppose u is positive and harmonic in $B_n \setminus \{0\}$. Then there exists a function v harmonic in B_n and a constant $a \geqslant 0$ such that*

(i) $u(x) = a \log(1/|x|) + v(x)$ *for all* $x \in B_2 \setminus \{0\}$ (*if* $n = 2$);

(ii) $u(x) = a|x|^{2-n} + v(x)$ *for all* $x \in B_n \setminus \{0\}$ (*if* $n > 2$).

The usual proofs of Bôcher's Theorem rely either on the theory of superharmonic functions ([**4**], Theorem 5.4) or series expansions using spherical harmonics ([**5**], Chapter X, Theorem XII). (The referee has called our attention to the proof given by G. E. Raynor [**7**]. Raynor points out that the original proof of Maxime Bôcher [**2**] implicitly uses some non-obvious properties of the level surfaces of a harmonic function.) In this paper we offer a different and simpler approach to this theorem. The only results about harmonic functions needed are the minimum principle, Harnack's Inequality, and the solvability of the Dirichlet problem in B_n.

We will investigate a harmonic function by studying its dilates. For u a function defined on $B_n \setminus \{0\}$ and $r \in (0,1)$, the *dilate* u_r is the function defined on $(1/r)B_n \setminus \{0\}$ by

$$u_r(x) = u(rx).$$

Note that every dilate of a harmonic function is harmonic.

For convenience, we assume in the rest of this paper that $n > 2$; all statements and proofs will easily carry over to the $n = 2$ case (with $\log(1/|x|)$ in place of $|x|^{2-n}$).

SPHERICAL AVERAGES. Let S denote the unit sphere in $\mathbf{R}^n$. Given a continuous function u defined in $B_n \setminus \{0\}$, we define $A[u](x)$, the average of u over the sphere of radius $|x|$, by

$$A[u](x) = \frac{1}{\sigma(S)} \int_S u(|x|\zeta)\, d\sigma(\zeta) \qquad (x \in B_n \setminus \{0\});$$

here σ denotes surface-area measure.

The following lemma is well known to potential theorists. The elementary proof given here was suggested by the referee.

Lemma 1. *If u is harmonic in $B_n \setminus \{0\}$, then there are constants a and b such that*

$$A[u](x) = a|x|^{2-n} + b$$

for all $x \in B_n \setminus \{0\}$. In particular, $A[u]$ is harmonic in $B_n \setminus \{0\}$.

Proof: Define f on $(0,1)$ by

$$f(r) = \frac{1}{\sigma(S)} \int_S u(r\zeta)\, d\sigma(\zeta);$$

so $A[u](x) = f(|x|)$. Because u is continuously differentiable on $B_n \setminus \{0\}$, we can compute f' by differentiating under the integral sign, obtaining

$$f'(r) = \frac{1}{\sigma(S)} \int_S \zeta \cdot (\nabla u)(r\zeta)\, d\sigma(\zeta) = \frac{r^{-n}}{\sigma(S)} \int_{rS} \tau \cdot (\nabla u)(\tau)\, d\sigma(\tau).$$

Let $0 < r_0 < r_1 < 1$, and let $\Omega = \{x \in \mathbf{R}^n : r_0 < |x| < r_1\}$. The divergence theorem, applied to ∇u, shows that

$$\int_{\partial\Omega} \mathbf{n} \cdot (\nabla u)(\tau)\, d\sigma(\tau) = \int_\Omega (\Delta u)(\tau)\, dV(\tau);$$

here $\mathbf{n}$ denotes the outward unit normal on $\partial\Omega$, σ denotes surface-area measure on $\partial\Omega$, and V denotes Lebesgue volume measure on $\mathbf{R}^n$. Because u is harmonic on Ω, the right hand side of the equation above is 0. Note also that $\partial\Omega = r_0 S \cup r_1 S$ and that $\mathbf{n} = -\tau/r_0$ on $r_0 S$ and $\mathbf{n} = \tau/r_1$ on $r_1 S$. Thus the equation above implies that

$$\frac{1}{r_0} \int_{r_0 S} \tau \cdot (\nabla u)(\tau)\, d\sigma(\tau) = \frac{1}{r_1} \int_{r_1 S} \tau \cdot (\nabla u)(\tau)\, d\sigma(\tau),$$

which means $f'(r)$ is a constant multiple of r^{1-n} (for $0 < r < 1$). Hence $f(r)$ is of the form $ar^{2-n} + b$, as desired. ■

Remark. Lemma 1 shows that every radial harmonic function in $B_n \setminus \{0\}$ has the form $a|x|^{2-n} + b$ (a function is called *radial* if its value at x depends only on $|x|$).

Lemma 2. *There exists a positive constant α such that for every positive harmonic u in $B_n \setminus \{0\}$,*

$$\alpha u(y) < u(x) \text{ whenever } 0 < |x| = |y| \leqslant 1/2.$$

Proof: Harnack's Inequality (see [4], Theorem 2.16) states that if Ω is a connected open subset of $\mathbf{R}^n$ and K is a compact subset of Ω, then there is a positive

constant α such that

$$\alpha u(y) < u(x)$$

for every positive harmonic function u in Ω and all $x, y \in K$. Thus there exists $\alpha > 0$ such that for all positive harmonic u in $B_n \setminus \{0\}$, $\alpha u(y) < u(x)$ whenever $|x| = |y| = 1/2$. Applying this result to the dilates u_r, $0 < r < 1$, gives the desired conclusion. ■

Lemma 3. *If u is positive and harmonic in $B_n \setminus \{0\}$ and $u(x) \to 0$ as $|x| \to 1$, then there is a positive constant a such that*

$$u(x) = a(|x|^{2-n} - 1)$$

for all $x \in B_n \setminus \{0\}$.

Proof: By Lemma 1, we need only show that $u = A[u]$ in $B_n \setminus \{0\}$. Suppose we could show that $u \geqslant A[u]$ in $B_n \setminus \{0\}$. Then if there were a point $x \in B_n \setminus \{0\}$ such that $u(x) > A[u](x)$, we would have

$$A[u](x) > A[A[u]](x) = A[u](x),$$

a contradiction. Thus we need only prove that $u \geqslant A[u]$ in $B_n \setminus \{0\}$, which we now do.

Let α be the constant of Lemma 2. Then by Lemma 1, $u - \alpha A[u]$ is harmonic in $B_n \setminus \{0\}$. By Lemma 2, $u(x) - \alpha A[u](x) > 0$ if $0 < |x| \leqslant 1/2$, and clearly $u(x) - \alpha A[u](x) \to 0$ as $|x| \to 1$ by our hypothesis on u. The minimum principle for harmonic functions thus shows that $u - \alpha A[u] > 0$ in $B_n \setminus \{0\}$.

We wish to iterate this result. For this purpose, define

$$f(t) = \alpha + t(1 - \alpha), \qquad t \in [0,1].$$

Suppose we know

$$w = u - tA[u] > 0 \quad \text{in } B_n \setminus \{0\} \tag{$*$}$$

for some $t \in [0,1]$. Since $w(x) \to 0$ as $|x| \to 1$, the above argument may be applied to w, yielding

$$w - \alpha A[w] = u - f(t)A[u] > 0 \quad \text{in } B_n \setminus \{0\}.$$

This process may be continued. Letting $f^{(m)}$ denote the m^{th} iterate of f, we see that $(*)$ implies

$$u - f^{(m)}(t)A[u] > 0 \quad \text{in } B_n \setminus \{0\}$$

for $m = 1, 2\ldots$. But $f^{(m)}(t) \to 1$ as $m \to \infty$, for every $t \in [0,1]$, so that $(*)$ holding for some $t \in [0,1]$ implies $u - A[u] \geqslant 0$ in $B_n \setminus \{0\}$. Since $(*)$ obviously holds when $t = 0$, we have $u - A[u] \geqslant 0$ in $B_n \setminus \{0\}$, as desired. ■

PROOF OF BÔCHER'S THEOREM. We first assume that u is positive and harmonic on a neighborhood of $\overline{B}_n \setminus \{0\}$. For $x \in B_n \setminus \{0\}$, define

$$w(x) = u(x) - P[u|_S](x) + |x|^{2-n} - 1;$$

here $P[u|_S]$ denotes the Poisson integral of $u|_S$ (the unique harmonic function in B_n that extends continuously to $\overline{B}_n$ with boundary values $u|_S$). As $|x| \to 1$, we have $w(x) \to 0$, and as $|x| \to 0$, we have $w(x) \to +\infty$. By the minimum principle, w is positive and harmonic in $B_n \setminus \{0\}$. Lemma 3, applied to w, shows that $u(x) = a|x|^{2-n} + v(x)$ in $B_n \setminus \{0\}$ for some v harmonic in B_n and some constant a. To finish the proof of Bôcher's Theorem in this special case, note that a must

be nonnegative, because otherwise $u(x) \to -\infty$ as $x \to 0$, which would violate the positivity of u.

For the general positive harmonic u in $B_n \setminus \{0\}$, we may apply the above result to $u_{1/2}$, so that

$$u(x/2) = a|x|^{2-n} + v(x) \quad \text{in } B_n \setminus \{0\}$$

for some v harmonic in B_n and some constant $a \geqslant 0$. This implies

$$u(x) = a2^{2-n}|x|^{2-n} + v(2x) \quad \text{in } (\tfrac{1}{2})B_n \setminus \{0\},$$

which shows that $u(x) - a2^{2-n}|x|^{2-n}$ extends harmonically to $(\frac{1}{2})B_n$, and hence to B_n. The proof of Bôcher's Theorem is complete. ■

POSITIVE HARMONIC FUNCTIONS ON $\mathbf{R}^n \setminus \{0\}$. We conclude this note by characterizing the positive harmonic functions on $\mathbf{R}^n \setminus \{0\}$. The proof uses Bôcher's Theorem and the well known result that a positive harmonic function on all of $\mathbf{R}^n$ is constant (see Note 1 below).

Corollary.
(i) *If u is positive and harmonic on $\mathbf{R}^2 \setminus \{0\}$, then u is constant.*
(ii) *If u is positive and harmonic on $\mathbf{R}^n \setminus \{0\}$ $(n > 2)$, then there are nonnegative constants a and b such that*

$$u(x) = a|x|^{2-n} + b$$

for all $x \in \mathbf{R}^n \setminus \{0\}$.

Proof: (i). If u is positive and harmonic on $\mathbf{R}^2 \setminus \{0\}$, then the function $u(e^z)$ is positive and harmonic on $\mathbf{R}^2$ $(= \mathbf{C})$ and hence is constant. This proves u is constant.

(ii). If u is positive and harmonic on $\mathbf{R}^n \setminus \{0\}$, we may write

$$u(x) = a|x|^{2-n} + v(x)$$

in $B_n \setminus \{0\}$, as in (ii) of Bôcher's Theorem. The function v extends harmonically to all of $\mathbf{R}^n$ by setting $v(x) = u(x) - a|x|^{2-n}$ for $x \in \mathbf{R}^n \setminus B_n$. We may thus apply the minimum principle to v: For any fixed $x \in \mathbf{R}^n$ and every $r > |x|$ we have

$$v(x) \geqslant \min\{v(\zeta)\colon |\zeta| = r\} > -a|r|^{2-n},$$

where the positivity of u gives the second inequality. Letting $r \to \infty$, we see that v is nonnegative and harmonic on $\mathbf{R}^n$ and hence is constant. This completes the proof. ■

Notes. 1. For the convenience of the reader, we sketch a simple proof (inspired by Nelson [**6**]) that a positive harmonic function v on $\mathbf{R}^n$ is constant; for the standard proof see [**3**], Theorem 1.19. Let $B(x, r)$ denote the open ball in $\mathbf{R}^n$ with center x and radius r. Fix $x \in \mathbf{R}^n$, $x \neq 0$, and let $r > |x|$. The volume version of the mean value property shows that $(v(x) - v(0))V(B(0, r))$ equals the difference of the integrals of v over $B(x, r)$ and $B(0, r)$. In this difference the integral of v over $B(x, r) \cap B(0, r)$ cancels, making $|(v(x) - v(0))V(B(0, r))|$ less than the integral of v over the symmetric difference of these balls (we have used the positivity of v here). This integral is less than the integral of v over $B(0, r + |x|) \setminus B(0, r - |x|)$, which we may compute exactly using the volume mean value property. It follows

that

$$|\nu(x) - \nu(0)| < \frac{(r + |x|)^n - (r - |x|)^n}{r^n}\nu(0).$$

The last term tends to zero as $r \to \infty$, and thus $\nu(x) = \nu(0)$, proving that ν is constant.

2. Another proof of Bôcher's Theorem, again quite different from the classical proofs, will appear in [**1**].

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On the Determination of the Intermediate Point in Taylor's Theorem

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1. INTRODUCTION. Where is the intermediate point in Taylor's Theorem exactly located? Let $f\colon I \to R$ be a n-times differentiable function in the open interval I containing the point a. In elementary calculus courses it is taught that for each $x \in I$ there is a point ξ between a and x satisfying

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n)}(\xi)}{n!}(x-a)^n. \tag{1}$$

But no additional information is given about the location of ξ within (a, x) (suppose momentarily $a < x$). S. Haber and O. Shisha [3] showed that under suitable conditions the point ξ lies in the left half of (a, x). Here a method is shown of approximating ξ by means of a sequence converging to it.

By Rolle's Theorem it is a simple matter to see that if $f^{(n+1)}$ exists and does not vanish in I then the point ξ that solves (1) is unique; in this case, ξ is a well defined single valued function of x, $\xi = \xi(x)$. We make the convention $\xi(a) = a$ so that the function ξ is continuous in I.

Let n be a positive integer, which will be fixed throughout this article. Given a function f as above, we shall denote $\mathscr{F}(f)$ the set $\{k \geq 1\colon f^{(n+k)}(a) \neq 0\}$. Suppose that $\mathscr{F}(f)$ is not empty for some f. The minimum of $\mathscr{F}(f)$ will be denoted by ν. Finally, λ will be the number defined by $\lambda = \binom{n+\nu}{n}^{-1/\nu}$.

2. RESULTS. If $\mathscr{F}(f) = \phi$, then f is a polynomial of degree less than or equal to n and the problem becomes trivial.

Theorem 1. *Let f be a function such that $\mathscr{F}(f) \neq \phi$. Assume that $f^{(n+\nu)}$ exists, is continuous, and does not vanish in I. Then $\xi(x)$ is differentiable at a and $\xi'(a) = \lambda$.*

Proof: If $x \in I$, applying Taylor's Theorem to the functions f and $f^{(n)}$,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+\nu)}(\tau)}{(n+\nu)!}(x-a)^{n+\nu}$$

and

$$f^{(n)}(\xi) = f^{(n)}(a) + \frac{f^{(n+\nu)}(\sigma)}{\nu!}(\xi-a)^\nu.$$

With simple manipulations the last two expressions, together with (1), yield

$$\frac{\xi(x)-a}{x-a} = \lambda \left\{ \frac{f^{(n+\nu)}(\tau)}{f^{(n+\nu)}(\sigma)} \right\}^{1/\nu}.$$

Letting $x \to a$ the theorem is established.

It may be surprising that the value of $\xi'(a)$ does not depend on the particular form of f but on the number of consecutive zeros of $f^{(k)}(a)$ for $k > n$. The higher derivatives $\xi^{(k)}(a)$, however, if they exist, do depend on the values of $f^{(k)}(a)$, $k > n$. By a similar approach we can find all the values of the successive derivatives of ξ at a whenever they are defined. To this end, the following formula for the nth derivative of a composite function will be useful [2, p. 19, formula 0.430].

$$\frac{d^n}{dx^n}F[u(x)] = \sum \frac{n!}{i!j!h!\cdots k!}\frac{d^mF}{du^m}\left(\frac{u'}{1!}\right)^i\left(\frac{u''}{2!}\right)^j\left(\frac{u'''}{3!}\right)^h\cdots\left(\frac{u^{(l)}}{l!}\right)^k.$$

Here Σ indicates summation over all solutions in non-negative integers of the equations $i + 2j + 3h + \cdots lk = n$ and $m = i + j + h + \cdots + k$. Suppose for simplicity that f belongs to $C^\infty(I)$, and that the Taylor series of f converges uniformly in I. Assume for one moment the existence of $\xi^{(k)}(a)$, $k = 2, 3, \ldots$. From (1) and

$$f(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k,$$

it follows at once

$$f^{(n)}[\xi(x)] = n!\sum_{k=n}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^{k-n}. \tag{2}$$

Differentiating ν times gives

$$f^{(n+\nu)}[\xi(x)](\xi'(x))^\nu + \frac{\nu!}{(\nu-2)!}f^{(n+\nu-1)}[\xi(x)](\xi'(x))^{\nu-2}\frac{\xi''(x)}{2!} + \cdots$$
$$= \frac{n!\nu!}{(n+\nu)!}f^{(n+\nu)}(a) + \frac{n!(\nu+1)!}{(n+\nu+1)!}f^{(n+\nu+1)}(a)(x-a) + \cdots.$$

Evaluating at $x = a$ we come out with the same result as before: $\xi'(a) = \lambda$. We can keep on with this process to determine the next values of $\xi^{(k)}(a)$. One more differentiation, for instance, leads to

$$f^{(n+\nu+1)}[\xi(x)](\xi'(x))^{\nu+1} + \frac{(\nu+1)!}{(\nu-1)!}f^{(n+\nu)}[\xi(x)](\xi'(x))^{\nu-1}\frac{\xi''(a)}{2!} + \cdots$$
$$= \frac{n!(\nu+1)!}{(n+\nu+1)!}f^{(n+\nu+1)}(a) + \frac{n!(\nu+2)!}{(n+\nu+2)!}f^{(n+\nu+2)}(a)(x-a) + \cdots. \tag{3}$$

Making $x = a$ the value of $\xi''(a)$ can be determined. By recurrence, the higher derivatives $\xi^{(k)}(a)$ can be calculated. We still need to prove the existence of $\xi^{(k)}(a), k = 2, 3, \ldots$.

Theorem 2. *If f is real analytic in I and not a polynomial of degree less than or equal to n, then $\xi^{(k)}(a)$ exists for all $k \geq 1$.*

Proof: In view of (2) we know that $f^{(n)}[\xi(x)]$ belongs to $C^\infty(I)$. On the other hand, since $\mathcal{F}(f) \neq \phi$, $f^{(n+1)}$ does not vanish in some deleted neighborhood $\mathcal{N}$ of a. It is not difficult to see that these two properties imply the existence of $\xi'(x)$ in $\mathcal{N}$. Similarly, the existence of $\xi^{(k)}(x)$ in some deleted neighborhood of a, for all $k \geq 2$, can be easily established by induction. Now, under the assumption of uniform convergence we know that the right-hand side of (3) has a limit as $x \to a$, and hence, $\xi''(x)$ has a limit as $x \to a$ as well. It is well known [**1**, p. 143, exercise 21.11] that if the limit of $\xi''(x)$ as $x \to a$ exists, so does $\xi''(a)$. The proof is now achieved by recurrence.

3. EXAMPLE. Let us compute the value of π by means of $\pi = 6\arcsin(1/2)$. For this, let $f(x) = \arcsin x$, $a = 0$, $I = (-1, 1)$ and $n = 2$:

$$\arcsin x = x + \frac{f''(\xi)}{2!}x^2 = x + \frac{\xi}{2(1-\xi^2)^{3/2}}x^2. \tag{4}$$

The first 9 derivatives of ξ are:

$$\xi'(0) = 1/3, \quad \xi'''(0) = 17/30, \quad \xi^{(5)}(0) = 779/3^3 7, \quad \xi^{(7)}(0) = 447739/2^2 3^5 5,$$
$$\xi^{(9)}(0) = 99671768/3^4 5^2 11, \quad \xi^{(11)}(0) = 5351859467/2 \cdot 3^4 \cdot 7 \cdot 13$$

(the even derivatives are all zero). But

$$\xi(x) \simeq \sum_{k=0}^{11} \frac{\xi^{(k)}(a)}{k!}x^k. \tag{5}$$

Letting $x = 1/2$ in (4) and (5) we obtain the value of π within an approximation of 9.3×10^{-7}. Substituting x by $x_0 = \sqrt{2 - \sqrt{2 + \sqrt{3}}}\,/2$, since $\pi = 24\arcsin x_0$, we obtain an approximation within 10^{-16} of π. We remark that the results derived from (4) relying in Taylor's Theorem solely (neglecting the remainder), for the same values of x and x_0 as above, give 3.0 and 3.132..., respectively, as the approximations to π.

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10190. *Proposed by Peter J. Ferraro, Roselle Park, NJ.*

Suppose t is a positive integer congruent to 1 modulo 4 but not a perfect square. Put $\alpha = (1+\sqrt{t})/2$.

(a) Prove that if n is a positive integer, then

$$1 \leqslant \lfloor \alpha^2 n \rfloor - \lfloor \alpha \lfloor \alpha n \rfloor \rfloor \leqslant \lfloor \alpha \rfloor.$$

(b) Does every integer in the interval $[1, \lfloor \alpha \rfloor]$ occur as such a difference for some positive integer n?

10191. *Proposed by Dragomir Ž. Đoković, University of Waterloo, Ontario, Canada.*

Let G be the group of $\mathbb{C}$-automorphisms of the function field $\mathbb{C}(z)$ and Σ the set of involutory automorphisms of $\mathbb{C}(z)$ which extend the complex conjugation on $\mathbb{C}$. Show that Σ splits into two orbits under the action $G \times \Sigma \to \Sigma$, $(\alpha, \beta) \mapsto \alpha \circ \beta \circ \alpha^{-1}$. (Thus there are only two essentially different ways of extending the complex conjugation to an involutory automorphism of $\mathbb{C}(z)$.)

10192. *Proposed by Paul Erdős, Hungarian Academy of Sciences, Budapest.*

Let $L(n)$ denote the least common multiple of the positive integers not exceeding n. For $n \geqslant 2$ let $g(n)$ denote the largest positive integer k such that $n^k | L(n)$. For example, $g(1) = 1$, $g(30) = 2$, $g(420) = 3$. Prove that for x large

$$\max_{2 \leqslant n \leqslant x} g(n) = \log x / \{\log\log x + o(1)\}.$$

NOTES

(10187) Notation and terminology for simple continued fractions and their sequences of convergents are fairly standard in Number Theory texts. Our source is Hardy and Wright, "Introduction to the Theory of Numbers", ch. X. **(10188)** The Stirling numbers of the second kind, $S(n,k)$ may be defined as the number of ways of partitioning a set of n distinguishable elements into k non-empty subsets. Further properties may be found in Riordan, "An Introduction to Combinatorial Analysis". **(10189)** A random variable, Z, is said to have a Cauchy distribution if $Z = \tan(T)$ with T uniformly distributed on $(-\pi/2, \pi/2)$. **(10191)** The terminology is explained in a sufficiently comprehensive introduction to abstract algebra, such as Lang, "Algebra". **(10192)** In this context, $o(1)$ is the customary way of denoting a function of x whose only interesting property is that it approaches zero as x tends to infinity.

SOLUTIONS

The Sign of a Special Function

E 3366 [1990, 64]. *Proposed by the editors.*

For $n = 1, 2, 3, \ldots,$ determine the subset of $(0,1)$ on which

$$\left(\frac{d}{dx}\right)^n \{\log x \cdot \log(1-x)\} < 0.$$

Solution I by Howard Morris, Chatsworth, CA. If $f(x) = \log x \cdot \log(1-x)$, then

$$f'(x) = \frac{\log(1-x)}{x} - \frac{\log x}{1-x} = \sum_{m=1}^{\infty} \frac{(1-x)^{m-1} - x^{m-1}}{m}.$$

From this it is evident that for n even $f^{(n)}(x)$ is always negative because all terms are negative, while for n odd $f^{(n)}(x)$ is negative if and only if $1 - x < x$, or $x > 1/2$.

Solution II by W. O. Egerland and C. E. Hansen, BRL, Aberdeen Proving Ground, MD. Letting $u = x - 1/2$, we have

$$\begin{aligned}\log x \cdot \log(1-x) &= [-\log 2 + \log(1+2u)] \cdot [-\log 2 + \log(1-2u)] \\ &= \log^2 2 - \log 2 \cdot \log(1-4u^2) + \log(1+2u)\log(1-2u).\end{aligned}$$

This is an even function of u, and its Taylor series has only even terms. Using the expansion for the log function and the identity

$$\frac{1}{l(2k-l)} = \frac{1}{2k}\left(\frac{1}{l} + \frac{1}{2k-l}\right),$$

we obtain $\log x \cdot \log(1-x) = \log^2 2 - \sum_{k=1}^{\infty} a_k (x - 1/2)^{2k}$, where $a_k = (4^k/k)\sum_{l=2k}^{\infty}(-1)^l/l$. With each a_k positive, we conclude that the nth derivative is negative in the interval $(1/2, 1)$ if n is odd, and in the interval $(0,1)$ if n is even. (Note: Since $\log x \log(1-x) \to 0$ as $x \to 0$, we obtain the formula $\log^2 2 = \sum_{k=1}^{\infty}\sum_{l=2k}^{\infty}(-1)^l/(kl)$.)

Editorial comment. Many solvers showed first that $f^{(n)}(x)$ is always negative for positive even n and observed that $f^{(n)}(x)$ is therefore strictly decreasing for odd n. Since $f^{(n)}(1/2 + u) = -f^{(n)}(1/2 - u)$ for odd n, we have $f^{(n)}(1/2) = 0$, which completes the proof.

All but two readers used suitable power series expansions. I. E. Leonard and J. F. McDonald derived the interesting closed form expression

$$f^{(n)}(x) = -\frac{(n-1)!}{x^n(1-x)^n}\left[(-1)^n x^n \int_0^{(1-x)/x} \frac{y^{n-1}}{1+y}\,dy + (1-x)^n \int_0^{x/(1-x)} \frac{y^{n-1}}{1+y}\,dy\right].$$

Solved also by 24 other readers and the proposer. One incorrect solution was received.

A Higher-Degree Binomial Coefficient Identity

E 3376 [1990, 240]. *Proposed by Robert J. Blodgett, Morningside, MD.*

Prove that

$$\sum_{i=0}^{N}\sum_{j=0}^{N}\binom{i+j}{j}^2\binom{4N-2i-2j}{2N-2j} = (2N+1)\binom{2N}{N}^2$$

for any positive integer N.

Solution I by George E. Andrews, IBM, Yorktown Heights, NY, and Peter Paule, Johannes Kepler Universität, Linz, Austria. We prove more generally that

$$\sum_{i=1}^{\lfloor m/2\rfloor}\sum_{j=0}^{\lfloor n/2\rfloor}\binom{i+j}{j}^2\binom{m+n-2i-2j}{n-2j} = \frac{\lceil (n+m)/2\rceil!\,(1+\lfloor (n+m)/2\rfloor)!}{\lfloor n/2\rfloor!\,\lfloor m/2\rfloor!\,\lceil n/2\rceil!\,\lceil m/2\rceil!},$$

where $n \geqslant 0$ and $m \geqslant 0$. This reduces to the desired result when we set $m = n = 2N$. The assertion follows immediately from the fact that each side satisfies the same initial conditions $f(n,0) = f(0,n) = \lfloor n/2\rfloor + 1$ and recurrence

$$f(n,m) - f(n-1,m) - f(n,m-1) = \begin{cases} \dbinom{n/2+m/2}{n/2}^2 & \text{if both } m \text{ and } n \text{ are even} \\ 0 & \text{otherwise.} \end{cases}$$

The initial conditions agree by inspection. The fact that the right side satisfies the recurrence can be seen by liberal use of the identities $\lceil (k-1)/2\rceil = \lfloor k/2\rfloor$ and $\lfloor (k+1)/2\rfloor = \lceil k/2\rceil$. To prove it for the left side we combine the sums term by term and use Pascal's rule $\binom{k+l}{k} = \binom{k+l-1}{k-1} + \binom{k+l-1}{l-1}$ to annihilate everything except the one term $i = n/2$, $j = m/2$ that arises when both m and n are even.

Solution II by G. W. Peck, Massachusetts Institute of Technology, Cambridge, MA. Let $P(n,m)$ denote the double sum in solution I. We give a combinatorial proof of the identity there when n and m are even, with $n = 2r$ and $m = 2s$. There are analogous arguments when n and/or m is odd, but we omit them here.

Let S be the collection of binary sequences with $2r$ ones and $2s$ zeros. For $\sigma \in S$, let $w_k(\sigma)$ denote the sum of the first k entries in σ. Let T be the set of pairs (σ, k) such that $\sigma \in S$, $k \in [0, r+s]$, and $w_{k+r+s}(\sigma) = w_k(\sigma) + r$. Since there are $r+s+1$ possible values for k and $\binom{r+s}{r}^2$ sequences σ completing a pair with each value of k, we have $|T| = (r+s+1)\binom{r+s}{r}^2$.

We want to count the pairs in another way to show also that $|T| = P(2r, 2s)$. The statement $w_{k+r+s}(\sigma) = w_k(\sigma) + r$ is the statement that the half of σ following position k has half the ones (and half the zeros). Let $\nu(\sigma)$ be the minimum value of k at which this occurs. To verify its existence, note that if the first half of σ has more than [fewer than] r ones, then the last half has fewer than [more than] r ones. Each time the window of length $r+s$ slides by one unit, the number of ones changes by 0 or 1, so there must be a first time where the window has exactly r ones.

For each $(\sigma, k) \in T$, we have $k \geqslant q = \nu(\sigma)$, and we will count T by grouping together the pairs with specified values of $k - q$ and $w_k(\sigma) - w_q(\sigma)$. Given $(\sigma, k) \in T$, define i and j by $i = w_k(\sigma) - w_q(\sigma)$ and $j = k - q - i$. Note that i

counts the ones and j the zeros in a sequence of length $k - q \leqslant r + j$; hence $0 \leqslant i \leqslant r$ and $0 \leqslant j \leqslant s$.

Given $(\sigma, k) \in T$, we extract three sequences from σ; let α be the subsequence in positions $q + 1$ to k (length $i + j$), let β be the subsequence in positions $q + r + s + 1$ to $k + r + s$ (length $i + j$), and let γ be the remaining entries of σ in order (length $2r + 2s - 2i - 2j$). By the fact that both (σ, q) and (σ, k) satisfy the window condition, we have $w_{k+r+s}(\sigma) - w_{q+r+s}(\sigma) = w_k(\sigma) - w_q(\sigma) = i$; hence α and β each have i ones, and γ has $2r - 2i$ ones. Note that the half of σ following q ends at $q + r + s$. Hence we have removed i ones from each of these "halves", which implies $\nu(\gamma) \leqslant q$. We cannot have $\nu(\gamma) < q$, because any earlier window of size $r + s - i - j$ must include position q and omit position $q + r + s$, so that reinsertion of α and β would yield $\nu(\sigma) < q$.

With i, j fixed, the number of ways to choose α, β, γ satisfying the conditions on length and weight is $\binom{i+j}{i}^2\binom{2r+2s-2i-2j}{2r-2i}$. Hence it suffices to show that there is a bijection between such triplets (α, β, γ) and pairs $(\sigma, k) \in T$. We prove this by reconstructing from an arbitrary (α, β, γ) satisfying $0 \leqslant i \leqslant r$ and $0 \leqslant j \leqslant s$ the unique pair $(\sigma, k) \in T$ from which they could be extracted. The needed observation was made above; we must have $q = \nu(\gamma)$. Hence we insert α after position q in γ and β after position $q + r + s - i - j$ of γ to obtain σ, and we set $k = q + i + j$. The fact that the half of σ following k has half the ones in σ follows from the fact that the half of γ following q has half the ones in γ. This inverts the extraction.

Editorial comment. R. J. Chapman (England) proved the identity of Solution I in the case where $n = 2r$ and $m = 2s$ by observing that the double sum can be read as a convolution in two variables. By equating corresponding coefficients, the identity for $P(2r, 2s)$ then becomes equivalent to the statement that the generating function $\sum\sum(r + s + 1)\binom{r+s}{s}^2 x^r y^s$ is the product of the generating functions $\sum\sum\binom{2r+2s}{2s}x^r y^s$ and $\sum\sum\binom{r+s}{r}^2 x^r y^s$. He computed these generating functions explicitly to verify this, a formidable task.

Peter Paule submitted another solution applying the method of D. Zeilberger ("A fast algorithm for proving terminating hypergeometric identities," to appear). He cleverly turned the original double summation into the single summation $\sum_{k=0}^{N}(2k + 1)\binom{2N+1}{N-k}^2$, and then he applied the algorithm of Zeilberger to discover a recurrence in N satisfied by this. The proof is completed by verifying that the right side also satisfies this recurrence and the same initial conditions.

Ira Gessel observed that the method of Solution I yields the more general identity

$$\sum_{i=0}^{\lfloor m/2\rfloor}\sum_{j=0}^{\lfloor n/2\rfloor}\frac{(2x)^{(i+j)}(x+1/2)^{(i+j)}}{i!\,j!\,(x+1/2)^{(i)}(x+1/2)^{(j)}}\binom{m+n-2i-2j}{n-2j}$$
$$=\frac{(2x+1)^{(\lfloor (n+m)/2\rfloor)}(x+1/2)^{(\lceil (n+m)/2\rceil)}}{\lfloor n/2\rfloor!\,\lfloor m/2\rfloor!\,(x+1/2)^{(\lceil m/2\rceil)}(x+1/2)^{(\lceil n/2\rceil)}},$$

where $x^{(n)} = x(x + 1)\cdots(x + n - 1)$. The identity of Solution I results when $x = 1/2$.

A. A. Jagers (Netherlands) mentioned an identity reminiscent of the original identity of the problem. This may have similar generalizations and proofs, though

Jagers proof of it was by eigenvalue methods:

$$\sum_{i=0}^{N}\sum_{j=0}^{N}\binom{i+j}{j}^2\binom{2N-i-j}{N-j}^2=\binom{4N+1}{2N+1}.$$

John Henry Steelman gave a solution similar to Solution I.

Monochromatic Polygon with Centroid

E 3378 [1990, 240]. *Proposed by Miklòs Bòna (student), Budapest, Hungary.*

Suppose the points of Z^2 (the set of points in the plane with integer coordinates) are colored with a finite number of colors. For every $n \geqslant 3$, prove that there exists a convex n-gon with vertices and centroid in Z^2 such that all $n+1$ points have the same color.

Solution by Edward R. Scheinerman, Johns Hopkins University, Baltimore, MD. We invoke van der Waerden's Theorem, which states that any finite coloring of the positive integers has arbitrarily long monochromatic arithmetic progressions. The finite version of the theorem asserts the existence of $W = W(r, l)$ such that every r-coloring of $[W] = \{1, \dots, W\}$ has a monochromatic l-term arithmetic progression. We will need a "product" version of this theorem for pairs of integers. We define an *arithmetic 2-grid* of size l to be a collection of integer points in the plane of the form $\{(a_1 + id_1, a_2 + jd_2)\colon\ 0 \leqslant i,\ j < l\}$, where a_1, a_2, d_1, d_2 are fixed positive integers.

Product van der Waerden Theorem: For all positive integers r, l, there exists $W' = W'(r, l)$ such that every r-coloring of $[W']^2$ has a monochromatic arithmetic 2-grid of size l.

Proof: Let $W' = W(r^{W(r,l)}, l)$, and let f be an arbitrary r-coloring of $[W']^2$. To each $s \in [W']$, assign a "vector color" $F(s) = (f(1,s), f(2,s), \dots, f(W(r,l), s)) \in [r]^{W(r,l)}$. By the choice of W', there exist $a_2, d_2 > 0$ such that $F(a_2) = F(a_2 + d_2) = \cdots = F(a_2 + (l-1)d_2)$ and each $a_2 + jd_2 \in [W']$. In this fixed r-ary vector $F(a_2 + jd_2)$ of length $W(r, l)$, there must be an l-term monochromatic arithmetic progression of positions with initial position a_1 and constant difference d_1. Thus $\{(a_1 + id_1, a_2 + jd_2)\colon 0 \leqslant i,\ j < l\} \subset [W']^2$ is monochromatic.

This theorem readily solves the problem. Given $n \geqslant 3$ and an r-coloring of Z^2, let $l = 6n^2$. Let S be a monochromatic arithmetic 2-grid $\{(a_1 + id_1, a_2 + jd_2)$: $0 \leqslant i,\ j < l\}$. Choose the n points $\{(a_1 + 2kd_1, a_2 + 6k^2 d_2)$: $0 \leqslant k < n\}$. Being in S, these points have the same color; being on a parabola, they form a convex n-gon. Since $(1/n)\sum_{k=0}^{n-1} 2k = n - 1$ and $(1/n)\sum_{k=0}^{n-1} 6k^2 = (n-1)(2n-1)$, the centroid is also in S and has the same color as the vertices.

Editorial comment. Most solvers noted the relationship of this problem to Gallai's Theorem, which states that if V is a finite collection of points in $\mathbb{R}^m$, then any r-coloring of the points of $\mathbb{R}^m$ contains a monochromatic W that is "similar without rotation" to V (scaling and translation are allowed). This can be proved using van der Waerden's Theorem, and the proof can be rephrased to prove the integer analogue of Gallai's Theorem, which is much stronger than the result requested here.

For $r = 2$, the proposer provided an elementary proof that a 2-coloring of the integer lattice contains n points of the same color whose centroid also has that color. (Note that any n points are the vertices of a simple polygon, not necessarily convex.) Given a 2-coloring, let $a_1, \dots, a_n$ be the vectors of n red points, and let $C = (1/n)\Sigma a_i$ be their centroid. If C is blue, let $b_j = (n+1)a_j - \Sigma a_i$ for $1 \leqslant j \leqslant n$. Then a_j is the centroid of the set obtained from $\{a_1, a_2, \dots, a_n\}$ by replacing a_j by b_j. If any b_j is red, we have the desired red set. Otherwise, $\{b_j\}$ is a blue set with blue centroid C, by straightforward computation. Note that this proof yields the desired monochromatic figure within a very small grid.

Solved also by A. Bialostocki, R. J. Chapman (Great Britain), R. High, L. Piepmeyer (Germany), and B. Reznick.

Some Strange 3-adic Identities

6625 [1990, 252]. *Proposed by Nicholas Strauss, Pontificia Universidade Católica do Rio de Janeiro, Brasil, and Jeffrey Shallit, Dartmouth College.*

If k is a positive integer, let $3^{\nu(k)}$ be the highest power of 3 dividing k. Put

$$r(n) = \sum_{i=0}^{n-1} \binom{2i}{i}$$

for positive integers n. Prove that

(i) $\nu(r(n)) \geqslant 2\nu(n)$,

(ii) $\nu(r(n)) = \nu\left(\binom{2n}{n}\right) + 2\nu(n)$.

Solution by Don Zagier, University of Maryland, College Park, and Max-Planck-Institut für Mathematik, Bonn, Germany. The assertion of the problem may be stated in the form:

$$v\left(\sum_{k=0}^{n-1} \binom{2k}{k}\right) = v\left(n^2 \binom{2n}{n}\right) \quad \text{for all } n \geqslant 1; \tag{1}$$

here, and throughout this solution, $v(\cdot)$ denotes the 3-adic valuation. We give a simple proof of (1) and of various other 3-adic identities related to it.

If we set

$$f(n) \doteq \frac{\sum_{k=0}^{n-1} \binom{2k}{k}}{n^2 \binom{2n}{n}} \qquad (n \geqslant 1), \tag{2}$$

then (1) says that $f(n)$ is a 3-adic unit for all $n \in \mathbb{N}$. In fact, a calculation of the first few values suggests that in fact

$$f(n) \equiv -1 \pmod 3 \quad \forall n \tag{3}$$

and a more extensive calculation suggests the more precise congruences

$$n \equiv m \pmod{3^j} \Rightarrow f(n) \equiv f(m) \pmod{3^{j+1}}. \tag{4}$$

This says that the function $f\colon \mathbb{N} \to \mathbb{Q} \subset \mathbb{Q}_3$ extends to a 3-adic continuous map $\mathbb{Z}_3 \to -1 + 3\mathbb{Z}_3$. The range studied ($n \leqslant 2200$) permits one to check these congruences for $j \leqslant 7$ (since $3^7 < 2200$) and hence to interpolate $f(n)$ with accuracy

$O(3^8)$. The interpolated values found in this way for negative integers and half-integers are equal, to this accuracy, to simple rational numbers, suggesting the further identities

$$f(-1) = -1, \qquad f(-2) = -\frac{7}{4}, \qquad f(-3) = -4, \ldots, \tag{5}$$

$$f\left(-\frac{1}{2}\right) = -4, \qquad f\left(-\frac{3}{2}\right) = -4, \qquad f\left(-\frac{5}{2}\right) = -\frac{196}{25}, \ldots. \tag{6}$$

We now state a result which includes all of these experimental observations.

Theorem. *The function f extends to a 3-adic analytic function from $\mathbb{Z}_3$ to $-1 + 3\mathbb{Z}_3$. Its values at negative integers and half-integers are rational numbers, given by*

$$f(-n) = -\frac{(2n-1)!}{n!^2} \sum_{k=0}^{n-1} \frac{k!^2}{(k-1)!} \qquad (n \geqslant 1), \tag{7}$$

$$f\left(-n-\frac{1}{2}\right) = -\frac{2^{4n+2}}{(2n+1)^2\binom{2n}{n}} \sum_{k=0}^{n} 2^{-4k}\binom{2k}{k} \qquad (n \geqslant 0). \tag{8}$$

As a corollary, we get the identities analogous to (1)

$$v\left(\sum_{k=0}^{n-1} \frac{k!^2}{(2k+1)!}\right) = v\left(\frac{n!^2}{(2n-1)!}\right) \qquad (n \geqslant 1), \tag{9}$$

$$v\left(\sum_{k=0}^{n} 2^{-4k}\binom{2k}{k}\right) = v\left((2n+1)^2\binom{2n}{n}\right) \qquad (n \geqslant 0). \tag{10}$$

Proof: Equation (2) implies that $f(n)$ satisfies the recursion relation

$$(2n+1)(2n+2)f(n+1) = 1 + n^2 f(n) \tag{11}$$

for $n \in \mathbb{N}$. If f has an extension to a 3-adic continuous function from $\mathbb{Z}_3$ to $\mathbb{Z}_3$, then this functional equation must hold for all $n \in \mathbb{Z}_3$. Since the left-hand side vanishes at $n = -1$ and $n = -1/2$, we must have $f(-1) = -1$ and $f(-1/2) = -4$; the further values in (7) and (8) then follow by induction on n using the functional equation (11). Thus we need only prove the first statement of the theorem.

Set $g(n) = 2nf(n)$; we show first that g extends to a 3-adic analytic function of n, and then that $g(x)$ is divisible by x. For g the recursion (11) becomes

$$2(2n+1)g(n+1) = 2 + ng(n). \tag{12}$$

Define rational numbers $a_0 = 1, a_1 = -1/2, \ldots$ by requiring that

$$g(n) = \sum_{r=0}^{\infty} a_r \binom{n-1}{r} \tag{13}$$

for $n = 1, 2, \ldots$ (note that the sum is finite for each n). If we show that $v(a_r) \to \infty$ as $r \to \infty$, then (13) will converge 3-adically for all $n \in \mathbb{Z}_3$ and give the desired continuation. Substituting (13) into (12) gives

$$2 + \sum_{r=0}^{n-1} (r+1)a_r\binom{n}{r+1} = \sum_{r=0}^{n} \left[2(2r+1)\binom{n}{r} + 4(r+1)\binom{n}{r+1}\right]a_r.$$

Comparing coefficients of $\binom{n}{r}$ in this gives $2(2r+1)a_r = -3ra_{r-1}$ for $r \geqslant 1$,

whence

$$a_r = \frac{(-3)^r r!^2}{(2r+1)!} \qquad (r \geqslant 0). \tag{14}$$

The 3-adic valuation of this does indeed tend to infinity with r (since $v(3^r/(2r+1)!) \geqslant 0$ and $v(r!) \to \infty$), so (13) gives the analytic continuation of g.

Lemma. *The series* $\sum_{r=0}^{\infty}(3^r r!^2/(2r+1)!)$ *converges 3-adically to* 0.

We will prove the lemma in a moment. Assuming it, we find

$$\begin{aligned} g(n) &= \sum_{r=0}^{n-1}(-3)^r \frac{r!}{(2r+1)!}(n-1)(n-2)\cdots(n-r) \\ &= \sum_{r=0}^{n-1}\frac{3^r r!^2}{(2r+1)!} - \frac{1}{2}n \\ &\quad + \sum_{r=2}^{n-1}(-3)^r\frac{r!}{(2r+1)!}\left[(n-1)(n-2)\cdots(n-r) - (-1)^r r!\right]. \end{aligned} \tag{15}$$

By the lemma, the first term in (15) has valuation

$$v\left(\sum_{r=0}^{n-1}\frac{3^r r!^2}{(2r+1)!}\right) = v\left(\sum_{r=n}^{\infty}\frac{3^r r!^2}{(2r+1)!}\right) \geqslant 2\frac{n-2}{3} \geqslant v(n)+1 \qquad (n \geqslant 4)$$

since $v((3^r r!^2)/(2r+1)!) \geqslant 2v(r!) \geqslant 2(r-2)/3$ for all r. Also,

$$(n-1)(n-2)\cdots(n-r) - (-1)^r r!$$

is divisible by n and $(-3)^r r!/(2r+1)!$ is divisible by 3 for all $r \geqslant 2$, so (15) gives

$$g(n) \equiv -\tfrac{1}{2}n \pmod{3^{v(n)+1}},$$

whence $f(n) = g(n)/2n$ is 3-integral and congruent to -1 modulo 3. Thus the theorem is proved.

Proof of Lemma: We have the power series identity

$$\begin{aligned} \sum_{r=0}^{\infty}\frac{r!^2}{(2r+1)!}x^r &= \sum_{r=0}^{\infty}\left(\int_0^1 t^r(1-t)^r\,dt\right)x^r \quad \text{(beta integral)} \\ &= \int_0^1 \frac{dt}{1+xt+xt^2} \\ &= \frac{1}{\sqrt{x^2-4x}}\log\frac{2-x+\sqrt{x^2-4x}}{2-x-\sqrt{x^2-4x}} \\ &= \frac{1}{3\sqrt{x^2-4x}}\log\frac{\left(2-x+\sqrt{x^2-4x}\right)^3/4}{\left(2-x-\sqrt{x^2-4x}\right)^3/4} \\ &= \frac{1}{3\sqrt{x^2-4x}}\log\frac{2-x(3-x)^2+(3-x)(1-x)\sqrt{x^2-4x}}{2-x(3-x)^2-(3-x)(1-x)\sqrt{x^2-4x}} \\ &= \frac{2}{3}\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{x^n(x-4)^n(3-x)^{2n+1}(1-x)^{2n+1}}{\left[2-x(3-x)^2\right]^{2n+1}} \end{aligned}$$

in $\mathbb{Q}[[x]]$. Both sides converge 3-adically if $v(x) > 0$, and the right-hand side vanishes for $x = 3$. This completes the proof of the lemma.

Finally, we remark that the computer calculations to $n = 2200$ suggested the further congruence

$$n \equiv m \equiv 0 \pmod{3^j} \Rightarrow f(n) \equiv f(m) \pmod{3^{2j+1}},$$

analogous to (4). If true, this says that the derivative of f at 0 vanishes. From what we have done we find that the Taylor series of f around the origin is given by

$$\begin{aligned} f(n) &= \frac{1}{2n} \sum_{r=0}^{\infty} \frac{3^r r!^2}{(2r+1)!} \left[(1-n)\left(1 - \frac{n}{2}\right) \cdots \left(1 - \frac{n}{r}\right) - 1 \right] \\ &= A + Bn + Cn^2 + \cdots \end{aligned}$$

with

$$\begin{aligned} A &= -\frac{1}{2} \sum_{r=1}^{\infty} \frac{3^r r!^2}{(2r+1)!} \left(1 + \frac{1}{2} + \cdots + \frac{1}{r}\right), \\ B &= \frac{1}{2} \sum_{r=2}^{\infty} \frac{3^r r!^2}{(2r+1)!} \sigma_2\left(1, \frac{1}{2}, \ldots, \frac{1}{r}\right), \end{aligned}$$

etc. (σ_2 = second elementary symmetric function). The assertion that $f'(0)$ vanishes is thus equivalent to the following statement, which is similar to but more complicated than our lemma above:

Conjecture. *The series* $\sum_{r=0}^{\infty}((3^r r!^2)/(2r+1)!)\sigma_2(1, 1/2, \ldots, 1/r)$ *converges* 3*-adically to* 0.

Another interesting problem would be to evaluate in closed form the 3-adic number A. To thirty 3-adic digits, A equals $\ldots 110000102110002221022212000212$.

Part (i) was solved also by Derek Hacon and Nicholas Strauss.
Part (ii) was solved also by Jean-Paul Allouche and Jeffrey Shallit.

A Convergent Sequence

E 3388 [1990, 428]. *Proposed by Matthew Cook (student), University of Illinois, Urbana, IL, Walther Janous, Ursulinengymnasium, Innsbruck, Austria, and Marcin E. Kuczma, University of Warsaw, Warsaw, Poland.*

Let x_1 and x_2 be arbitrary positive numbers. Suppose we define a sequence $\{x_n\}_{n=1}^{\infty}$ by putting $x_{n+2} = 2/(x_{n+1} + x_n)$ for $n = 1, 2, 3, \ldots$. Prove that the sequence converges.

Solution by David Borwein, University of Western Ontario, London, Ontario, Canada. We first prove that the sequence is bounded. If both x_n and x_{n-1} are between a^{-1} and a, then $a^{-1} \leqslant (x_n + x_{n-1})/2 \leqslant a$, so x_{n+1} is between the same bounds.

Now let $l = \liminf x_n$ and $L = \limsup x_n$. Since L is finite, for any $\varepsilon > 0$ there is an integer n_0 such that $x_n < L + \varepsilon$ for $n > n_0$. Hence $x_{n+2} = 2/(x_{n+1} + x_n) > 1/(L + \varepsilon)$ for $n > n_0$. It follows that $l \geqslant 1/L > 0$. Similarly, $x_n > l - \varepsilon > 0$ for $n > n_1$ implies $x_{n+2} < 1/(l - \varepsilon)$ for $n > n_1$, whence $L \leqslant 1/l$. Therefore $l = 1/L$.

Let $S = \{n_i\}_{i=0}^{\infty}$ be an infinite sequence of positive integers such that $x_{n_i+2} \to L$. By taking subsequences, if necessary, we may assume that x_{n_i+1}, x_{n_i} and x_{n_i-1} approach l_1, l_2, and l_3, respectively. Since $x_{m+1} + x_m = 2/x_{m+2}$ and $x_m + x_{m-1} = 2/x_{m+1}$, we have $l_1 + l_2 = 2/L = 2l$ and $l_2 + l_3 = 2/l_1$. Since $l \leqslant l_1, l_2, l_3 \leqslant L$, it follows that $l_1 = l_2 = l$ and $l_2 = l_3 = L$. Hence $l = L$, and $x_n \to 1$.

Editorial comment. A number of solvers provided generalizations. F. Brulois and also W. O. Egerland and C. E. Hansen showed that the conclusion remains true if x_1 and x_2 are complex with positive real part. D. Laugwitz (Germany), G. Karakostas and C. Petalas (Greece), and O. Saleh and T. Walters observed that the denominator can be generalized to $px_n + qx_{n+1}$, where $p + q = 2, p > 0, q > 0$. W. Janous (Austria) noted that the solution of the original problem extends to the sequence defined by $x_{n+k} = k/\sum_{i=0}^{k-1} x_{n+i}$. Finally, J. H. Lindsey II asserted that the result also holds if $x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1})$, where f is an continuous function from $(\mathbb{R}^+)^n$ to $\mathbb{R}^+$ that is nonincreasing in each variable and decreasing in the last two, and in addition $f(x, x, \dots, x)$ as a one-variable function $g(x)$ satisfies $g(1) = 1$ and $g(g(x)) = x$.

Solved by 30 readers and the proposers.

The Smallest Trisection of the Perimeter of a Triangle

E 3397 [1990, 611]. *Proposed by Ji Chen and Cheng-Hui Lo, University of Science and Technology, Hefei, Anhui, China.*

The perimeter of a triangle ABC is divided into three equal parts by three points P, Q, R. Show that

$$\text{Area}(PQR) > \tfrac{2}{9}\,\text{Area}(ABC)$$

and that the constant $2/9$ is best possible.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. We shall prove the stronger result that $\text{Area}(PQR) \geqslant (2/9)(1 + F^2/(abcs))F$, where a, b, c are the sides, s the semiperimeter, and F the area of triangle ABC. In the extremal situation, no two of the points P, Q, R can be interior points of the same side of ABC, because shifting them by a constant distance in the appropriate direction will lower the height (and reduce the area) of PQR. Hence we may assume that P, Q, R lie on BC, CA, AB, respectively.

Let us put $p = (s - a)/3$, $q = (s - b)/3$, $r = (s - c)/3$. Then p, q and r are all non-negative, and $a = 3q + 3r$, $|b| = 3r + 3p$, $|c| = 3p + 3q$. The location of P, Q, R can now be parametrized by x; set $|BP| = q + 2r + x$, $|CQ| = r + 2p + x$, and $|AR| = p + 2q + x$. Let us define $H(x) = 1 - \text{Area}(PQR)/\text{Area}(ABC)$. Now

$$\frac{\text{Area } ARQ}{\text{Area } ABC} = \frac{(1/2)|AR| \cdot |AQ| \sin A}{(1/2)|AB| \cdot |AC| \sin A} = \frac{|AR| \cdot |AQ|}{bc}$$

$$= \frac{(p + 2q + x)(p + 2r - x)}{9(p + q)(p + r)}.$$

Similarly for BPR and CQP. Hence

$$H(x) = \frac{1}{9}\left[\frac{(q+2r+x)(q+2p-x)}{(q+r)(q+p)} + \frac{(r+2p+x)(r+2q-x)}{(r+p)(r+q)} + \frac{(p+2q+x)(p+2r-x)}{(p+q)(p+r)}\right].$$

This is a quadratic expression in x, where the coefficient of x is 0 and the coefficient of x^2 is negative. Hence its maximum is attained at $x = 0$!

By substituting $x = 0$, we obtain

$$H(0) = \frac{7}{9} - \frac{2pqr}{9(p+q)(q+r)(r+p)} < \frac{7}{9}.$$

We observe that $H(0)$ approaches $7/9$ from below as p goes to 0 with q and r fixed. In other words, very "flat" triangles show that the constant is best possible. To obtain the stronger formula, we can express $H(0)$ in terms of the sides, the semiperimeter, and the area of ABC as $H(0) = (7/9) - (2/9)F^2/(abcs)$, which yields the lower bound claimed for the ratio of the areas.

Editorial comment. J. G. Mauldon also obtained the above inequality and proved in addition that the best upper bound for the area of the smallest trisecting triangle PQR is $(1/4)$Area ABC, attained if and only if ABC is equailateral, and that the best lower bound for the area of the largest trisecting triangle PQR is $(4/9)$Area ABC, which is unattainable.

Solved also by M. Abért (student, Hungary), J. Balogh (student, Hungary), P. Dubovshy & Z. Nasirov (USSR), J. S. Frame, J. Fukuta (Japan), J. F. Goehl Jr., E. Lee, J. H. Lindsey II, J. G. Mauldon, Victor Pambuccian, A. Pedersen (Demark), J. H. Steelman, J. S. Sumner, J. M. Weinstein, and the proposers. Partial solutions were received from L. Kuipers, S. Kung, H. Lipman, V. Schindler, and an anonymous contributor. One incorrect solution was received.

Numbers Related by the Totient Function

E 3398 [1990, 611]. *Proposed by Alan H. Stein, University of Connecticut, Waterbury, CT.*

Find all pairs of positive integers m, n such that $\phi(m)|n$ and $\phi(n)|m$, where ϕ denotes Euler's function.

Solution by Thomas Honold and Hubert Kiechle, Technische Universität München, Munich, Germany. Call a pair (m, n) *primitive* if $\gcd(m, n)$ is squarefree. There are exactly eleven primitive pairs of solutions, namely

$$(1,1)(1,2),(2,2),(2,3),(2,4),(2,6),(4,6),(4,10),(6,6),(6,14)\ (6,18).$$

Because $\phi(pn) = p\phi(n)$ when p is prime dividing n, all other solutions can be obtained from primitive solutions by the following rule: If p is a prime dividing both m and n, then the pair (m, n) is a solution if and only if the pair (pm, pn) is a solution. Thus, for example, the pair $(6, 18)$ yields the infinite family $\{(2^r3^s, 2^r3^{s+1}): r, s \geqslant 1\}$, and $(4, 10)$ yields $\{(2^{r+1}, 2^r5): r \geqslant 1\}$, while $(2, 3)$ yields no other pairs.

Hence we may assume that (m, n) is primitive. If $m = 2^r$, then $r \leqslant 2$, because $\phi(m) = 2^{r-1}$ divides n but 4 does not divide $\gcd(m, n)$. Now $\phi(n)|m$ requires that (m, n) is one of the first eight pairs listed above.

By symmetry, we may now assume that both m and n have odd prime divisors. Hence $\phi(m), \phi(n)$ and therefore also m, n are even. If $4|m$, then $4|\phi(m)|n$, which contradicts primitivity. By symmetry, neither m nor n is divisible by 4. Now m cannot have two different odd prime factors, because this would imply $4|\phi(m)|n$. We conclude that $m = 2p^r$ and $n = 2q^s$, where p, q are odd primes and $r, s \geqslant 1$. Assuming $p \leqslant q$, from $(p-1)|n$ we conclude $p = 3$. If $q = 3$, then we obtain the pairs $(6,6)$ and $(6,18)$. If $q \neq 3$, then it follows from the hypothesis that $r = s = 1$ and $q - 1 = 2p$, and hence $(m, n) = (6, 14)$.

Editorial comment. R. J. Chapman and J. H. Steelman each proved that the conditions of the problem imply that $\phi(\phi(n))|n$, and that this is the case if and only if $n \in \{1, 2^r, 3, 2^r 3^s, 2^r 5, 2^r 7\colon r, s \geqslant 1\}$.

Solved by 34 readers and the proposer. Three solutions omitted one or more classes of solutions.

The First Third

6637 [1990, 621]. *Proposed by Herbert S. Wilf, University of Pennsylvania, Philadelphia, PA.*

Let $f(n)$ be the sum of the first one-third of the coefficients in the expansion of $(1+x)^{3n}$, i.e.,

$$f(n) = \sum_{k=0}^{n} \binom{3n}{k} \qquad (n = 0, 1, 2, \dots).$$

Prove that

$$\sum_{n=0}^{\infty} f(n) \left(\frac{4u^2}{27}\right)^n = \frac{u}{u - 2\sin(\frac{1}{3}\arcsin u)} - \frac{2u}{2u - 3\sin(\frac{1}{3}\arcsin u)}.$$

Solution by Ira Gessel, Brandeis University, Waltham, MA. First we show that

$$\sum_{n=0}^{\infty} f(n) \frac{\alpha^n}{(1+\alpha)^{3n+1}} = \frac{1}{(1-\alpha)(1-2\alpha)}. \tag{1}$$

The coefficient of α^m in the left-hand side of (1) is

$$\sum_{n=0}^{m} f(n)(-1)^{m-n}\binom{m+2n}{m-n} = \sum_{n=0}^{m}\sum_{k=0}^{n}(-1)^{m-n}\binom{3n}{k}\binom{m+2n}{m-n}. \tag{2}$$

If we set $n = k + i$, (2) becomes

$$\sum_{i \geqslant 0,\, k \geqslant 0,\, i+k \leqslant m} (-1)^{m-k-i}\binom{3k+3i}{k}\binom{m+2k+2i}{m-k-i}. \tag{3}$$

Expressing the binomial coefficients in (3) in terms of factorials and rearranging them, we obtain

$$\sum_{i=0}^{m}\sum_{k=0}^{m-i}(-1)^{m-k-i}\binom{m-i}{k}\binom{m+2k+2i}{m-i}. \tag{4}$$

For fixed i,

$$\sum_{k=0}^{m-i}(-1)^{m-k-i}\binom{m-i}{k}\binom{m+2k+2i}{m-i}$$

is the $(m-i)$th difference of a polynomial in k of degree $m-i$ with leading coefficient $2^{m-i}/(m-i)!$, and is therefore equal to 2^{m-i}. Thus

$$\sum_{n=0}^{\infty} f(n) \frac{\alpha^n}{(1+\alpha)^{3n+1}} = \sum_{m=0}^{\infty} \alpha^m \sum_{i=0}^{m} 2^{m-i} = \sum_{i,j\geq 0} \alpha^{i+j} 2^j = \frac{1}{(1-\alpha)(1-2\alpha)}.$$

It follows from (1) that

$$\sum_{n=0}^{\infty} f(n) \left(\frac{\alpha}{(1+\alpha)^3}\right)^n = \frac{1+\alpha}{(1-\alpha)(1-2\alpha)} = \frac{3}{1-2\alpha} - \frac{2}{1-\alpha}. \tag{5}$$

Now let $\theta = (1/3)\arcsin u$ and let

$$\alpha = \frac{3\sin(1/3 \arcsin u)}{u} - 1 = \frac{3\sin\theta}{\sin 3\theta} - 1.$$

From the identity $3\sin\theta - \sin 3\theta = 4\sin^3\theta$ it follows that

$$\frac{\alpha}{(1+\alpha)^3} = \frac{4u^2}{27}. \tag{6}$$

Then expressing (5) in terms of u yields the desired identity. (To justify the identity, we may interpret θ and α as formal power series in u, or we may take θ, α, and u to be sufficiently small complex numbers to guarantee convergence.)

Editorial comment. The solutions received were roughly of three types according to the main tool used: The Cauchy integral formula, the Lagrange inversion formula, or binomial coefficient combinatorics (as in Gessel's solution). Cecil Rousseau observed that the first type, in combination with steepest descent, will also yield:

$$f(n) \sim 3^{1/2}(n\pi)^{-1/2}(27/4)^n,$$

while the second type is available on pp. 159–160 and pp. 179–180 of H. S. Wilf, *Generating functionology*, Acad. Press, 1990. David Callan remarked that his combinatorial approach also yields a solution of E 3415 by P. Flajolet and D. E. Knuth.

Solved also by P. J. Bushell (U.K.), David Callan, Kevin Ford (student), L. Van Hamme (Belgium), Kee-Wai Lau (Hong Kong), Rolf Richberg (Germany), Cecil Rousseau, Frank W. Schmidt, James A. Wilson, and the proposer.

UNSOLVED PROBLEMS

In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial or related results. Typescripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Alberta, Canada T2N 1N4.*

What Divisibility Properties Do Generalized Harmonic Numbers Have?

Yuri Matiyasevich

Harmonic numbers H_n are defined by

$$H_n = 1 + \frac{1}{2} + \cdots \frac{1}{n}. \tag{1}$$

Let N_n denote the numerator of H_n. It is easy to see (e.g., by calculating the reciprocals in (1) in the finite field $\mathscr{F}_p$) that for a prime p greater than 2,

$$p \mid N_{p-1}. \tag{2}$$

Moreover it is known (see [1]) that a stronger divisibility property holds, namely, for a prime p greater than 3,

$$p^2 \mid N_{p-1}. \tag{3}$$

Harmonic numbers can be generalized in many ways. The definition we give here may look unnatural, but it seems to lead to numbers with interesting divisibility properties. We define **generalized harmonic numbers** $H_n^{(r)}$ **of rank** r by

$$H_n^{(r)} \coloneqq \sum_{n_0 + \cdots + n_r \le n} \frac{1}{n_0 n_1 \dots n_r} \tag{4}$$

so that in particular $H_n = H_n^{(0)}$. We will denote the numerator of $H_n^{(r)}$ by $N_n^{(r)}$.

It is easy to calculate $H_n^{(r)}$ for small values of r and n by using any modern computer algebra package capable of performing exact rational arithmetic. However, for large values of r and n direct use of the definition (4) is not effective. In such a case one can use an equivalent definition:

$$H_n^{(r)} = \frac{-(-1)^{n+r}}{n!}\left(\frac{d^n}{(dt)^n} \frac{(\ln t)^{r+1}}{t}\bigg|_{t=1} \right). \tag{5}$$

Numerical calculations of $H_n^{(r)}$ give strong evidence in favor of several conjectures. An analog of (2) seems to be true for generalized harmonic numbers of any rank.

Conjecture 1. *For any r, any n and any prime p greater than $r+2$,*

$$p \mid N_{p-1}^{(r)}.$$

It is easy to find counterexamples to an analog of (3), but all of them seem to be for generalized harmonic numbers of odd rank.

Conjecture 2. *For any even r, any n and any prime p greater than $r+3$,*

$$p^2 \mid N_{p-1}^{(r)}.$$

The second prime factor p seems not to disappear entirely in the case of odd rank, but just moves to the next number.

Conjecture 3. *For any odd r, any n and any prime p greater than $r+2$,*

$$p \mid N_p^{(r)}.$$

Also, in the case of odd rank, a new divisibility seems to arise.

Conjecture 4. *For any odd r, any n and any prime p greater than $(r+1)/2$,*

$$p \mid N_{2p-1}^{(r)}.$$

The author made sample calculations of the generalized harmonic numbers using *Mathematica* during his stay at the Mathematical Sciences Research Institute in Berkeley and at Stanford University.

REFERENCES

1. Z. I. Borevich & I. R. Shafarevich, Number Theory, translated from the Russian by Newcomb Greenleaf, *Pure & Appl. Math.*, 20, Academic Press, New York–London, 1966, Chap. 5, §8, Ex. 5.
2. A. Gardiner, Four problems on prime power divisibility, this MONTHLY, 95 (1988) 926–931.

LETTERS

Just for the record I think it should be pointed out to readers that the idea in the attractive and pedagogically useful note:

Sandy Grabiner, "The Tietze extension theorem and the open mapping theorem," *Monthly* 93 (1986), 190–199, *MR*88a:54034

was anticipated in a similar such note:

M. C. McCord, "A theorem on linear operators and the Tietze extension theorem," *Monthly* 75 (1968), 47–48, *MR*37#2018.

Both Professor Grabiner and *Math Reviews* overlooked this connection.

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In the January 1991 issue of the monthly Stephen Kuhn mentions in his article "The Derivative a la Cartheodry", that he is disappointed about not finding more than 1 or 2 texts mentioning Cartheodry's definition of the derivative, and then also proceeds to show it power by proving the chain rule as an example. However what we were surprised to find was that his search did not include that absolutely excellent text by Tom Apostol: "Mathematical Analysis" in which the first few pages of Chapter 5 not only states Cartheodry's definition but also gives the exact same proofs of some of the examples in Kuhn's article. Also Kuhn need not regret that it has remained obscure, since an entire generation of Caltech juniors has grown up on Apostol's texts.

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R. P. Boas has kindly pointed out to me that the proof of the ∞/∞ version of L'Hôpital's rule that appears in my *Monthly* note of February, 1991 (Vol. 98, No. 2, 156–57) can be traced back to Stoltz in the 1890's and has been rediscovered several times since then. Interested readers will find more information in Professor Boas' article in the Mathematics Magazine 63, no. 3 (1990), 155–159.

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The argument presented in *A Simple Proof of Zorn's Lemma*, by Jonathan Lewin in the April, 1991, MONTHLY, is based on the same line of reasoning used by Hellmuth Kneser in *Das Auswahlaxiom und das Lemma von Zorn*, which appeared in Mathematische Zeitschrift, 96 (1967), pages 62–63. In fact, Kneser uses the argument to prove the following slightly sharpened version of Zorn's Lemma: *If X is a partially ordered set in which every well ordered subset has an upper bound, then X has a maximal element.*

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The discoveries of Newton have done more for England and for the race, than has been done by whole dynasties of British monarchs; and we doubt not that in the great mathematical birth of 1853, the Quaternions of Hamilton, there is as much real promise of benefit to mankind as in any event of Victoria's reign.

—*Thomas Hill*

REVIEWS

Visions of Symmetry: Notebooks, Periodic Drawings, and Related Work of M. C. Escher. By Doris Schattschneider, W. H. Freeman and Company, New York, 1990, xiii + 354 pp.

Douglas J. Dunham

"How did he do it?" is a question raised in *Visions of Symmetry* that comes naturally to mind when one views the art of M. C. Escher (1898–1972). Most of his work after 1936 had a distinctly mathematical flavor, mainly dealing with periodic patterns of the Euclidean plane. Visions of Symmetry explains "*how* he did it"—i.e. how he created periodic patterns—both with biographical information and with color reproductions of Escher's 1941–42 "theory" and "abstract motif" notebooks. The centerpiece of *Visions of Symmetry* which illustrates *what* Escher did, is the superb full color reproduction of all 137 of Escher's periodic patterns from his "regular division drawings" notebooks (plus 14 additional patterns not in the notebooks). The large format of *Visions of Symmetry* (about 280 × 230 mm) displays Escher's patterns at nearly full scale. *Visions of Symmetry* is the *only* book in which all of these patterns are reproduced, dozens of them appearing in print for the first time. Finally, *Visions of Symmetry* contains notes on all 137 + 14 patterns and a separate chapter that shows how Escher *used* the patterns in his prints. In addition to the periodic patterns, *Visions of Symmetry*, contains 200 other Escher illustrations (mostly in color), three indexes of drawings, a bibliography, and a concordance.

For the Escher fan, *Visions of Symmetry* fills a gap in the literature by showing all of his notebook patterns, answering the question "how did he do it?", and relating the patterns to his prints. For the person interested in tilings and patterns, *Visions of Symmetry* provides many beautiful examples (which illustrate the theory expounded in Grünbaum and Shepard's *Tilings and Patterns* [1987]). Escher's colored periodic patterns can even be used to visually illustrate elementary concepts in group theory, as explained by Marjorie Senechal's article "The Algebraic Escher" [1988].

To make more precise the kind of patterns Escher created, we review some of the terminology. A *pattern* on the Euclidean plane is a collection of congruent copies of one or more basic subpatterns or *motifs*. Each motif is a nonempty subset of the plane. With a few exceptions, Escher's patterns used one or two motifs. For the moment, we visualize the pattern by coloring copies of the motif black and leaving the rest of the plane white. So, by placing copies of a unit square motif at alternate locations on the lattice with integer coordinates, one obtains the familiar black-and-white (1-motif) checkerboard pattern.

A *symmetry* of a pattern is a Euclidean isometry that maps the pattern onto itself with each motif copy being mapped onto another copy (of the same motif if there is more than one motif). Thus, there are four possible types of symmetries of a pattern: translations, rotations, reflections, and glide-reflections. The checkerboard pattern exhibits all four types of symmetries. The set of symmetries of a

pattern forms a group called the *symmetry group* of the pattern. We say that a pattern is *periodic* if its symmetry group contains translations in two linearly independent directions, but no translations by arbitrarily small amounts (thus excluding "strip" patterns and the pattern of points with rational coordinates). Up to isomorphism, there are just 17 symmetry group of periodic patterns, the *plane symmetry groups*.

In addition to being periodic, Escher's patterns have a second characteristic property: copies of the motif(s) *tile* the plane, that is, they cover it without gaps or overlaps. Such a pattern is called a *tiling*. Escher's motifs are always closed topological disks, resulting in "nice" (nonpathological) tilings. Different motif copies can intersect either at isolated points or along arcs, called *vertices* or *edges* (respectively) of the tiling. If we use the method above to visualize a pattern of tiles, we would only see a solid black plane since copies of the motif cover it. So, we visualize tilings differently: we only color black the boundaries of motif copies —the edges and vertices of the tiling.

A third characteristic of Escher's pattern is that the interiors of motif copies are colored according to the *map-coloring principle*: two copies that share an edge receive different colors. In the terminology of color symmetry, if each motif copy receives one of n colors, we say the pattern is *n-colored*. If we color the plane lattice for unit squares black and white alternately, we obtain the usual 2-colored checkerboard pattern (a different interpretation than in the third paragraph of this review). If, for each color, a symmetry of the (uncolored) pattern sends all motif copies of that color to motif copies of a single color, we call that symmetry a *color symmetry* of the (colored) pattern. In other words, a color symmetry induces an associated permutation of the colors.

Now suppose that the black squares of alternate rows of the checkerboard pattern are colored red instead. The translation diagonally by one square is a color symmetry of the new pattern (red and black are interchanged), whereas translation horizontally or vertically by one square is not a color symmetry. So, the new pattern is less regularly colored than the original checkerboard. We say that a pattern is *perfectly colored* if every symmetry is a color symmetry and the associated permutations form a transitive subgroup of S_n, the permutation group of n colors. Thus, the black-white-red checkerboard is not perfectly colored. However, if we also color the white squares of alternate rows green, the resulting black-white-red-green checkerboard is perfectly 4-colored. With two exceptions, Escher's patterns are perfectly colored. We define an *n-color group* to be an isomorphism class of symmetry groups (together with their associated permutation subgroups) of perfectly colored patterns.

Now that we have an idea of what Escher did, we can begin to answer the question "How did he do it?" To set the stage, Escher's interest in creating periodic patterns increased considerably after his second visit, in 1936, to the Alhambra palace in Spain, which is decorated extensively with periodic patterns having abstract motifs. In 1937, Escher's half-brother, B. G. Escher, referred him to an article by George Pólya [1924], which included sample (abstract) periodic patterns corresponding to each of the 17 plane symmetry groups. Escher's goal was to create periodic patterns with recognizable animal motifs, not just abstract motifs. He took the obvious route: he modified the boundaries of the abstract motifs that he had collected (but this process did not always prove to be easy for him). As Escher created more and more motifs, he developed his own rules for motif creation and his own periodic pattern classification system, which is recorded

in his 1941–42 "theory" notebook. His system includes 2- and 3-color periodic patterns with one or two motifs based on the 7 plane symmetry groups not containing reflections.

While Escher was developing his theory, he was recording his periodic patterns in his "regular division drawings" notebooks. These were the examples that corresponded to his theory. For the rest of Escher's life, he continued to add to his "drawings" notebooks, producing 137 numbered periodic patterns, each pattern being classified according to his system.

In addition to developing a classification system and creating patterns, Escher's work also progressed in a third direction: the design of the graphic prints for which he became famous. About 60 of these prints used a periodic pattern as an integral part of its composition. All of Escher's prints, including these 60, appear in the book, *M. C. Escher: His Life and Complete Graphic Work* [Bool, 1982].

Another question asked about Escher is: "How did Escher's work fit in with the development of mathematical (color) symmetry theory?" The 17 plane symmetry groups were first classified by E. S. Federov [1891] a hundred years ago, and rediscovered by Pólya [1924] and others. As mentioned above, Escher was aware of this classification and focused on the 7 groups not containing reflections. While Escher was formulating his system, he was not aware of the first developments in 2-color symmetry that occurred in the late 1920's and mid 1930's. Escher created patterns with 3- and 4-color symmetry in the late 1930's. Mathematicians began their investigations of n-color symmetry ($n \geq 2$) in the 1950's and in 1961 van der Waerden and Burckhardt [1961] formulated the concepts of color symmetry in terms of symmetry groups as outlined above. The 23 3-color groups were first classified by Grünbaum in 1976 [1976] (Figure 8.2.2 of Grünbaum and Shepard [1987] shows sample patterns); the n-color groups for $2 \leq rn \leq 15$ were determined by Jarratt and Schwarzenberger in 1980 [1980], and for $2 \leq n \leq 60$ by Wieting in 1981 [1981]. Thus Escher was definitely a pioneer in 3- and 4-color symmetry. He was also a pioneer in his work that considered patterns with more than one motif.

A final question that we consider is: "Was Escher a mathematician?" (or the related question "What was Escher's mathematical background?"). Escher's academic mathematical background was not particularly impressive, though geometry seemed to agree with him better than algebra. In deriving his pattern classification system, Escher went through the familiar mathematical cycle: work out some examples, form a hypothesis, work more examples, revise the hypothesis, work more examples, etc. He also correctly conjectured two theorems in plane geometry (his diagrams "proved" them to his satisfaction)—see pages 88–90 of *Visions of Symmetry*. In fact, one can consult *Visions of Symmetry* for more details on each of the "Escher" questions raised above (and other such questions).

Escher's works have inspired considerable mathematical activity—for example, many of the articles in the book *M. C. Escher: Art and Science* [Coxeter, et al., 1986]. We mention two open areas of research suggested by Escher's works. The first is the classification of all hypersymmetric tiles, that is tiles possessing a symmetry that is not a symmetry of the periodic tiling.

A second open area involves periodic 2-colored 2-motif patterns. There are two possibilities: the copies of each motif are all of one color, or some copies of each motif are black and others are white, as in Figure 1. Patterns of the first type are called "Heaven and Hell" patterns after Escher's periodic pattern of this type with white angels and black devils. This is the only pattern that Escher adapted to each of the three "classical geometries": the sphere, the Euclidean plane, and the

hyperbolic plane (where the pattern is named *Circle Limit IV*). If the bounding circular arcs in Figure 1 are made to bulge to the left instead of the right in alternate rows, one obtains a "Heaven and Hell" pattern. Andreas Dress has classified the 37 kinds of "Heaven and Hell" patterns [1986]. However, the classification of the 20-colored 2-motif patterns of the second type remains open. There appear to be seven patterns of this type among Escher's periodic pattern drawings.

While reading *Visions of Symmetry*, this reviewer learned a considerable amount about Escher and his periodic patterns, and even discovered a few previously overlooked subtleties of Escher's prints. I trust that other readers of this book will have a similarly pleasant experience.

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Precalculus, T(13). *Precalculus, Algebra and Trigonometry.* Sharon Cutler Ross, Linda Hawkins Boyd. Brooks/Cole, 1991, xi + 481 pp, $46. [ISBN: 0-534-14550-7] Written with an emphasis on problem solving and on awareness of the graphing calculator, this text spends time rehashing the number system, little time on long drill work, and moves quickly to topics of mathematical substance: analytic geometry, inequalities, a heavy dose of functions, a treatment of systems of equations that includes Gauss-Jordan elimination, and topics from discrete mathematics. AWR

Finite Mathematics, T(13-14: 1, 2). *Finite Mathematics.* Roland E. Larson, Bruce H. Edwards. DC Heath, 1991, xii + 564 pp, $37.50 net. [ISBN: 0-669-16801-7] Suitable for business, economics, and social science students. Covers elementary linear algebra, combinatorics and probability, simplex method, probability distributions, Markov chains, game theory, and finance topics such as interest, annuities, and amortization. AD

Finite Mathematics, T(13: 1). *Essentials of Finite Mathematics: Matrices, Linear Programming, Probability, Markov Chains.* Robert F. Brown, Brenda W. Brown. Ardsley House, 1990, ix + 454 pp, $45.95. [ISBN: 0-912675-78-0] Covers standard topics in finite mathematics. Each chapter begins with a brief essay on some application and ends with review exercises. LC

Education, P, L. *The Development of Elementary Mathematical Concepts in Preschool Children.* A.M. Leushina. Soviet Stud. in Math. Educ., V. 4. Transl: Joan Teller. NCTM, 1991, xxiv + 481 pp, $25 (P). [ISBN: 0-87353-299-6] Translation of a Russian monograph first published in 1974 that offers a thorough analysis of how young children develop notions of number, counting, volume, shape, time, and spatial orientation. Concludes with a series of chapters on curriculum and teaching methods for three-, four-, five-, and six-year-old children. Introduced by a new Preface by Leslie Steffe relating the work to more recent studies. LAS

Education, P, L. *Curriculum Evaluation Standards for School Mathematics Addenda Series, Grades 5–8: Patterns and Functions.* Elizabeth Phillips, *et al.* NCTM, 1991, viii +

72 pp, $13 (P). [ISBN: 0-87353-324-0] One in a series of "addenda" that provide teachers with ideas and examples to support implementation of the NCTM *Standards*. This volume contains fifteen "investigations" of patterns in numbers, growth, measurement, and graphs to enrich middle school mathematics instruction. Emphasize appropriate use of manipulatives, technologies, and cooperative learning. LAS

Education, P, L. *The Development of Spatial Thinking in Schoolchildren.* I.S. Yakimanskaya. Soviet Stud. in Math. Educ., V. 3. Transl: Robert H. Silverman. NCTM, 1991, xv + 239 pp, $25 (P). [ISBN: 0-87353-298-8] The author introduces a model for levels of development of spatial thinking, different from the van Hiele levels, to guide teaching. Examples from industry underscore the importance of spatial thinking; reports of investigations with children in grades 4-8 document that children can develop sophistication in spatial thinking at early ages. LAS

Education, P. *A Guide for Reviewing School Mathematics Programs.* Eds: Glendon W. Blume, Robert F. Nicely, Jr. NCTM and ASCD, 1991, ix + 65 pp, $8 (P). [ISBN: 0-87353-334-8] Checklists of critical elements of school mathematics programs identified by NCTM, MAA, MSEB, and NRC documents. Enables analysis of both current implementation and perception of importance of elements of goals, curriculum, instruction, evaluation, and administrative responsibility. Guidelines for using in internal or external review processes, or to aid in textbook selection. MW

Education, S(15-17). *Mathematics Homework on a Micro.* G.T. Wain, S.M. Flower. Mathematical Assoc (259 London Road, Leicester LE2 3BE), 77 pp, (P). Seventy-two simple BASIC program listings ($\leq$ 20 lines each) and assorted question sets for homework tasks in number, geometry, graphs, algebra, statistics, and general investigations. Tasks differentiated by difficulty level. Translation from British to American computers and language may require reworking of some worksheets, but the ideas are worthwhile for pre-service and in-service secondary mathematics teachers. MW

Education, P. *Children Reading Mathematics.* Eds: Hilary Shuard, Andrew Rothery. John Murray (50 Albmarle St., London W1X 4BD), 1988, 170 pp, $17 (P). [ISBN: 0-7195-4093-3] Conclusions of a British research and discussion group, the Language and Reading Mathematics Group. Purposes of mathematical writing, categories of text types, and application of readability tests. Factors affecting readability: graphs, charts, diagrams, and symbols; page layout; and suitability to reader level. Suggestions for writing readable mathematics texts and for helping students improve mathematical reading skills. MW

History, S, P***, L***.** *The Man Who Knew Infinity: A Life of the Genius Ramanujan.* Robert Kanigel. Charles Scribner's, 1991, ix + 438 pp, $27.95. [ISBN: 0-684-19259-4] An extraordinary compelling biography, richly textured with social, psychological, personal, and mathematical details, both of Ramanujan and of his mentor Hardy. Prize-winning science journalist Kanigel creates with his two utterly contrasting mathematician-protagonists a warm, romantic tale that would be beyond the scope of imagination were it not, after all, true. Reviewers justly praise this book as one of the best scientific biographies ever written. LAS

History, S(16-17), P. *Geschichte der Algebra: Eine Einführung.* Erhard Scholz, *et al.* Bibliographisches Institut, 1990, 506 pp. [ISBN: 3-411-14411-4] A history of algebra from the earliest times into the twentieth century. Extensive bibliography. JD-B

Graph Theory, P. *Contemporary Methods in Graph Theory.* Rainer Bodendiek. Bibliographisches Institut, 1990, xxii + 676 pp. [ISBN: 3-411-14301-0] Forty-six research papers in honor of Klaus Wagner on topics including topological graph theory, coloring, hamiltonicity, and infinite graphs. JPH

Graph Theory, T(16-17: 2), L? *Eulerian Graphs and Related Topics, Part 1, Volume 2.* Herbert Fleischner. Annals of Disc. Math., V. 50. North-Holland (US Distr: Elsevier Science), 1991, x + 323 pp, $100. [ISBN: 0-444-89110-2] Continues discussion of Eulerian graphs, including chapters on various types of closed covering walks, Eulerian trails, and algorithms for Eulerian trails. Note price. LC

Combinatorics, P. *Geometry and Combinatorics: Selected Works of J.J. Seidel.* Eds: D.G. Corneil, R. Mathon. Academic Pr, 1991, xix + 410 pp, $69.50. [ISBN: 0-12-189420-7] Twenty-eight research and survey papers highlighting Seidel's work on the interplay of combinatorics, algebra, and geometry. Topics include graphs and designs, lines with few angles, matrices and forms, and non-Euclidean geometry. JPH

Combinatorics, S(18), P. *The Dilworth Theorems: Selected Papers of Robert P. Dilworth.* Eds: Kenneth P. Bogart, Ralph Freese, Joseph P.S. Kung. Birkhäuser, 1990, xxvi + 465 pp, $59.50. [ISBN: 0-8176-3434-7] Dilworth's important papers on ordered sets and lattice theory. Sections preceded by background exposition by Dilworth and followed by articles on later influences with extensive references. JPH

Discrete Mathematics, T(14-18), S, L. *Difference Equations: An Introduction with Applications.* Walter G. Kelley, Allan C. Peterson. Academic Pr, 1991, xi + 455 pp, $44.50. [ISBN: 0-12-403325-3] Assuming a good calculus background (and a little sophistication), this book provides a nice overview. Many examples illustrate diversity of uses: statistics,

computing, electrical circuit analysis, dynamical systems, economics, biology. Style and numerous exercises should make this a good course text. KS

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Calculus, P, L*.** *The Laboratory Approach to Teaching Calculus.* Eds: L. Carl Leinbach, *et al.* Notes No. 20. MAA, 1991, ix + 264 pp, $20 (P). [ISBN: 0-88385-074-5] Thirty-three class-tested examples of diverse strategies for using calculators, computers (or even paper-and-pencil) to teach calculus with a laboratory approach—the study of mathematical phenomena through observation, exploration, and analysis. Institutions are diverse—large and small, public and private—as are approaches (e.g., programming, CAS, spreadsheets). A valuable source of "what works" to motivate those waiting in line to join the calculus reform movement. LAS

Complex Analysis, T(15-16: 1), L. *Complex Variables for Mathematics and Engineering, Second Edition.* John H. Mathews. Wm C Brown, 1988, x + 358 pp. [ISBN: 0-697-06764-5] Minor changes from the *First Edition* (TR, October 1982) include additional exercises, examples, and a couple of added sections: a section on harmonic functions and the Dirichlet problem, and a section on the argument principle and Rouché's Theorem. LC

Differential Equations, P. *Lecture Notes in Mathematics-1473: Functional Differential Equations with Infinite Delay.* Y. Hino, S. Murakami, T. Naito. Springer-Verlag, 1991, x + 317 pp, $33 (P). [ISBN: 0-387-54084-9] Intended as a "unified theory of this field in terms of functional analysis and dynamical systems." MLR

Differential Equations, T(16-17), L. *Lectures on Differential and Integral Equations.* Kôsaku Yosida. Dover, 1991, ix + 220 pp, $6.95 (P). [ISBN: 0-486-66679-4] The basic theory, including a discussion of the boundary-value problem of second order linear differential equations and the theory of Weyl-Stone eigenfunction expansions. "Self-contained." MLR

Differential Equations, T*(14-15), S, L**.** *Ordinary Differential Equations.* George F. Carrier, Carl E. Pearson. Classics in Appl. Math., V. 6. SIAM, 1991, x + 220 pp, $25.50 (P). [ISBN: 0-89871-265-3] Reprint of 1968 original (TR, February 1969); minor revisions. A refreshing change from the omnipresent "cookbook" approach; heuristic arguments and beautiful, open-ended problems drive the discussion. Most problems end with a question forcing the solver to think about what he or she just did. Covers all the usual topics; great source of challenging problems for standard course. SK

Differential Equations, P. *Vector Lyapunov Functions and Stability Analysis of Nonlinear Systems.* V. Lakshmikantham, V.M. Matrosov, S. Sivasundaram. Math. & Its Applic., V. 63. Kluwer Academic, 1991, x + 172 pp, $79. [ISBN: 0-7923-1152-3] From the Preface: "Lyapunov functions and the so-called Lyapunov second method are now well-established as the most powerful technique for the analysis of the stability and qualitative properties of (systems of) differential equations. The trouble, especially in concrete situations, is finding Lyapunov functions ... Thus it makes sense to weaken the requirements and to look for several functions which together give enough control and insight; i.e., investigate vector Lyapunov functions ... This is the first book that deals with the method of vector Lyapunov functions." AWR

Differential Equations, P. *Algebraic Methods in Nonlinear Perturbation Theory.* V.N. Bogaevski, A. Povzner. Appl. Math. Sci., V. 88. Springer-Verlag, 1991, xii + 265 pp, $59. [ISBN: 0-387-97491-1] The authors answer the question, "Why another book on the perturbation theory of differential equations?" by listing four goals: to develop, making use of a change of variables, a formalism generalizing the Poincaré-Bogolyubov-Krylov-Mitropolsky notion of normal form, and to give a method for calculating the asymptotics without having to guess their form; to propose an effective approach to singular perturbation problems and a satisfactory matching procedure; to discuss a possibility of its minimization; and to show possible ways to extend the formalism to partial differential equations. AWR

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With these functions defined as hypergeometric functions, properties are studied (recursion formulas, orthogonality relations, etc.). Prerequisite: three semesters of calculus and knowledge of modern physics. LC

Differential Equations, P. *Lecture Notes in Mathematics-1475: Delay Differential Equations and Dynamical Systems.* Eds: S. Busenberg, M. Martelli. Springer-Verlag, 1991, viii + 249 pp, $28 (P). [ISBN: 0-387-54120-9] Proceedings of a conference in honor of Kenneth Cooke in Claremont, California, January 1990. Contains nineteen research articles and three short surveys on equations with piecewise continuous delays (K. Cooke and J. Wiener); equations with several delays (J. Hale); and persistence in dynamical systems (P. Waltman). SK

Partial Differential Equations, T(18), S, P. *Analytic Pseudo-Differential Operators and their Applications.* Julii A. Dubinskii. Math. & Its Applic., V. 68. Kluwer Academic, 1991, xii + 252 pp, $98. [ISBN: 0-7923-1296-1] Develops theory of pseudo-differential operators with analytic symbols covering basic and distribution spaces, Fourier transforms of analytic functions, and the complex Fourier method. Continues with the Cauchy problem in complex domain and pseudo-differential operators with real argument. Note price. KS

Numerical Analysis, T(15-16: 1), L. *Numerical Linear Algebra.* Willy Brandal. BCS Assoc, 1991, viii + 204 pp, $30 (P). [ISBN: 0-914351-05-2] Studies numerical methods for obtaining solutions to linear systems $A\overline{x} = \overline{b}$ where A is invertible and of finding the eigenvalues and eigenvectors of a square matrix A with real entries. Covers finding eigenvalues by the power and QR method, discusses Gaussian and Gauss-Jordan elimination, pivoting and scaling, LU decomposition, etc. Prerequisite of calculus and linear algebra. LC

Numerical Analysis, P. *Numerical Methods for Free Boundary Problems.* Ed: P. Neittaanmäki. ISNM, V. 99. Birkhäuser, 1991, xv + 439 pp, $99. [ISBN: 0-8176-2641-7] Thirty-nine papers from a conference held in Finland in July 1990. Includes results on Stefan-like problems, optimal control, identification, and fluid flow problems. RWN

Numerical Analysis, P. *Mixed and Hybrid Finite Element Methods.* Franco Brezzi, Michel Fortin. Ser. in Computat. Math., V. 15. Springer-Verlag, 1991, ix + 350 pp, $59. [ISBN: 0-387-97582-9] A review and development of non-standard finite element methods. Includes foundations of variational and approximation theory as well as examples and applications to a good variety of problems. RWN

Operator Theory, P. *Selfadjoint and Nonselfadjoint Operator Algebras and Operator Theory.* Ed: Robert S. Doran. Contemp. Math., V. 120. AMS, 1991, xxi + 215 pp, $49 (P). [ISBN: 0-8218-5127-6] Proceedings of the CBMS Regional Conference held at Texas Christian University, May 1990. Contains twenty-nine lectures and a collection of problems on operator algebras. MLR

Operator Theory, S(18), P. *Spinor Construction of Vertex Operator Algebras, Triality, and $E_8^{(1)}$.* Alex J. Feingold, Igor B. Frenkel, John F.X. Ries. Contemp. Math., V. 121. AMS, 1991, ix + 146 pp, $34 (P). [ISBN: 0-8218-5128-4] Vertex operator algebras have their origin in recent efforts to unify and understand the mathematics necessary for dealing with string theory in physics along with several exotic algebras from mathematics as exemplified by the Chevalley and Griess algebras. Assumes a background in the classical Lie algebra theory, then proceeds to develop spinor constructions for various algebras. References. JS

Operator Theory, P. *Lecture Notes in Mathematics-1465: Wavelets and Singular Integrals on Curves and Surfaces.* Guy David. Springer-Verlag, 1991, x + 107 pp, $16 (P). [ISBN: 0-387-53902-6] Transcripts of lectures given at the Nankai Institute of Mathematics, June 1988. MLR

Operator Theory, S(18), P. *Lecture Notes in Mathematics-1472: Bose Algebras: The Complex and Real Wave Representations.* Torben T. Nielsen. Springer-Verlag, 1991, 132 pp, $16 (P). [ISBN: 0-387-54041-4] Written for mathematicians; requires no background in mathematical physics. Presents an algebraic axiomatic formalization of Bose-Fock spaces (one-boson-spaces) growing out of work of Irving Segal and others in the sixties. References, subject index. JS

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Functional Analysis, T(18: 2), S, P, L? *Theory of Orlicz Spaces.* M.M. Rao, Z.D. Ren. Pure & Appl. Math., V. 146. Marcel Dekker, 1991, ix + 449 pp, $145. [ISBN: 0-8247-8478-2] First seven chapters introduce fundamental structure of Orlicz spaces including Young's functions, Orlicz function spaces, linear functionals and weak topologies, analysis of linear operators between Orlicz spaces, and geometry and smoothness of these spaces. Final three chapters contain further and recent developments at accelerated pace. Note price. KS

Functional Analysis, T(17-18: 2), P, L. *Functional Analysis, Second Edition.* Walter Rudin. Intern. Ser. in Pure & Appl. Math. McGraw-Hill, 1991, xv + 424 pp, $51.65. [ISBN: 0-07-054236-8] Few changes from the *First Edition* (TR, May 1973). Additional topics include the mean ergodic theorem of von Neumann, the Hille-Yosida theorem on

semigroups of operators, and fixed point theorems. MLR

Analysis, P. *Orthogonal Functions, Revised English Edition.* G. Sansone. Dover, 1991, xix + 411 pp, $9.95 (P). [ISBN: 0-486-66730-8] Unabridged republication of a volume first published by Interscience in 1959 as Volume IX of the series Pure and Applied Mathematics. Covers the theory of orthogonal series, including Fourier series, Legendre series, and spherical harmonics. LC

Geometry, S*, L.** *The Penguin Dictionary of Curious and Interesting Geometry.* David Wells. Penguin Books, 1991, xiv + 285 pp, $20 (P). [ISBN: 0-14-011813-6] Apollonian gasket, Brocard points, cycloids, dragon curves, Euler line, Fatou dust, geodesic domes, Hénon attractor, Islamic tessellations ... Steiner networks, Thébault's theorem, unilluminable room, Voderberg tilings, wallpaper patterns, sonahedra. Hundreds of shapes, famous and obscure, ancient and modern, each with a brief description and illustration. A fascinating book for browsing; excellent stimulation for math club projects. LAS

Topology, S(18), P. *Topology of Lie Groups, I and II.* Mamoru Mimura, Hirosi Toda. Transl. of Math. Mono., V. 91. AMS, 1991, iv + 451 pp, $192. [ISBN: 0-8218-4541-1] *Part I* covers the classical groups as examples of Lie groups. Includes fast-paced introductions to theories of topological groups, fibre bundles, homotopy, and (co)homology groups. *Part II* covers the general theory, especially of compact Lie groups. Includes integration, Bott-Morse theory, cohomology of exceptional groups. Good reference for the mathematically sophisticated reader. No exercises. Note price. MC

Mathematical Modelling, T(18), S, P. *Interactive System Identification: Prospects and Pitfalls.* Torsten Bohlin. Communic. & Control Engin. Ser. Springer-Verlag, 1991, xii + 365 pp, $98. [ISBN: 0-387-53636-1] Intended as a guide for someone making choices of methods to solve subproblems in the modelling of dynamic systems, this text tries to emphasize what goes into the theoretic assumptions that underlie identification methods. Derivations used by various methods enter in only as they are needed to intelligently discuss how one's assumptions enter into the choices to be made. AWR

Probability, S(18), P. *Intersections of Random Walks.* Gregory F. Lawler. Prob. & Its Applic. Birkhäuser, 1991, 219 pp, $49.50. [ISBN: 0-8176-3557-2] Assumes a standard measure theoretic course in probability including Martingales and Brownian motion. Begins by developing standard results of simple random walks and probabilistic tools for analyzing walks. Subsequent subjects include harmonic measure, the probability that paths of independent random walks intersect (four, three, and two dimensions), and self-avoiding walks (random walks conditioned to have few or no self-intersections). Last chapter presents Laplacian random walks. Only chapters one and two contain exercises; bibliography. KB

Stochastic Processes, T(16-18: 1, 2), L. *Stochastic Models in Queueing Theory.* J. Medhi. Academic Pr, 1991, xiii + 444 pp, $63.50. [ISBN: 0-12-487550-5] Covers stochastic processes, birth-death and non-birth-death queueing systems, network of queues, non-Markovian queueing systems, queues with general arrival and service distributions, queues with vacations, and asymptotic methods. Each chapter includes exercises and references. Very readable; assumes a previous course in applied probability and advanced calculus. KB

Elementary Statistics, T(13: 1). *Understandable Statistics: Concepts and Methods, Fourth Edition.* Charles Henry Brase, Corrinne Pellillo Brase. DC Heath, 1991, xvii + 702 pp, $33.50 net. [ISBN: 0-669-24477-5] Revision of the authors' 1987 *Third Edition* (TR, December 1987). Significant change is the addition of computer displays of Minitab and ComputerStat output in the "Using Computers" sections. (ComputerStat is an interactive software package designed to accompany the text.) Also contains several new sections, new supplementary materials, and many new problems. RSK

Elementary Statistics, S(13-17), L. *Multivariate Statistical Analysis: A Conceptual Introduction, Second Edition.* Sam Kash Kachigan. Radius Pr, 1991, xiv + 303 pp, $12.95 (P). [ISBN: 0-942154-91-6] Essentially a paperback re-issue of the 1982 edition (TR, November 1989) with an additional chapter on multidimensional scaling. A conceptual introduction emphasizing rationale, applications and interpretations of multivariate statistical methods: correlation analysis, regression analysis, analysis of variance, discriminant analysis, factor analysis, cluster analysis, and multidimensional analysis. Assumes minimal mathematical background; contains no formal exercises. KB

Mathematical Statistics, T(18: 1), P. *Point Processes and Their Statistical Inference, Second Edition Revised and Expanded.* Alan F. Karr. Prob.: Pure & Appl., V. 7. Marcel Dekker, 1991, xiv + 490 pp, $110. [ISBN: 0-8247-8532-0] Features a complete reorganization and rewriting of material pertaining to the multiplicative intensity model and stationary point processes. Additional material includes the Cox regression model and expanded explanations of many fundamental statistical concepts. Goal is to present a unified description of inference for point processes. Contains exercises, appendices, and an expanded, updated (and extensive) bibliography. KB

Statistical Methods, P. *Data Quality Control: Theory and Pragmatics.* Eds: Gunar E. Liepins, V.R.R. Uppuluri. Stat.: Textbooks &

Mono., V. 112. Marcel Dekker, 1990, xii + 360 pp, $89.75. [ISBN: 0-8247-8354-9] Contains sixteen articles, some detail actual quality control practices that have been successful in industry and government. Other articles address editing, imputation, statistical matching, and error localisation. Contains several hard-to-read pages where print shows through from reverse sides. RWJ

Statistics, S?(17-18), P, L. *Lecture Notes in Statistics-66: Exact Confidence Bounds when Sampling from Small Finite Universes.* Tommy Wright. Springer-Verlag, 1991, xvi + 431 pp, $54 (P). [ISBN: 0-387-97515-2] Given a population of units of which an unknown number A have a particular attribute, the author considers estimation of A from a sample. A variety of exact and conservative confidence bounds are given for A along with extensive tables. Exact tests and sample size determination are also discussed. RWJ

Statistics, T(15-17: 1). *Statistical Process Control: Theory and Practice.* G. Barrie Wetherill, Don W. Brown. Chapman & Hall, 1991, xiv + 400 pp, $65. [ISBN: 0-412-35700-3] Revision of Wetherill's 1977 book *Sampling Inspection and Quality Control* (TR, April 1978), with modifications based on experiences in industry. Presents techniques of statistical process control, together with some theory. Roughly three-fourths of the text deals with charting and one-fourth with sampling inspection. RSK

Computer Systems, P, L*. *TEX for the Impatient.* Paul W. Abrahams, Karl Berry, Kathryn A. Hargreaves. Addison-Wesley, 1990, xix + 357 pp, $27 (P). [ISBN: 0-201-51375-7] A very useful supplement to (but no substitute for) *The TEXbook.* A variety of brief illustrative examples is followed by an alphabetic glossary of TEX concepts, then by discussions of various commands, grouped by type, usually illustrated by helpful examples showing typical use. Includes tips, useful macros, and hints for deciphering error messages. Concludes with a brief summary of commands, each with one-line descriptions. LAS

Computer Systems, S(16-17), P. *Warren's Abstract Machine: A Tutorial Reconstruction.* Hassan Aït-Kaci. MIT Pr, 1991, xvii + 114 pp, $17.50 (P). [ISBN: 0-262-51058-8] Prolog is the most widely used logic programming language in the world, being implemented on a wide array of different computer systems. Most of those implementations are based on WAM—The Warren Abstract Machine. WAM is an idealised model of an abstract computer system consisting of a memory architecture and instruction set that has been optimised for the execution of Prolog programs. It is the *de facto* standard for all Prolog implementations. This text describes WAM and its internal structure. GMS

Computer Graphics, P. *Curves and Surfaces.* Eds: Pierre-Jean Laurent, Alain Le Méhauté, Larry L. Schumaker. Academic Pr, 1991, xviii + 514 pp, $49.95. [ISBN: 0-12-438660-1] Contains seventy-seven papers dealing with computer imaging of curves and surfaces from an international conference held June 1990 in Chamonix-Mont-Blanc, France. OJ

Computer Science, S(16-17), P. *A Unifying Framework for Structured Analysis and Design Models: An Approach Using Initial Algebra Semantics and Category Theory.* T.H. Tse. Tracts in Theoret. Comput. Sci., V. 11. Cambridge Univ Pr, 1991, xi + 179 pp, $34.50. [ISBN: 0-521-39196-2] Structured analysis and design (SAAD) is an approach to designing and implementing large complex software systems. There are many SAAD techniques currently in use, including methods developed by Jackson, Yourdon, and DeMarco. This work, growing out of the author's Ph.D. thesis, attempts to describe the common theoretical basis of all of these methods using the mathematics of algebra and category theory. He also describes a prototype system that implements and demonstrates his ideas. GMS

Applications (Physical Science), S(17-18), P. *Mathematical Approaches in Hydrodynamics.* Ed: Touvia Miloh. SIAM, 1991, xxi + 517 pp, $68.50 (P). [ISBN: 0-89871-277-7] A collection of papers solicited in honor of Marshall P. Tulin. Categories covered include hydrodynamics of deformable bodies, cavity flow, linearised free-surface flows, non-linear waves, wave diffraction, ship hydrodynamics, vorticity and stratified flows, flows in porous-media, invicid fluids and magnetohydrodynamics, Stokes flows, and lifting surfaces. MU

Applications (Physics), S(15-17), L.** *Quantum Profiles.* Jeremy Bernstein. Princeton Univ Pr, 1991, viii + 178 pp, $17.95. [ISBN: 0-691-08725-3] Three delightful essays on the joys and mysteries of physics conveyed by means of personal sketches of physicists John Stewart Bell and John Archibald Wheeler, based on extensive conversations with each, and of the Swiss engineer Michele Angelo Besso, based on a fifty-two year correspondence with Einstein. Science writing at its best, by one of the masters. LAS

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Abe Shenitzer received his B.Sc. degree from Brooklyn College and his Ph.D. degree from NYU. He is emeritus professor of mathematics at York University in Canada. His previous articles have appeared in *The Mathematical Intelligencer*. Dr. Shenitzer has translated a number of Russian and German mathematical books and papers into English. He is interested in the history of mathematics and its use in the teaching of mathematics.

Nathaniel A. Friedman initially studied engineering at the University of Michigan where he received a B.S.E. (1959) and M.S.E. (1960) before taking a great course in measure theory from Fred Gehring. He switched to mathematics and Brown University where he received a Ph.D. in 1964 under the supervision of Rafael V. Chacon. He spent three years at the University of New Mexico and one year visiting at Westfield College, University of London before coming to SUNY-Albany in 1968, where he has been since. His research interests are in measure-theoretic ergodic theory and he has authored the book *Introduction to Ergodic Theory*, published by Van Nostrand Reinhold Company (1970).

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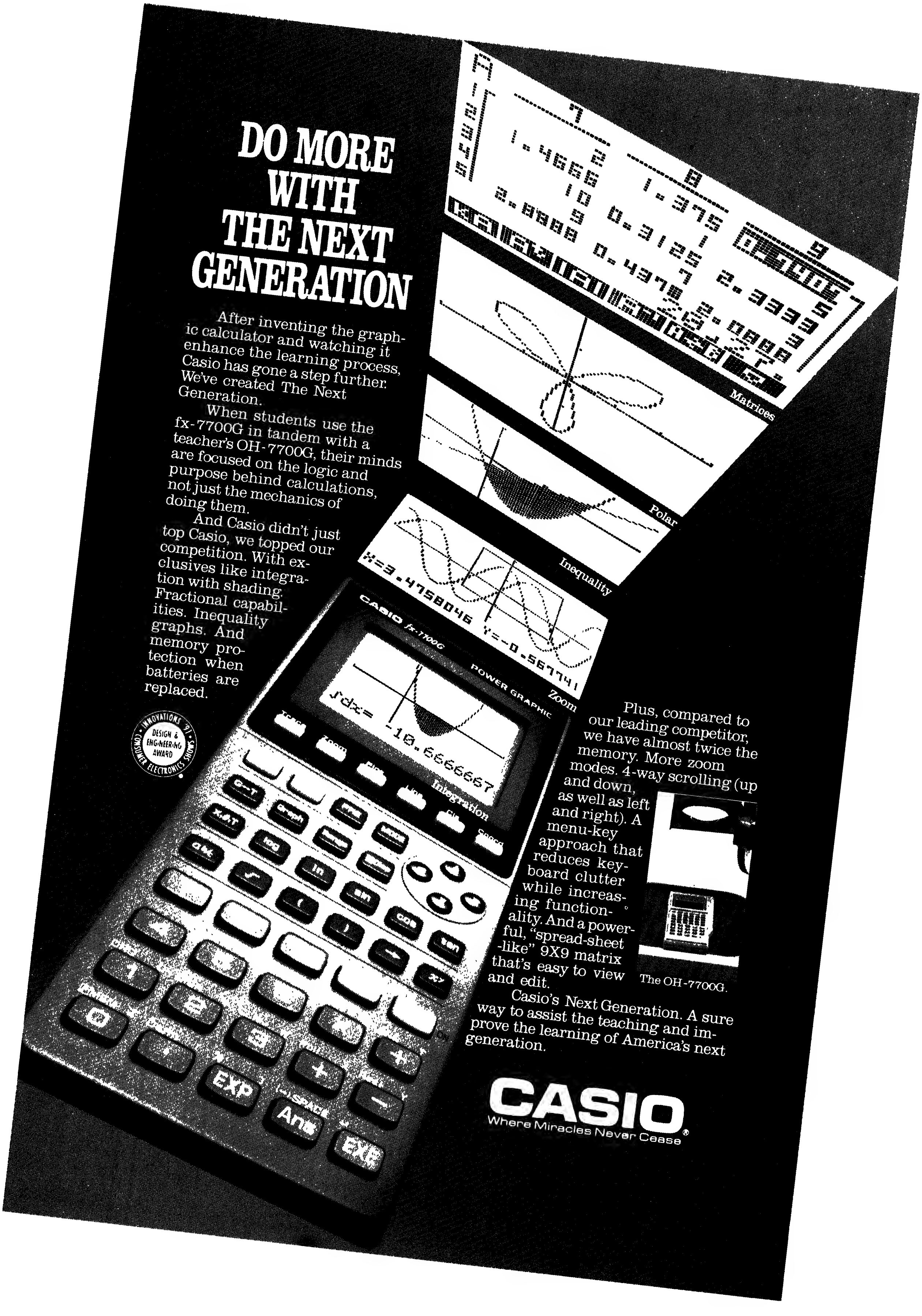
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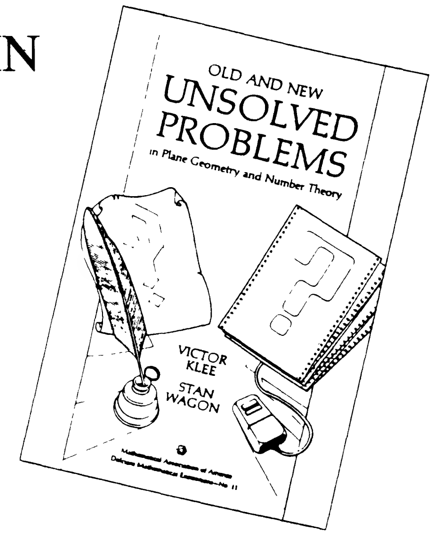

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The American Mathematical Monthly

Volume 99, Number 2 / FEBRUARY 1992

LYNN A. STEEN

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Articles may be expositions of old results or presentations of new ones They may concern all of mathematics or one small area, a broad development or a single application, historical reminiscences or one important event While some articles may contain the author's new research, the novelty of material and generality of the results is far less important than the clarity of exposition and general interest Discussing one illuminating case of a well known result is far better than providing all the details of an obscure but new proposition Articles in the *Monthly* are supposed to inform and to entertain, they are meant to be read rather than archived.

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Pythagoras used to say life resembles the Olympic Games; a few men strain their muscles to carry off a prize; others bring trinkets to sell to the crowd for a profit; and some there are who seek no further advantage than to look at the show and see how and why everything is done. They are spectators of other men's lives in order better to judge and manage their own.

—*Michel de Montaigne (1533–1592)*

There is more than one way to win the prize. In mathematics, of course, we think of prizes as big theorems and the recognition that goes with them. Being a great mathematician means doing great mathematics. Mathematics, however, is rather stingy in awarding such acclaim. Few win prizes—that way.

Such a simple view of prizes does a good deal of harm to our subject. It makes young mathematicians set one-dimensional goals: Prove great theorems, write great papers, win prestigious grants. Because mathematics is stingy in awarding talent as well as acclaim, all too often measuring success becomes slightly distorted: Prove theorems, write papers, and win grants. Young mathematicians (and old ones too) confuse form for substance, measuring the value of research by the number of pages it consumes or the dollars it delivers. They believe they are competing for the prize, while most are selling trinkets for only slight profit.

The tragedy in this is not that so much minor mathematics is published—trinkets have some value after all—but rather that so many mathematicians have a narrow vision of mathematics. They view mathematics as research alone (*their* research), and they equate their own ability to contribute with their ability to publish papers. Instead of seeing mathematics as a broad cultural enterprise, that includes research and teaching and scholarship and history, they see mathematics as a single important but limited activity. They fail to provide service to mathematics because that's not what *real* mathematicians do.

Service is not just sitting on committees. (Surely *some* committees should be classified as "disservice.") Service can be as simple as explaining an intriguing bit of mathematics to a student (or a colleague), or it can be as complicated as setting up a new program on a national level. Service comes from an attitude about mathematics, a sense of history and culture, a passion for the subject rather than its rewards.

Occasionally everyone ought to become a spectator, to step away from the busy crowd, and to look at other ways to compete. Looking at the lives of others shows there are many ways to be a mathematician. There are many ways to win a prize.

—John Ewing

Award for Distinguished Service to Dr. Lynn Arthur Steen

Kenneth M. Hoffman and James R. C. Leitzel

The nation's spotlight is focused on mathematics and the efforts to reform and revitalize its teaching and learning at all levels. Mobilizing the mathematical community and the nation behind one common plan of action has required tremendous effort to build consensus and increase public awareness of the underlying issues. Throughout this process, which has spanned nearly two decades, Lynn Arthur Steen has provided distinguished leadership in formulating national strategies and communicating the issues to the various concerned communities.

Lynn Steen must be considered among the world's preeminent scientists who use their minds and talents to transmit to broad audiences the basic understandings of the nature of their discipline and the surrounding issues in research and education. Through books, magazine articles, original news articles, op-ed pieces, lectures, congressional testimony, and presidential leadership of scientific organizations, he has established a new standard for the intelligible scientific scholar.

A hallmark of Lynn Steen's prolific writing is the lack of wasted breath—every word counts in communicating to his audience. Whether using techniques of analogy, allusion, or alliteration, he conveys a spritely style which captures the imagination of the reader. In all his work there is evidence of his closely coupled intellectual and writing skills. An important gauge of Lynn Steen's work in promoting and capturing public understanding of mathematics is his 1988 article in *Science* magazine—"The Science of Patterns." This article, written for the scientific public, takes the reader on a tour of present-day mathematics, placing the subject in an historical and scientific context, while conveying forcefully the idea that mathematics remains a dynamic discipline—still growing and changing after 7,000 years.

While Lynn Steen's record of publications is extremely lengthy, the January 1989 publication of *Everybody Counts* (National Academy Press) must be considered the premier piece of writing he has done for the general public. The richness of its language and exposition, its pervasive quotability, and its ability to capture and hold its reading audience make it stand out boldly in a landscape flooded with reports on education.

Lynn Arthur Steen graduated from Luther College with a double major in mathematics and physics. He immediately entered the graduate program at the Massachusetts Institute of Technology, where he received his doctorate in mathematics in 1965. In that same year he joined the faculty of St. Olaf College in Northfield, Minnesota, where he has taught for 26 years and is currently Professor of Mathematics and Director of Academic Computing.

His professional activity spans the full spectrum from school mathematics, through deep involvement in issues of undergraduate education, to leadership positions at the national level in the broader mathematical community.

—In 1985–6 he served as President of the Mathematical Association of America, stimulating many new projects, forging linkages with other organizations, coordinating MAA's first long range planning effort, and greatly strengthening MAA's national leadership position.

—He was a founding member of the Mathematical Sciences Education Board, serving both as a member of the Board and its Executive Committee from 1985–1991.

—He has served as codirector of the Minnesota Mathematics Mobilization, the prototype of the Mathematical Sciences Education Board's State Coalition initiatives, and a member of the National Council of Teachers of Mathematics Commission on Standards for School Mathematics.

—He has served on the Conference Board of the Mathematical Sciences (CBMS) for five years and was chair for the period 1988–90. As chair, he was instrumental in securing

external funding for workshops on strategic planning and graduate education in the mathematical sciences. Under the auspices of CBMS, he served as editor of *Mathematics Today: Twelve Informal Essays*, which was published by Springer-Verlag.

—He served as chair of the Council of Scientific Society Presidents in 1989. Under his leadership, that council initiated several activities that are still having direct effect in the broad mathematics and scientific community.

—He served as chair of a joint task force of the American Association of Colleges and the MAA. The report of that task force, *Challenges for College Mathematics: An Agenda for the Next Decade*, outlines significant opportunities for changing the complete undergraduate mathematics environment.

The skill with which he has carried out these varied roles has made Lynn Steen a significant voice in Washington policy circles. His advice and counsel are frequently sought by high ranking officials of such organizations as the National Science Foundation and the Office of Science and Technology Policy.

He was an active and productive member of the Committee on the Mathematical Sciences in the Year 2000 (MS 2000), a joint committee of the National Research Council's Board on Mathematical Sciences and Mathematical Sciences Education Board. He played a major role in the development of the final report of MS 2000, *Moving Beyond Myths: Revitalizing Undergraduate Mathematics*, in which the following observation is made:

> Responses to the problems facing undergraduate mathematics must occur on many fronts, including faculty members and their departments, colleges and universities, business and industry, professional societies, and government agencies. All those with a stake in mathematics must reassert the vital importance of effective undergraduate education in the mathematical sciences. Over the next decade, the mathematical community must restructure fundamentally the culture, content, and context of undergraduate mathematics education.

In characteristic fashion, Lynn Steen, as chairman of the MAA Committee on the Undergraduate Program in Mathematics (CUPM), is already actively pursuing that task. Under his leadership, CUPM has already undertaken several initiatives to accomplish a variety of goals. Recently CUPM has published a report on the Undergraduate Major in the Mathematical Sciences, has working subcommittees addressing issues in the use of technology, requirements for quantitative literacy, new modes of assessment, and various aspects of service courses.

He has an international reputation, serving frequently on program, advisory, or planning groups for various international meetings. He has been involved in one way or another with several International Congresses of Mathematicians and International Congresses on Mathematics Education. Most recently he gave a plenary address at the China regional meeting of the International Council on Mathematics Instruction in Beijing, China.

From 1982–1988 he served as Secretary of Section A (Mathematics) for the American Association for the Advancement of Science. He has also been a member of the Council of the American Mathematical Society, been twice awarded the Lester R. Ford Award for Expository Writing by the MAA, and has been recognized with honorary Doctor of Science degrees by Luther College and Wittenberg University.

While we celebrate and recognize Lynn Steen for his significant works as a scholar, writer, and editor, let us also pay him respect for his unsurpassed leadership as president and chairman of various professional societies in the mathematical sciences. He has challenged the community to consider thoughtfully its role in developing and revitalizing mathematics education at all levels. His ability to conceptualize and organize a task, and then see to its effective conclusion, has enabled the professional community to move to its current place of preeminence.

Whether we are dealing with issues in mathematics today or mathematics tomorrow, the influence of Lynn A. Steen will have continuing impact. There is every evidence that he is just reaching his stride, and we can all look forward to still more significant contributions in the years ahead. It is especially appropriate at this time and for this professional association to honor and celebrate the contributions of Lynn A. Steen by presenting to him the Yueh-Gin Gung and Dr. Charles Y. Hu Award for Distinguished Service.

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Strang's Strange Figures

Norman Richert

Pictures are playing an increasingly important role both in mathematical research and in the teaching of mathematics. Consider the current interest in fractals and in computer programs such as *Mathematica*. Textbooks, particularly at the beginning undergraduate level, need to provide more hooks into this world of pictures. A browse through current calculus texts reveals many computer generated images. In most cases these are not really "interesting" pictures, but only pictures now generated by computer that were formerly created by skilled artists—for example, surfaces. But the revolution in graphics is not the ability to draw pictures that once were very difficult to draw by hand, but rather the ability to draw pictures that were effectively *impossible* to draw by hand.

Why be timid? Let us challenge the students to confront really interesting problems with pictures. For example, in third-semester calculus a battery of techniques for describing the behavior of functions of two variables is developed. The application of these techniques could be viewed as a way of answering questions about pictures. However, their application tends to be trivial. Why? Partly because of the traditional emphasis of quadric surfaces—which happen to be easy to sketch.

A pair of interesting pictures is presented by Gilbert Strang on the cover of his new calculus book **[10]**. Professor Strang presented these plots during the panel discussion, "Calculus for the Twenty-First Century," at the 1990 AMS/MAA meeting in Louisville. They were created by Doug Hardin of Vanderbilt University and they are easy to define yet impossible to draw by hand (no one has the patience). They present mysterious behavior and their "solution" is not really calculus, at least not traditional calculus. Perhaps a "Lean" calculus should stick to business, but part of a "Lively" calculus should be interesting problems.

Figure 1 shows the sine function plotted at integers $n = 1, \dots, 10{,}000$. Figure 2 is an enlarged piece of the same plot, with $n = 1, \dots, 1{,}000$. The first figure seems to be sinusoidal, but it is not $\sin x$. There are too many curves, and their period is wildly wrong (over 15,000). Why doesn't the second plot look more like the first? After all, it is the same function, with the x-scale enlarged.

What is seen could be passed off as the effect of discretization. Is discussion of these effects important in a calculus course? Certainly it cannot be a major part (it takes two pages in Strang's book). On the other hand, part of the philosophy of the current calculus curriculum initiatives implies breaking out of some of the old ruts about the proper content of calculus. Discussion of images generated by computers is an appealing way to implement this philosophy. This note will explore one line of explanation of the plots.

Some very interesting questions can be raised as to what it *means* to plot a function, questions that traditionally have been brushed aside, with cases such as

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational,} \end{cases}$$

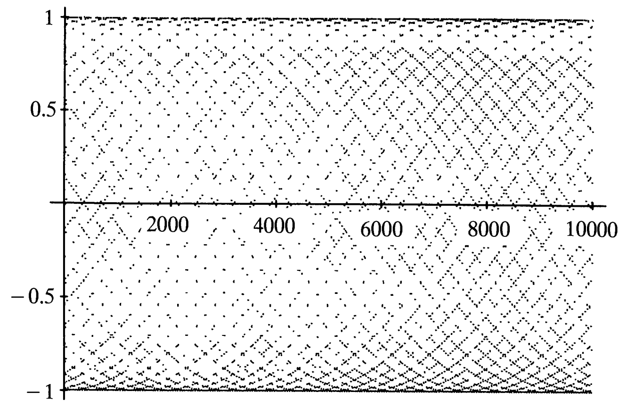

FIG. 1. 10,000 points of $\sin n$, $n = 1, 2, 3, \ldots$. What is the explanation of the many periodic curves?

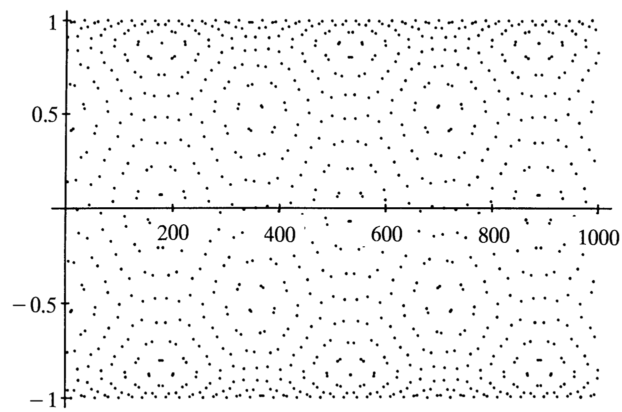

FIG. 2. 1,000 points of $\sin n$. What happened to the periodic curves? Why the hexagonal patterns?

simply viewed as pathological. Yet that example is not so far from what would happen to Figure 2 if more points were plotted: at the scale and dot size of these plots, the graph would look completely black. Computer plotting has made hand plots of points passé in the search for information about a graph. But it has made the *meaning* of plots more pertinent, not less so. Thoughtful students have always had nagging doubts as to why we can blithely play "connect the dots" after a few measly points are plotted. Computer generated plots simply up the ante on these doubts, as these pictures nicely illustrate. This should be a powerful new incentive to ask more interesting questions about functions and their graphs. Many current programs will do a nice job with these plots.

It is not hard to see that the difference between the two figures has to do with *scale*. The current interest in fractals makes such issues topical. Computer programs have become increasingly sophisticated in dealing with scaling issues automatically. Yet scaling issues will not simply go away, as every user of graphical software knows. It seems quite appropriate to begin to discuss rescaling in the context of calculus.

How does the apparently nested family of sine curves arise in Figure 1? They are in some sense optical illusions, like the apparent spirals of seeds in the

sunflower. The seeds actually grow in a single tight spiral out of the center. The adjacencies in these figures and in sunflowers defeat any mental effort to see the "real" curve. The key to Figure 1 must lie in a consideration of adjacency, or "nearness." The subinterval size of 1 is a substantial fraction of the total period of 2π, so $\sin k$ and $\sin(k+1)$ are usually quite far apart (when are they close?) in the y-direction. How small can $\sin(k+p) - \sin k$ get for p an integer? Because sine is periodic, this is directly related to how close p is to a multiple of 2π. That is, how small can $|p - q(2\pi)|$ be for p and q positive integers? Because the sine function is continuous and periodic with period 2π, for $|p - q(2\pi)|$ small enough, $|\sin(k+p) - \sin k|$ will be small, independently of k. In fact, a little extra attention paid to the derivative will show that $|\sin(k+p) - \sin k| \le |p - q(2\pi)|$. The graph points $(k, \sin k)$ and $(k+p, \sin(k+p))$ will then be close, so the collection of points $(k + mp, \sin(k+mp))$, $m = 0, 1, 2 \dots$, will appear to form a curve. For $k = 0$ this is the curve that corresponds to a rescaling of the x-axis by a factor of $|(p - q(2\pi))/p|$, namely,

$$f_0(x) = \sin\left(\frac{p - q(2\pi)}{p}x\right).$$

The whole family of curves is

$$f_k(x) = \sin((p - q(2\pi))x/p + 2\pi qk/p), \qquad k = 0, \dots, p-1.$$

But how small is "small"? This is where scale comes in. The figures at Louisville measured roughly 10×14 cm. At this scale, small would seem to be less than 1.0 mm, which is to say less than 70 (Figure 1 units) on the horizontal scale and less than 0.02 on the vertical scale. So we now have detailed specifications: find positive integers $p \le 70$ and q so that $|p - q(2\pi)| \le 0.02$.

Now we arrive at a question that is not really calculus, but number theory. How small can we make $|p - q(2\pi)|$ for p and q integral? A related, though weaker question is how small we can make $|p/q - 2\pi|$, a question probably more suggestive to most students. We have meandered into the area called *diophantine approximation*, a piece of which is the study of rational approximations to irrational numbers. Most students know that $22/7$ is a good approximation to π, so they all know a tiny bit of diophantine approximation. It is not implausible to suppose that $44/7$ is a good approximation to 2π. In fact, $44 - 7(2\pi) \approx 0.018$, meeting both the smallness measures estimated above. So $\sin(k + 44) - \sin k$ will be relatively small for each k. In fact, a new plot of $\sin(44n)$ quickly shows that one of the family of nested curves has been identified. It is the increasing curve through the origin in Figure 1.

But there are lots of other good approximations to 2π, say $628/100 = 157/25$. Why don't they show up in the picture? We shouldn't stray too far from calculus, except to point out that there are not *lots* of other good approximations to 2π, at least not with numerators less than 70, or what comes to the same thing, denominators less than 11. (The references **[5, 6, 9]** contain further reading.) A lot can be learned with a calculator by hunting for values p and q to make $|p - q(2\pi)|$ small. In particular, $|157 - 25(2\pi)| = 0.08$, so we do worse by a factor of 3.6 horizontally and a factor of as much as 4.9 vertically. This fraction simply does not yield the strong adjacency patterns that $44/7$ does. In fact, considering all values of p and q, the next smallest value of $|p - q(2\pi)|$ is 0.009, using the fraction $333/53$. But 333 is way off the "smallness" scale in the x-direction.

Finding the Good Approximations to 2π

In the discussion of the figures, we have used particular approximations to 2π, the *best approximations*. To be precise, the best approximations to a real number x are the rationals p/q so that $|p - qx| < |p' - q'x|$ for $p', q' \in \mathbf{Z}$, $0 < q' \leq q$ and $p'/q' \neq p/q$. Clearly, given q, we can take p to be the nearest integer to qx. For x irrational, this uniquely defines a sequence $p_0/q_0, p_1/q_1, p_2/q_2, \dots$, of best approximations to x, with $q_0 < q_1 < q_2 < \cdots$. In fact, an initial segment of the sequence can be calculated by trial and error from the definition simply by considering increasing q. The table illustrates this procedure for $x = 2\pi$.

q	1	2	3	4	5	6	7
qx	6.283	12.566	18.850	25.133	31.416	37.699	43.982
$p - qx$	0.283	−0.434	−0.150	0.133	0.416	−0.301	−0.018

Examining the table, we see that $p_0/q_0 = 6/1$, $p_1/q_1 = 19/3$, $p_2/q_2 = 25/4$, and $p_3/q_3 = 44/7$.

The *continued fraction* algorithm provides a mechanism for directly calculating these approximations, without the need for trial and error calculations. A number of recent Monthly pieces have treated continued fractions, for example [4, 7]. A very beautiful older piece by L. R. Ford is [3]. Let $[x]$ denote the *greatest integer* of x, the largest integer not larger than x. Set $x_0 = x$ and $a_0 = [x_0]$. Then recursively calculate a pair of sequences: $\{x_k\}$ of real numbers and $\{a_k\}$ of positive integers, $k = 1, 2, 3, \dots$, with

$$x_k = \frac{1}{x_{k-1} - a_{k-1}} \quad \text{and} \quad a_k = [x_k].$$

The a_k's are the *partial quotients*. Then the sequence of *convergents* $\{p_k/q_k\}$ is calculated recursively by

$$\begin{aligned} &p_{-1} = 1, \quad p_0 = a_0, \quad q_{-1} = 0, \quad q_0 = 1, \quad \text{and} \\ &p_k = a_k p_{k-1} + p_{k-2}, \\ &q_k = a_k q_{k-1} + q_{k-2}, \quad \text{for } k = 1, 2, 3, \dots. \end{aligned}$$

This algorithm can easily be programmed. The convergents converge fairly rapidly to x (hence the name). This set of convergents is, with the possible exception of p_0/q_0, identical to the set of best approximations to x. For $x = 2\pi$, we have $[a_0; a_1, a_2, a_3, \dots] = [6; 3, 1, 1, 7, 2, 146, 3, \dots]$. Complete understanding of these figures, particularly Figures 4 and 5, requires the use of a larger set of approximating fractions, the *best approximations of the first kind*. Those which are not best approximations are *intermediate fractions*, that is, of the form

$$\frac{p}{q} = \frac{a p_{k-1} + p_{k-2}}{a q_{k-1} + q_{k-2}}, \qquad a = 1, 2, \dots, a_k - 1.$$

Where did the 44 sine curves go in Figure 2? One answer involves the scale change. The "small" distance in Figure 1 now corresponds to a range of 7 in n values, so 44 has become "large." The scale can be restored by tilting the page and viewing from the side so as to compress x distances. Violà, the curves reappear. But what about the hexagonal pattern?

In Figure 2 we are seeing an interference pattern created by the interaction of three distinct approximations to 2π: $44/7$, $25/4$ and $19/3$. Such patterns result when regular patterns of dots are overlaid on each other and are known as Moiré patterns [**1**, **2**, **8**]. Of interest here is the case when the patterns are *regular screens of dots*, that is, lattices of points formed by the vertices of a tessellation of the plane by regular polygons. Regular screens of dots produce these Moiré interference patterns in a limited number of ways, one of which consists of roughly hexagonal regions. This is an important issue in printing and textile manufacture. These patterns are the telltale sign that a printed photograph, as in a newspaper, has been reproduced from another printed photograph, rather than from an original. To uncover the roughly regular screens of points producing hexagonal Moiré patterns in Figure 2, we must do a bit more analysis.

As is discussed in the accompanying box, there is an infinite sequence of *best approximations* to 2π (or any irrational number). The first few terms in the sequence are $6/1$, $19/3$, $25/4$, $44/7$ and $333/53$. We have seen why $333/53$ is not prominent in Figure 1. It will simplify the discussion if we limit our analysis to points near the x-axis, where $|\Delta \sin x| \approx |\Delta x|$, so that we may take the $|p - q(2\pi)|$ values as exact vertical displacements. The features being discussed are most influenced by the periodicity and the period, rather than the shape of the graph. (We could even substitute a sawtooth function for $\sin x$. The sine function has the conceptual appeal that the value of its period is implicit rather than explicit.)

The previous discussion implicitly used the metric $d_{\max}(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$. Let us measure actual distances on a graph of N points. The plotting rectangle has height v and width av, with $v = 100$ mm and $a = 1.4$ on the Louisville figures. Then the distance d (in graph measurement units) between points on a member of the curve family associated with p/q is

$$d = \sqrt{\left(\frac{p - q(2\pi)}{2/v}\right)^2 + \left(\frac{p}{N/av}\right)^2} = \frac{v}{2}\sqrt{(p - q(2\pi))^2 + \left(\frac{2a}{N}\right)^2 p^2}.$$

In Figure 1, with $N = 10\,000$, the values associated with $44/7$, $25/4$ and $19/3$ respectively are roughly 1.1, 6.6 and 7.5 mm. So the eye finds the $44/7$ family to be a clear feature. In Figure 2 with $N = 1000$, these values become 6.2, 7.5 and 8.0, so three distances are comparable.

Consider a point $P_1(n_1, \sin n_1)$ near the x-axis. The next point to the right on the family of curves associated with $44/7$ is $P_2(n_1 + 44, \sin(n_1 + 44))$. Suppose that $\sin x$ is increasing at n_1, so that $\sin(n_1 + 44) - \sin n_1 \approx 44 - 7(2\pi) > 0$. The next point to the right on the family associated with $25/4$ is $P_3(n_1 + 25, \sin(n_1 + 25))$, for which $\sin(n_1 + 25) - \sin n_1 \approx 25 - 4(2\pi)) < 0$. Hence P_3 is downhill from P_1. Finally, since $44 - 25 = 19$, the $\overline{P_2 P_3}$ segment is associated with the $19/3$ family. Because the distances, calculated above, are roughly equal, the triangle $P_1 P_2 P_3$ is roughly equilateral. This forms the template of a regular screen with hexagonal symmetry, that is, six-fold rotational symmetry. The overlay of two hexagonal screens produces a Moiré pattern with hexagonal symmetry, which we see in Figure 2.

Because the Moiré pattern dominates our view of Figure 2, it is hard to see the regular screens which produce it. Separate the sine function into two functions: the function of increasing pieces, and that for decreasing pieces. Let

$$\operatorname{SinUp}(x) = \begin{cases} \sin x & \text{if } \sin x \text{ is increasing at } x, \\ 0 & \text{otherwise.} \end{cases}$$

Define SinDown similarly. If $p - q(2\pi) > 0$ then plotting SinUp instead of sine yields the increasing pieces of the corresponding family of curves. If $p - q(2\pi) < 0$ it yields the decreasing pieces.

In the case of $N = 1000$, plotting SinUp reveals the roughly regular screen of points, as Figure 3 shows. Plotting SinDown yields essentially the mirror image of this lattice across a vertical line. Plotting the sine function overlays these two lattices, and produces the hexagonal Moiré pattern of Figure 2. This can be checked directly by making a transparency of Figure 3, flipping it left to right, and overlaying it on Figure 3. The families of curves corresponding to $25/4$ and $19/3$ which are clear in Figure 3 can be seen in Figure 2 by tilting the page: by viewing at an angle roughly 10° away from the y-axis. The screens of SinUp and SinDown when $N = 10\,000$ are too far from regular for Moiré interference to be noticeable in Figure 1.

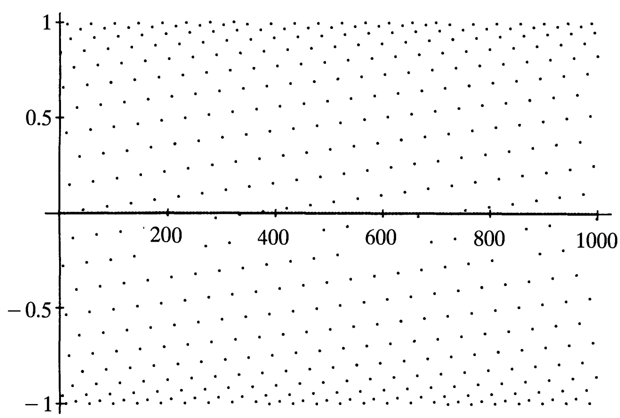

FIG. 3. 1,000 points of SinUp n. Hexagons vanish.

A good test of these observations might be to modify the sine function to yield a period different than 2π, and hence (presumably) different approximating fractions, and make some more plots. Plotting $\sin((2\pi)n/(6 + \varepsilon))$ for various ε is one way to modify the period. Setting $\varepsilon = \phi - 1 = (1 + \sqrt{5})/2 \approx 0.618034$, the fractional part of the golden ratio ϕ, yields some plots quite different than Figures 1 and 2, as Figure 4 illustrates. They are relatively unaffected by changes in scale. The continued fraction expansion of this period differs from that of 2π beginning with the first order partial quotient. In fact, the features of these figures are quite sensitive to the exact value of the period. Figure 5 illustrates what a more substantial modification of the period can produce, with period $2\pi - 5$.

The importance of computing in calculus cannot be overstated. The symbolic capabilities are forcing us to reevaluate the importance of rote techniques in differentiation and integration. Simultaneously the graphical capabilities allow us to discuss genuinely interesting graphs. I applaud Professor Strang for a step in the direction of interesting graphs.

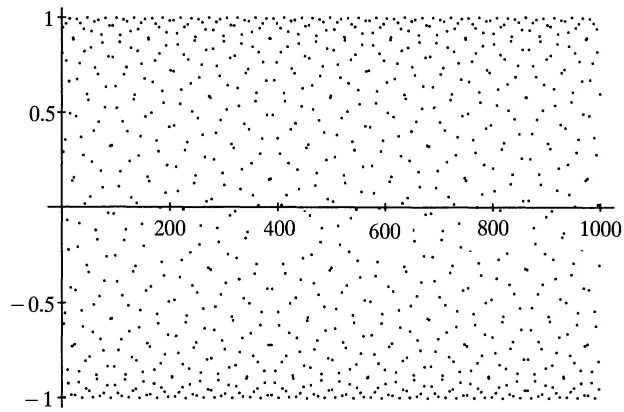

FIG. 4. No hexagons. No periodic curves. 1,000 points of $\sin(n(2\pi/(5+\phi))$.

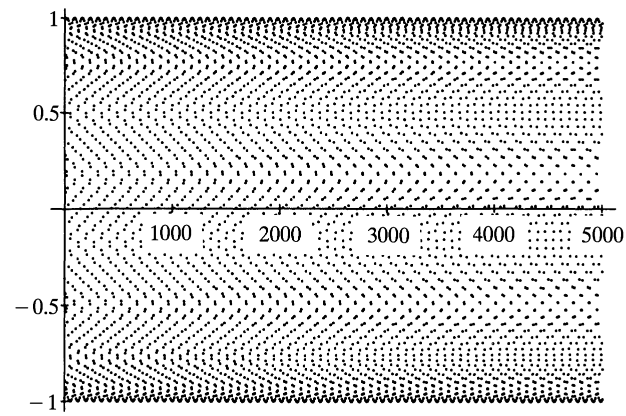

FIG. 5. A very different effect. 5,000 points of $\sin(n(2\pi/(2\pi-5))$.

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Zonohedra and Generalized Zonohedra

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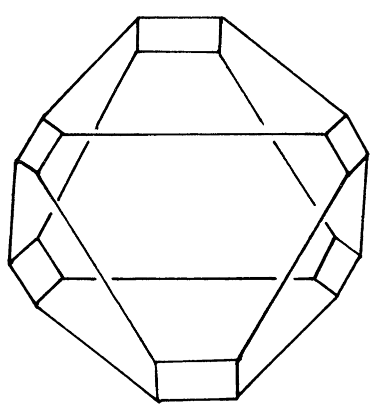
The purpose of this note is primarily to disclose the results of some historical sleuthing which has uncovered an old misreading that is now in widespread use. It will also say a little about why zonohedra are interesting, including why the original definition is useful.

We define a *generalized zonohedron* to be a polyhedron (in R^3) such that each face has an even number of edges and on each face, each edge is parallel to its opposite edge; an example is in Figure 1. This definition is equivalent to, and essentially the same as, the definition of a zonohedron given originally by the Russian crystallographer Fedorov [F].[1] Another equivalent definition would be to say that the union of the images of the edges of the polyhedron under the generalized Gauss map consists of complete great circles. This map sends each face of the polyhedron to its exterior unit normal, each edge to the shorter great circle segment connecting the exterior unit normals of the planes intersecting along that edge, and each vertex to the smaller spherical region bounded by the appropriate great circle segments. Thus the generalized Gauss map of a polyhedron induces a decomposition of the sphere, called the n-diagram of that polyhedron, which is a realization of the topological dual of the polyhedron.



Coxeter, in [**C1**, pp. 27–28], first considered zonohedra which are convex polyhedra bounded by parallelograms; opposite edges of a parallelogram are

[1]Fedorov's definition (page 688 of [F], section 65, definition 16) is as follows: "Unter Zonoëder versteht man ein Polyëder, dessen Flächen sammtlich im (primären) Zonenverbande stehen"; just prior to this (definition 13) is the definition "Unter einer primaären Zone versteht man eine Reihe von in parallelen Kanten der Figur sich schneidenden Flächen, sonst ist die Zone secundär."

necessarily of the same length. He then developed a construction for zonohedra (given below), which implicitly assumes that the opposite edges of each face, whether four-sided or not, are of equal length. He was apparently thereby led into believing that opposite edges of each face of *any* zonohedron must also be of the same length, stating on page 31 that "[Fedorov] does not seem to have realized, however, that a convex zonohedron is capable of such a simple definition as this: a convex polyhedron whose faces are centrally symmetrical polygons." In subsequent work [**C2**, p. 140] he in fact took that "simple definition" as the definition of a convex zonohedron and referred the reader to Fedorov without further explanation. However, whenever one of Coxeter's zonohedra C has a face all of whose vertices are trivalent, then there are generalized zonohedra with the same n-diagram as C but which are not zonohedra in Coxeter's sense, since such a face can be shifted parallel to itself maintaining the same adjacency relationships. Since Coxeter's redefinition is now in widespread use, it is probably less confusing to continue to use it as the definition of a zonohedron, and to use Fedorov's original definition, or one of its equivalent reformulations, as the definition of a generalized zonohedron, as above.

It should be immediately pointed out that any generalized zonohedron is isomorphic to a zonohedron in Coxeter's sense, as Fedorov himself more or less showed [**F**].[2] Two polyhedra are isomorphic if they have "the same abstract description," that abstract description being "assigning symbols to the vertices and writing down the cycles of vertices that belong to the various faces" [**C1**, 106–107]. For convex polyhedra, this means that their n-diagrams are topologically the same. To any generalized zonohedron there corresponds a collection of diameters of the sphere, namely those parallel to families of edges of the generalized zonohedron. The union of these diameters can be used as a star to construct, via Coxeter's star construction, a zonohedron where all edges have the same length: take the convex hull of the set of points which are sums of subsets of the set of ends of the line segments forming the star (a slight rephrasing of [**C1**], pp. 27–28; more briefly, and in a phrasing which does not require the line segments to contain the origin or any other common point, take the Minkowski sum of the line segments). This zonohedron will have edges and faces parallel to those of the original generalized zonohedron; in fact, both will have the same generalized Gauss map and thus the same n-diagram.

A convex body defines a surface energy function, namely the support function of that convex body. The convex body is then the Wulff shape (the equilibrium crystal shape) for that surface energy function ([**B1**, **T1**]). Suppose we want to solve the analog of a soap-film problem, i.e., to prescribe a boundary and to find surfaces of least surface energy having that boundary; in case of nonuniqueness, we look for the minimizing surface which has the most volume behind it. This problem is particularly nice when the Wulff shape is a generalized zonohedron (as it is for potassium aluminum alum [**B2**], for example) and the boundary consists of line segments parallel to edges of the generalized zonohedron. Then the resulting surface has all its tangent planes parallel to faces of the generalized zonohedron (at least if it is of finite topological type—an unsolved problem) [**T1**]. Furthermore, there is a construction for such minimizing surfaces in this case [**T2**]. It was the

[2]Part IV, Chapter 12, Section 69, Theorem 24, on page 689, says "Existirt ein convexes Zonoëder, so existirt auch ein gleichkantiges, sonst ihm morphologisch gleiches Zonoëder."

desire to give a name to these nice Wulff shapes that led the author to Fedorov and the discovery that his use of the word zonohedron was what was desired.

Fedorov introduced zonohedra as a step toward understanding tiling of 3-dimensional space, aiming towards the classification of symmetry groups of crystals. Zonohedra have appeared recently in the literature on the cut-and-project method for generating mathematical analogs of quasicrystals [**DK**, **ES**], since the projection of any n-cube (or more generally, the Voronoi polytope for any set of lattice points) into a lower dimensional space is a zonohedron (in Coxeter's sense). Zonohedra and their higher dimensional analogs are also the subject of some current research. For example, [**BM**] shows that the inradius and circumradius of a zonohedron provide bounds in the estimation of the surface area of a convex body by a finite number of projections, and it gives bounds on how closely zonohedra can approximate the unit ball as a function of the number of line segments which sum to make the zonohedra. Also, among all zonohedra which are the sum of 3, 4, or 6 line segments of unit length, those whose faces are congruent rhombi have the largest inradius, whereas whether they have the smallest circumradius is still open [**L**]. For some interesting applications of zonohedra in the spirit of Buckminster Fuller, see [**B3**].

Finally, there is a related question concerning graphs and convex bodies. The n-diagram of a convex polyhedron is essentially a particular embedding of a graph on a sphere and has the property that all edges are "short" segments of great circles ("short" meaning shorter than a semicircle). Is the converse true—given an embedding of a graph on the sphere with that property, is there a convex polyhedron with that embedded graph as its n-diagram? Coxeter's star construction shows that it is true if the union of the great circle segments consists of complete great circles; the existence of generalized zonohedra shows that in fact there can be whole families of such convex polyhedra associated to such a given embedded graph whenever the graph has a vertex surrounded by triangles. The general problem can be phrased in terms of linear programming, but the answer is still not obvious (to the author, at least!).

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How to Make Pi Equal to Three

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Pi is equal to the quotient obtained when the distance around a circle is divided by the distance across. For most circles, pi is a little bit bigger than three. But, for spinning circles, the Lorenz-Fitzgerald contraction must be taken into account. Since the circumference of a rotating circle lies in the direction of motion, its length decreases as the rate of rotation increases. Since the radius of a rotating circle lies perpendicular to the direction of motion, its length remains constant. Therefore, as the rate at which a circle rotates increases, pi decreases. A simple calculation shows that, for a circle one meter in radius, rotation at roughly ten million revolutions per second will bring about the desired value for pi.

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The Uniformization of Rectangles, an Exercise in Schwarz's Lemma

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INTRODUCTION. This note concerns Riemann mappings of rectangles onto the unit disk in the complex plane. Let's fix some notation for the discussion. For the unit circle and unit disk we'll use, respectively, $\mathbf{S}^1$ and $\mathbf{D}$. For the rectangle parallel to the coordinate axes, centered at the origin $0 \in \mathbf{C}$ with height 1 and base length $a \geq 0$, so that the corners are at $(\pm a/2, \pm 1/2)$, we'll use $\mathrm{R}(a)$. In particular, it will be convenient to have $\mathrm{R}(b) \subset \mathrm{R}(a)$ if $b \leq a$.

Riemann's mapping theorem says that every simply connected plane domain Ω not equal to the whole plane can be mapped conformally one-to-one onto $\mathbf{D}$. Thus this certainly holds for the interior of rectangles. These maps are thoroughly well understood from several points of view, primarily using the Schwarz-Christoffel formula (elliptic functions in a not too subtle disguise).

The purpose of this note is to show that rectangles with different modulus (different values of a) can be seen as conformally different and completely characterized by sets of four distinct points on $\mathbf{S}^1$ with very little machinery. In fact, our only tools will be the Schwarz reflection principle, the Schwarz lemma (in other words, Schwarz seemed to have a pretty good sense about this sort of thing), some elementary conformal mappings, and the concept of normal families of analytic functions. Perhaps then the real purpose of this note is to serve as a reminder of the power of the profound, beautiful, and essentially elementary maximum modulus principle.

The only reference necessary for this discussion is [A]. The theorem of Löwner may also be found in [C]. The proof given there is of a somewhat different flavor, and Löwner's proof [L] was roughly the same as the one given here.

1. THE PROBLEM. A Riemann mapping of a plane domain Ω is uniquely determined by giving a point $p \in \Omega$ whose image is $0 \in \mathbf{D}$ and a tangent direction at p whose image is in the positive real direction at 0. By Schwarz's lemma, the Riemann map satisfying these conditions is the unique $f\colon \Omega \to \mathbf{D}$ maximizing $|f'(p)|$.

(*) For the rectangle $\mathrm{R}(a)$ we denote by $\varphi_a\colon \mathrm{R}(a) \to \mathbf{D}$ the Riemann map such that $0 \mapsto 0$ and the positive real direction is preserved, and have that $\varphi_a'(0)$ is maximized over all such maps.

It is easy to see, using the reflection principle, that φ_a extends continuously to the boundary of $\mathrm{R}(a)$. This extension is one-to-one from $\partial\mathrm{R}(a)$ onto $\mathbf{S}^1$. It preserves sets of positive linear measure and sets of zero linear measure, and is analytic across the four bounding segments of $\mathrm{R}(a)$. If

$$\varphi_a\left(\frac{a}{2} + i\frac{1}{2}\right) = \zeta(a) \in \mathbf{S}^1$$

then we see, again via the reflection principle, that

$$\varphi_a\left(\frac{a}{2} - i\frac{1}{2}\right) = \overline{\zeta(a)}, \qquad \varphi_a\left(-\frac{a}{2} - i\frac{1}{2}\right) = -\zeta(a),$$

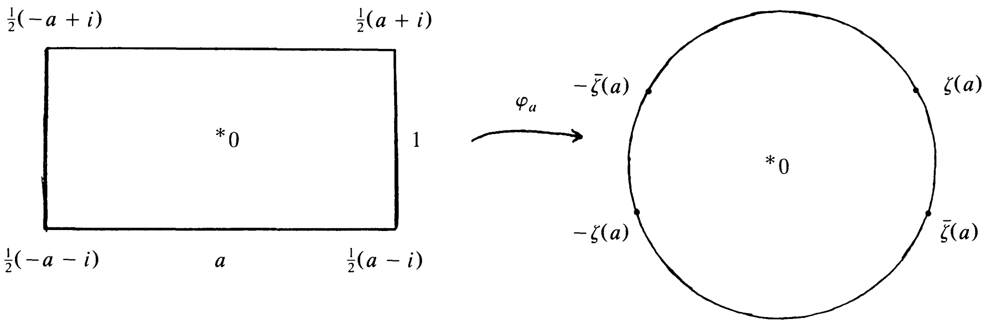
and

$$\varphi_a\left(-\frac{a}{2} + i\frac{1}{2}\right) = -\overline{\zeta(a)}.$$

In fact,

$$\varphi_a\left(\frac{a}{2} + i\frac{1}{2}\right) = \zeta = e^{i\theta} \quad \text{with } \theta \in \left(0, \frac{\pi}{2}\right).$$

Moreover, Schwarz's lemma implies that if $0 < b < a$ then $\varphi_b'(0) > \varphi_a'(0)$.



Observation. *Any ordered* 4*-tuple of points on* $\mathbf{S}^1$, $(\alpha, \beta, \gamma, \delta)$ *is determined, up to a Möbius transformation sending* $\mathbf{D}$ *to itself, by the cross ratio*

$$\chi(\alpha, \beta, \gamma, \delta) = \frac{\alpha - \gamma}{\alpha - \delta}\frac{\beta - \delta}{\beta - \gamma}.$$

If the 4-tuple is ordered cyclically on $\mathbf{S}^1$ then $\chi(\alpha, \beta, \gamma, \delta) > 1$ (remember, cyclic means with increasing θ). Thus we have a map $\mathbf{X}$: $(0, \infty) \to (1, \infty)$ defined by

$$\begin{aligned}\mathbf{X}(a) &= \chi\left(\varphi_a\left(\frac{a}{2} + i\frac{1}{2}\right), \varphi_a\left(-\frac{a}{2} + i\frac{1}{2}\right), \varphi_a\left(-\frac{a}{2} - i\frac{1}{2}\right), \varphi_a\left(\frac{a}{2} - i\frac{1}{2}\right)\right)\\ &= \chi\left(\zeta(a), -\overline{\zeta(a)}, -\zeta(a), \overline{\zeta(a)}\right).\end{aligned}$$

Problem. *Show that* $\mathbf{X}$ *is continuous, one-to-one, and onto* $(1, \infty)$.

We will do this in three steps. First we show that it is one-to-one and monotone. Then that it is continuous. And finally that it is onto. Schwarz's lemma allows us to

obtain from one-to-oneness the following classical corollary. The proof is left as an exercise.

Corollary. *Let* $\mathrm{R}_1, \mathrm{R}_2 \subset \mathbf{C}$ *be two rectangles in the plane with sides* $\mathrm{A}_i, \mathrm{B}_i, \mathrm{C}_i, \mathrm{D}_i$ *having lengths* $l(\mathrm{A}_i) = l(\mathrm{C}_i) = a_i$, $l(\mathrm{B}_i) = l(\mathrm{D}_i) = b_i$, $(i = 1, 2)$. *Then there is a conformal map from* R_1 *to* R_2 *with boundary correspondence sending* A_1 *to* A_2, *etc., if and only if* $a_1/b_1 = a_2/b_2$.

As the square, i.e. R(1), has symmetry with respect to its diagonals, we have that $\zeta(1) = e^{i(\pi/4)}$ and $\mathbf{X}(1) = 2$. In fact one sees readily that

$$\mathbf{X}\left(\frac{1}{a}\right) = \frac{\mathbf{X}(a)}{\mathbf{X}(a) - 1}$$

and thus when discussing onto-ness we restrict our attention to showing that $\mathbf{X}(a) \to \infty$ as $a \to \infty$.

2. MONOTONICITY.

Löwner's Theorem. *Let* $f\colon \mathbf{D} \to \mathbf{D}$ *with* $f(0) = 0$. *Assume that this function maps an arc* $A \subset \mathbf{S}^1$ *of lengths* s *onto an arc* $f(A) \subset \mathbf{S}^1$ *of length* σ. *Then we must have* $\sigma \geq s$ *with equality if and only if either* $s = \sigma = 0$, *or* f *is a pure rotation, i.e.,* $f(z) = e^{i\theta}z$ *for some* $\theta \in [0, 2\pi)$.

Corollary. *If simply connected domain* $\Omega \subset \mathbf{D}$ *contains* 0, *and if the boundary of* Ω *contains a circular arc on* $\mathbf{S}^1$ *with positive angular measure, then a Riemann map* f: $\Omega \to \mathbf{D}$ *with* $f(0) = 0$ *satisfies*

$$m(A) \geq m(f(A))$$

with equality if and only if $\Omega = \mathbf{D}$ *and* f *is a rotation.*

Proof of theorem. Here's the picture. Assume $s \geq 0$, so that we can reflect across A.

Let $f\colon \mathbf{D} \to \mathbf{D}$, and consider

$$F(z) = \log\frac{f(z)}{z} = \log\left|\frac{f(z)}{z}\right| + i\Theta\left(\frac{f(z)}{z}\right) = U + iV,$$

so that

$$\partial_r U = \frac{1}{r}\partial_\theta V$$

by the Cauchy-Riemann conditions. By reflection, f extends across A. Thus as $r = 1$ on A we have that $\partial_r U = \partial_\theta V$. Since $\log|f(z)/z| \leq 0$ in $\mathbf{D}$ (by Schwarz's lemma), and $= 0$ on A, we have that $\partial_r U \geq 0$ on A.

Thus $\partial_r \log|f(z)| - 1 \geq 0$, or $\partial_r \log|f(z)| \geq 1$, and hence $\partial_\theta\Theta(|f(z)|) \geq 1$ on A. It follows that $m(f(A)) \geq m(A)$. we have equality here if and only if $\partial_r \log|f(z)| \equiv 1 \equiv \partial_\theta\Theta(|f(z)|)$ on A, where F is analytic. Thus F is constant and f is a rotation. ◇

Now if we have $1 \leq b \leq a$ then $\varphi_a|_{R(b)}$ has the images of the top and bottom segments of R(b) strictly shorter than those for R(a). Hence by Löwner's theorem they are shorter still under φ_b. This implies strict monotonicity of **X**, and in fact that R(a) and R(b) are conformally distinct for $a \neq b$.

3. CONTINUITY OF X. We will show that if $a_n \nearrow a$ then $\mathbf{X}(a_n) \nearrow \mathbf{X}(a)$, as $a_n \searrow a$ implying $\mathbf{X}(a_n) \searrow \mathbf{X}(a)$ is similar and left as an exercise.

To this end, let $a_n > a/2$ so that on reflection of φ_{a_n} in the left and right sides of R(a_n) we obtain a bounded univalent holomorphic function on R(a). Moreover, as the a_n increase the images of R(a) are decreasing. More precisely, if we denote by φ_{a_n} (R(a)) the image of R(a) under the reflection of φ_{a_n} and $a/2 < a_m \lneq a_n < a$ then $\varphi_{a_n}(\mathrm{R}(a)) \subsetneq \varphi_{a_m}(\mathrm{R}(a))$ by Schwarz's lemma.

The point is that the φ_{a_n} form a normal family and hence converge to a function φ: R(a) $\to$ **D**, as any point in R(a) is eventually in the R(a_n) so that its image under the φ_{a_n} is eventually in **D**. This limiting function φ is not constant as $0 < \varphi_a'(0) < \varphi_{a_n}'(0)$ for all n implies $0 < \varphi_a'(0) \leq \varphi'(0)$. We can now argue that φ is univalent using Hurwitz's theorem, but it is easier to see directly that $\varphi = \varphi_a$ since, as we noted in the introduction, φ_a is the unique function from R(a) to **D** satisfying (*). Thus we have $\mathbf{X}(a_n) \nearrow \mathbf{X}(a)$ as desired.

4. ONTO-NESS. If $a_n \nearrow \infty$ then $\mathrm{R}(a_n) \to \Omega = \{z\colon |\Im z| < 1/2\}$. Then again the φ_{a_n} form a normal family on every subdomain of Ω with compact closure in **C**, and again we denote the limiting function by $\varphi\colon \Omega \to \mathbf{D}$. Now the uniformizing map for Ω satisfying (*) is $\varphi_\infty = (e^{\pi z} - 1)/(e^{\pi z} + 1)$ and we argue as in §3 that $\varphi = \varphi_\infty$.

Finally, onto-ness follows from a straightforward calculation that

$$\chi\left(\varphi_\infty\left(a + i\tfrac{1}{2}\right), \varphi_\infty\left(a - i\tfrac{1}{2}\right), \varphi_\infty\left(-a + i\tfrac{1}{2}\right), \varphi_\infty\left(-a - i\tfrac{1}{2}\right)\right) \to \infty \quad \text{as } a \to \infty.$$

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The Jordan-Schönflies Theorem and the Classification of Surfaces

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INTRODUCTION. The Jordan curve theorem says that a simple closed curve in the Euclidean plane partitions the plane into precisely two parts: the interior and the exterior of the curve. Although this fundamental result seems intuitively obvious it is fascinatingly difficult to prove. There are several proofs in the literature. For example, Tverberg [**12**] gave a proof involving only approximation with polygons. Here, we give a short proof based only on a trivial part of Kuratowski's theorem on graph planarity (see Lemma 2.5, below), namely, that $K_{3,3}$ is not planar.

Then we turn to another fundamental topological result: the classification of (compact) surfaces. A *surface* is a connected compact topological space which is locally homeomorphic to a disc (that is, the interior of a circle in the plane). The classification of surfaces says that every surface is homeomorphic to a space obtained from a sphere by adding handles or crosscaps. One of the first complete proofs was given by Kerékjártó [**4**] and there are several short proofs based on the assumption that every surface can be triangulated (see e.g. [**1, 2**]). Tutte [**11**] gave a proof in a purely combinatorial framework. In this paper we present a self-contained proof. The proof consists of two parts: a "topological" part and a "combinatorial" part. The combinatorial part (Section 5) is very short. It differs from other proofs in that it uses no topological results, not even the Jordan curve theorem. In particular, it does not use Euler's formula (which includes the Jordan curve theorem). Thus, the combinatorial part can be read independently of the previous results and it is of interest to those applications (for example to the Heawood problem mentioned below) where the surfaces under consideration are already triangulated.

The topological part is a proof of the fact that every surface S can be triangulated, i.e., S is homeomorphic to a topological space obtained by pasting triangles together. The idea behind this is simple: First we consider, for each point p in S, a small disc D_p around p. As S is compact, S is covered by a finite collection of the discs D_p. If S minus the boundaries of those discs consists of a finite number of connected components, then each of these is homeomorphic to a disc and it is then easy to triangulate S. However, the discs D_p may overlap in a complicated way. The previous proofs in the literature of the fact that every surface can be triangulated are complicated and appeal to geometric intuition. In Section 4 we present a short proof, which is perhaps not easy to follow, but which is simple in the sense that it merely consists of repeated use of the following extension of the Jordan curve theorem: If C_1 and C_2 are simple closed Jordan curves in the plane and f is a homeomorphism between them, then f can be extended to a homeomorphism of the whole plane. This extension, which is called the Jordan-Schönflies theorem is a classical result, which is of interest in its own

right. In the present paper it forms a bridge between the Jordan curve theorem and the classification theorem. Although the Jordan-Schönflies theorem may also seem intuitively clear, it does not generalize to sets homeomorphic to a sphere in R^3, as shown by the so-called Alexander's Horned Sphere, see [**5**]. (The Jordan curve theorem does generalize to spheres in R^3.) We present a new (graph-theoretic) proof of the Jordan-Schönflies theorem in Section 3. No previous knowledge of graph theory and only basic topological concepts will be assumed in the paper. In order to emphasize that the proofs are rigorous, no figures (which could be an excuse for lack of details) are included. Instead there are, inevitably, quite a number of technical details in the topological part (Sections 3 and 4). The difficulty in the topological part lies precisely in the details.

The classification of surfaces is not only a beautiful result of considerable independent interest. It has turned out to be a valuable tool in combinatorial analysis. Heawood [**3**] introduced the problem of determining the smallest number $h(S)$ such that every map on the surface S can be coloured in $h(S)$ colours in such a way that no two neighbouring countries receive the same colour. Heawood established an upper bound for $h(S)$. He claimed that his upper bound in fact equals $h(S)$ (except for the sphere) and that this follows by drawing a certain complete graph on S such that no two edges cross. While this claim, which became known as the Heawood conjecture, turned out to be correct, it took almost 80 years before Ringel and Youngs (see [**6**]) completed the proof. One of the main ideas behind the proof is the following: Instead of starting out with S and drawing the complete graph on S, we start out with the complete graph and "paste" discs on it such that we obtain a surface. By the classification theorem and Euler's formula, we know exactly which surface we get, and if we are clever enough, we get S.

The solution of the Heawood problem is an example where the classification theorem plays a role in reducing a problem with a topological content into a purely combinatorial one.

Recently, surfaces have also played a crucial role in a purely combinatorial result with far-reaching consequences in discrete mathematics and theoretical computer science. Let p be a graph property satisfying the following: If G is a graph with property p, then every graph obtained from G by deleting or contracting edges also has property p. The Robertson-Seymour theory [**7**] implies an efficient method (more precisely, a polynomially bounded algorithm) for testing if an arbitrary graph has property p. In particular, for any fixed surface S, there is an efficient algorithm for testing if an arbitrary graph G can be embedded into S, that is, drawn on S such that no two edges cross. In contrast to this, the problem of determining the smallest number of handles that must be added to the sphere in order to get a surface on which G can be embedded is a very difficult one. More precisely, it is NP-complete as shown by the author [**9**].

2. PLANAR GRAPHS AND THE JORDAN CURVE THEOREM. A *simple arc* in a topological space X is the image of a continuous $1-1$ map f from the real interval $[0,1]$ into X. We say that $f(0)$ and $f(1)$ are the *ends* of the arc and that the arc *joins* $f(0)$ and $f(1)$. A *simple closed curve* is defined analogously except that now $f(0)=f(1)$. We say that X is *connected* (more precisely, arcwise connected) if any two elements of X are joined by a simple arc. A *simple polygonal arc* or *closed curve* in the plane is a simple arc or closed curve which is the union of a finite number of straight line segments.

Lemma 2.1. *If Ω is an open connected set in the plane, then any two points in Ω are joined by a simple polygonal arc in Ω.*

Proof: Let p and q be any two points in Ω and let f be a continuous map from $[0, 1]$ to Ω such that $f(0) = p$ and $f(1) = q$. Let A consist of those numbers t in $[0, 1]$ such that Ω contains a simple polygonal arc from p to $f(t)$. Put $t_0 = \sup A$. We must have $t_0 = 1$ since otherwise it is easy to find a t_1 in A such that $t_1 > t_0$, a contradiction. □

A *region* of an open set in the plane is a maximal connected subset. A *graph* G is the union of two finite disjoint sets $V(G)$ and $E(G)$ (called the *vertices* and *edges*, respectively) such that, with every edge, there are associated two distinct vertices x and y, called the *ends* of the edge. We denote such an edge by xy and say that it *joins* x and y or that it is *incident* with x and y. If more than one edge joins x and y we speak of a *multiple* edge. An *isomorphism* between two graphs is defined in the obvious way. A *path* is a graph with distinct vertices $v_1, v_2, \dots, v_n$ and edges $v_1v_2, v_2v_3, \dots, v_{n-1}v_n$. If $n \geq 2$ and we add an edge v_nv_1 to this path we obtain a *cycle*. We denote both the above path and cycle by $v_1v_2 \dots v_n$. (It will always be clear from the context if we are talking about a path or a cycle.) If G is a graph and $A \subseteq V(G) \cup E(G)$, then $G - A$ is the graph obtained from G by deleting all vertices of A and all those edges which are in A or are incident with a vertex in A. We say that G is *connected* if every pair of vertices in G are joined by a path, and G is 2-*connected* if it is connected and, for every vertex v, $G - \{v\}$ (which we also denote by $G - v$) is connected. The graph G can be *embedded* in the topological space X if the vertices of G can be represented by distinct elements in X and each edge of G can be represented by a simple arc which joins its two ends in such a way that two edges have at most an end in common. If X is the Euclidean plane R^2, then a graph represented in X is a *plane graph*, and an abstract graph which can be represented in X is a *planar graph*.

Lemma 2.2. *If G is a planar graph, then G can be drawn (embedded) in the plane such that all edges are simple polygonal arcs.*

Proof: Let Γ be a plane graph isomorphic to G. Let p be some vertex of Γ, and let D_p be a closed disc with p as center such that D_p intersects only those edges that are incident with p. Furthermore, assume that $D_p \cap D_q = \emptyset$ for every pair of distinct vertices p, q of Γ. For each edge pq of Γ let C_{pq} be an arc contained in pq such that C_{pq} joins D_p with D_q and has only its ends in common with $D_p \cup D_q$. We can now redraw G such that all arcs C_{pq} are in the new drawing and such that the parts of the edges in the discs D_p are straight line segments. Using Lemma 2.1 it is now easy to replace each C_{pq} by a simple polygonal arc. □

A *subdivision* of a graph G is a graph obtained from G by "inserting vertices on edges." More precisely, some (or all) edges of G are replaced by paths with the same ends. Kuratowski's theorem says that a graph is nonplanar if and only if it contains a subdivision of one of the Kuratowski graph $K_{3,3}$ or K_5. K_5 is the graph on five vertices such that every pair of vertices are joined by exactly one edge. $K_{3,3}$ is the graph with six vertices $v_1, v_2, v_3, u_1, u_2, v_3$ and all nine edges v_iu_j, $1 \leqslant i \leqslant 3$,

$1 \leqslant j \leqslant 3$. A discussion of this fundamental result (including a short proof) can be found in [8]. We shall use here only the simple fact that $K_{3,3}$ is nonplanar. For this we need the following special case of the Jordan curve theorem.

Lemma 2.3. *If C is a simple closed polygonal curve in the plane, then $R^2 \setminus C$ has precisely two regions each of which has C as boundary.*

Proof: We first prove that $R^2 \setminus C$ has at most two regions. So suppose (*reductio ad absurdum*) that q_1, q_2, q_3 belong to distinct regions of $R^2 \setminus C$. Select a disc D such that $D \cap C$ is a straight line segment. For each $i = 1, 2, 3$ we can walk along a simple polygonal arc (close to C but not intersecting C) from q_i into D. Hence some two of q_1, q_2, q_3 are connected by a simple polygonal arc, a contradiction.

Next we prove that $R^2 \setminus C$ is not connected. For each point q in $R^2 \setminus C$ we consider a straight half line L starting at q. The intersection $L \cap C$ is a finite number of intervals some of which may be points. Consider such an interval Q. If C enters and leaves Q on the same side of L we will say that C *touches* L at Q. Otherwise C *crosses* L at Q. It is easy to see that the number of times that C crosses L (reduced modulo 2) does not change when the direction of L is changed. So that number depends only on q (and C) and is called the parity of q. Now, the parity is the same for all points on a simple polygonal arc in $R^2 \setminus C$ and hence it is the same for all points in a region of $R^2 \setminus C$. By considering a half line that intersects C precisely once we get points of different parity and hence in different regions. □

The unbounded region of a closed curve C is called the *exterior* of C and is denoted $\text{ext}(C)$. The union of all other regions is the *interior* and is denoted $\text{int}(C)$. Furthermore, we write

$$\overline{\text{int}}(C) = C \cup \text{int}(C) \quad \text{and} \quad \overline{\text{ext}}(C) = C \cup \text{ext}(C).$$

We shall extend Lemma 2.3.

Lemma 2.4. *Let C be a simple closed polygonal curve and P a simple polygonal arc in $\overline{\text{int}}(C)$ such that P joins p and q on C and has no other point in common with C. Let P_1 and P_2 be the two arcs on C from p to q. Then $R^2 \setminus (C \cup P)$ has precisely three regions whose boundaries are C, $P_1 \cup P$, $P_2 \cup P$, respectively.*

Proof: Clearly, $\text{ext}(C)$ is a region of $R^2 \setminus (C \cup P)$. As in the proof of Lemma 2.3 we conclude that the addition of P to C partitions $\text{int}(C)$ into at most two regions. So, we only need to prove that P partitions $\text{int}(C)$ into (at least) two regions. Let L_1, L_2 be crossing line segments such that L_1 is a segment of P, and L_2 has precisely the point in $L_1 \cap L_2$ in common with $C \cup P$. By the proof of Lemma 2.3, the ends of L_2 are in $\text{int}(C)$ and in distinct regions of $R^2 \setminus (P \cup P_1)$, hence also in distinct regions of $R^2 \setminus (P \cup C)$. □

Lemma 2.4 implies that, if r and s are points on $P_1 \setminus \{p, q\}$ and $P_2 \setminus \{p, q\}$, respectively, then it is not possible to join r and s by a simple polygonal arc in $\overline{\text{int}}(C)$ without intersecting P. These remarks also hold when ext and int are interchanged. Hence we get:

Lemma 2.5. *$K_{3,3}$ is nonplanar.*

Proof: $K_{3,3}$ may be thought of as a cycle C: $x_1x_2x_3x_4x_5x_6$ with three chords x_1x_4, x_2x_5, x_3x_6. Now if $K_{3,3}$ were planar we would have a plane drawing such that all edges are simple polygonal arcs, by Lemma 2.2. Then C would be a simple closed polygonal curve and two of the chords x_1x_4, x_2x_5, x_3x_6 would either be in $\operatorname{int}(C)$ or $\operatorname{ext}(C)$. But this would contradict the remark after Lemma 2.4. □

Everything so far is standard and trivial. Now we are ready for the Jordan curve theorem. We remark again that the proof uses only the nonplanarity of $K_{3,3}$.

Proposition 2.6. *If C is a simple closed curve in the plane, then $R^2 \setminus C$ is disconnected.*

Proof: Let L_1 (respectively, L_2) be a vertical straight line intersecting C such that C is entirely in the closed right (respectively, left) half plane of L_1 (respectively, L_2). Let p_i be the top point on $L_i \cap C$ for $i = 1, 2$, and let P_1 and P_2 be the two curves on C from p_1 to p_2. Let L_3 be a vertical straight line between L_1 and L_2. Since $P_1 \cap L_3$ and $P_2 \cap L_3$ are compact and disjoint, L_3 contains an interval L_4 joining P_1 with P_2 and having only its ends in common with C. Let L_5 be a polygonal arc from p_1 to p_2 in $\operatorname{ext}(C)$ consisting of segments of L_1, L_2 and a horizontal straight line segment above C. If L_4 is in $\operatorname{ext}(C)$, then there is a simple polygonal arc L_6 in $\operatorname{ext}(C)$ from L_4 to L_5. But then $C \cup L_4 \cup L_5 \cup L_6$ is a plane graph isomorphic to $K_{3,3}$, contradicting Lemma 2.5. Hence, the midpoint of L_4 does not lie in $\operatorname{ext}(C)$, so $\operatorname{int}(C)$ is nonempty. □

We shall also use the nonplanarity of $K_{3,3}$ to show that $\operatorname{int}(C)$ has only one region. For this we need some graph theoretic facts. First a result on abstract graphs.

Lemma 2.7. *If G is a 2-connected graph and H is a 2-connected subgraph of G, then G can be obtained from H by successively adding paths such that each of these paths joins two distinct vertices in the current graph and has all other vertices outside the current graph.*

Proof: The proof is by induction on the number of edges in $E(G) \setminus E(H)$. If that number is zero, that is, $G = H$, then there is nothing to prove. So assume that $G \neq H$. By the induction hypothesis, Lemma 2.7 holds when the pair G, H is replaced by another pair G', H' such that $E(G') \setminus E(H')$ has fewer edges than $E(G) \setminus E(H)$. Now let H' be a maximal 2-connected proper subgraph of G containing H. If $H' \neq H$ we apply the induction hypothesis to H', H and then to G, H'. So assume that $H' = H$. Since G is connected, there is an edge x_1x_2 in $E(G) \setminus E(H)$ such that x_1 is in H. Since $G - x_1$ is connected, it has a path P: $x_2x_3 \cdots x_k$ such that x_k is in H and all x_i, $2 \leqslant i < k$, are not in H. Possibly $k = 2$. Since $H \cup P \cup \{x_1x_2\}$ is 2-connected, we have $H \cup P \cup \{x_1x_2\} = G$ and the proof is complete. □

If S is a set, then $|S|$ will denote its cardinality.

Lemma 2.8. *If Γ is a plane 2-connected graph with at least three vertices, all of whose edges are simple polygonal arcs, then $R^2 \setminus \Gamma$ has $|E(\Gamma)| - |V(\Gamma)| + 2$ regions each of which has a cycle of Γ as boundary.*

Proof: Let C be a cycle in Γ. By Lemma 2.3, Lemma 2.8 holds if $\Gamma = C$. Otherwise, Γ can be obtained from C by successively adding paths as in Lemma 2.7. Each such path is added in a region. That region is bounded by a cycle and now we apply Lemma 2.4 to complete the proof. (Lemma 2.4 says that the number of regions is increased by 1 when a region is subdivided). $\square$

For a plane graph Γ, the regions of $R^2 \setminus \Gamma$ will also be called *faces* of Γ. The unbounded face is the *outer face* and, if Γ is 2-connected, then the boundary of the outer face is the *outer cycle*.

The union of two abstract graphs is defined in the obvious way. For plane graphs we shall make use of a different type of union.

Lemma 2.9. *If Γ_1 and Γ_2 are two plane graphs such that each edge is a simple polygonal arc, then the union of Γ_1 and Γ_2 is a graph Γ_3.*

Proof: First, let Γ_i' denote the plane graph such that Γ_i' is a subdivision of Γ_i and each edge of Γ_i' is a straight line segment for $i = 1, 2$. Secondly, let Γ_i'' be the subdivision of Γ_i' such that a point p on an edge a of Γ_i' is a vertex of Γ_i'' if either p is a vertex of Γ_{3-i}' or p is on an edge of Γ_{3-i}' that crosses a. Then the usual union of the graphs Γ_1'' and Γ_2'' can play the role of Γ_3. $\square$

If both Γ_1 and Γ_2 in Lemma 2.9 are 2-connected and have at least two points in common, then also Γ_3 is 2-connected.

Lemma 2.10. *Let $\Gamma_1, \Gamma_2, \dots, \Gamma_k$ be plane 2-connected graphs all of whose edges are simple polygonal arcs such that Γ_i has at least two points in common with each of Γ_{i-1} and Γ_{i+1} and no point in common with any other Γ_j $(i = 2, 3, \dots, k-1)$. Assume also that $\Gamma_1 \cap \Gamma_k = \varnothing$. Then any point which is in the outer face of each of $\Gamma_1 \cup \Gamma_2, \Gamma_2 \cup \Gamma_3 \cdots \Gamma_{k-1} \cup \Gamma_k$ is also in the outer face of $\Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_k$.*

Proof: Suppose p is a point in a bounded face of $\Gamma_1 \cup \cdots \cup \Gamma_k$. Since $\Gamma_1 \cup \cdots \cup \Gamma_k$ is 2-connected, it follows from 2.8 that there is a cycle C in $\Gamma_1 \cup \cdots \cup \Gamma_k$ such that $p \in \operatorname{int}(C)$. Choose C such that C is in $\Gamma_i \cup \Gamma_{i+1} \cup \cdots \cup \Gamma_j$ and such that $j - i$ is minimum. We shall show that $j - i \leqslant 1$. So assume that $j - i \geqslant 2$. Among all cycles in $\Gamma_i \cup \cdots \cup \Gamma_j$ having p in the interior we assume that C is chosen such that the number of edges in C and not in Γ_{j-1} is minimum. Since C intersects both Γ_j and Γ_{j-2}, C has at least two disjoint maximal segments in Γ_{j-1}; let P be one of these; let P' be a shortest path in Γ_{j-1} from P to $C - V(P)$; the ends of P' divide C into arcs P_1 and P_2, each of which contains segments not in Γ_{j-1}. One of the cycles $P' \cup P_1$ and $P' \cup P_2$ contains p in its interior; it has fewer edges not in Γ_{j-1} than C has. This contradicts the minimality of C, so a minimal C does not lie in a minimal union $\Gamma_i \cup \Gamma_{i+1} \cup \cdots \cup \Gamma_j$ with $i \leqslant j - 2$. $\square$

Proposition 2.11. *If P is a simple arc in the plane, then $R^2 \setminus P$ is connected.*

Proof: Let p, q be two points in $R^2 \setminus P$ and let d be a positive number such that each of p, q has distance $> 3d$ from P. We shall join p, q by a simple polygonal arc in $R^2 \setminus P$. Since P is the image of a continuous (and hence uniformly continuous) map we can partition P into segments $P_1, P_2, \dots, P_k$ such that P_i

joins p_i and p_{i+1} for $i = 1, 2, \ldots, k$ and such that each point on P_i has distance less than d from p_i ($i = 1, 2, \ldots, k-1$). Let d' be the minimum distance between P_i and P_j, $1 \leqslant i \leqslant j-2 \leqslant k-2$. Note that $d' \leqslant d$. For each $i = 1, 2, \ldots, k$, we partition P_i into segments $P_{i,1}, P_{i,2}, \ldots, P_{i,k_i}$ such that $P_{i,j}$ joins $p_{i,j}$ with $p_{i,j+1}$ for $j = 1, 2, \ldots, k_i - 1$ and such that each point on $P_{i,j}$ has distance less than $d'/4$ to $p_{i,j}$, and let Γ_i be the graph which is the union of the boundaries of the squares that consist of horizontal and vertical line segments of length $d'/2$ and have a point $p_{i,j}$ as midpoint. Then the graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ satisfy the assumption of Lemma 2.10. Hence both of p and q are in the outer face of $\Gamma_1 \cup \cdots \cup \Gamma_k$ (because they are outside the disc of radius $3d$ and with center p_i while $\Gamma_i \cup \Gamma_{i+1}$ is inside that disc) and P does not intersect that face. Therefore, p and q can be joined by a simple polygonal arc disjoint from P. □

If C is a closed subset of the plane and Ω is a region of $R^2 \setminus C$, then a point p in C is *accessible* from Ω if for some (and hence each) point q in Ω, there is a simple polygonal arc from q to p having only p in common with C. If C is a simple closed curve, then p need not be accessible from Ω. However, if P is any arc of C containing p, then Proposition 2.11 implies that $R^2 \setminus (C \setminus P)$ is connected and therefore contains a simple polygonal arc P' from q to a region of $R^2 \setminus C$ distinct from Ω. Then P' intersects C in a point on P. Since P can be chosen to be arbitrarily small we conclude that the points on C accessible from Ω are dense on C. We also get

Theorem 2.12 (*The Jordan Curve Theorem*). *If C is a simple closed curve in the plane, then $R^2 \setminus C$ has precisely two regions, each of which has C as boundary.*

Proof: Assume (*reductio ad absurdum*) that q_1, q_2, q_3 are points in distinct regions $\Omega_1, \Omega_2, \Omega_3$ of $R^2 \setminus C$. Let Q_1, Q_2, Q_3 be pairwise disjoint segments of C. By the remark following Proposition 2.11, Ω_i has a simple polygonal arc $P_{i,j}$ from q_i to Q_j for $i, j = 1, 2, 3$. We can assume that $P_{i,j} \cap P_{i,j'} = \{q_i\}$ for $j \neq j'$. (If we walk along $P_{i,2}$ from Q_2 towards q_i and we hit $P_{i,1}$ in $q_i' \neq q_i$, then we can modify $P_{i,2}$ such that its last segment is close to the segment of $P_{i,1}$ from q_i' to q_i and such that the new $P_{i,2}$ has only q_i in common with $P_{i,1}$. $P_{i,3}$ can be modified similarly, if necessary.) Clearly, $P_{i,j} \cap P_{i',j'} = \varnothing$ when $i \neq i'$. We can now extend (by adding a segment in each of Q_1, Q_2, Q_3) the union of the arcs $P_{i,j}$ ($i, j = 1, 2, 3$) to a plane graph isomorphic to $K_{3,3}$. This contradicts Lemma 2.5. Thus $R^2 \setminus C$ has precisely two regions $\operatorname{ext}(C)$ and $\operatorname{int}(C)$. As above, Proposition 2.11 implies that every point of C is a boundary point of $\operatorname{ext}(C)$ and $\operatorname{int}(C)$. □

The Jordan Curve Theorem is a special case of the Jordan-Schönflies theorem which we prove in the next section. For this we shall generalize some of the previous results. First, Lemma 2.4 generalizes as follows.

Lemma 2.13. *Let C be a simple closed curve and P a simple polygonal arc in* $\operatorname{int}(C)$ *such that P joins p and q on C and has no other point in common with C. Let P_1 and P_2 be the two arcs on C from p to q. Then $R^2 \setminus (C \cup P)$ has precisely three regions whose boundaries are C, $P_1 \cup P$, and $P_2 \cup P$, respectively.*

Proof: As in the proof of Lemma 2.4 the only nontrivial part is to prove that $\overline{\operatorname{int}}(C)$ is partitioned into (at least) two regions. If the ends of L_2 (defined as in the proof of Lemma 2.4) are in the same region of $R^2 \setminus (P \cup C)$, then that region contains

a polygonal arc P_3 such that $P_3 \cup L_2$ is a simple closed polygonal curve. By the proof of Lemma 2.3, the ends of L_1 are in distinct regions of $R^2 \setminus (P_3 \cup L_2)$. But they are also in the same region of $R^2 \setminus (P_3 \cup L_2)$ since they are joined by a simple arc (in $P \cup C$) not intersecting $P_3 \cup L_2$. This contradiction proves Lemma 2.13. □

We also generalize Lemma 2.8.

Lemma 2.14. *If Γ is a plane 2-connected graph containing a cycle C (which is a simple closed curve) such that all edges in $\Gamma \setminus C$ are simple polygonal arcs in $\overline{\mathrm{int}}(C)$, then $R^2 \setminus \Gamma$ has $|E(\Gamma)| - |V(\Gamma)| + 2$ regions each of which has a cycle of Γ as boundary.*

Proof: The proof is as that of Lemma 2.8 except that we now use Lemma 2.13 instead of Lemma 2.4. □

Finally, we shall use the fact that Lemma 2.9 remains valid if Γ_1 and Γ_2 are plane graphs whose intersection contains a cycle C such that all edges in Γ_1 or Γ_2 (not in C) are simple polygonal arcs in $\overline{\mathrm{int}}(C)$.

3. THE JORDAN-SCHÖNFLIES THEOREM. If C and C' are simple closed curves and Γ and Γ' are 2-connected graphs consisting of C (respectively, C') and simple polygonal arcs in $\overline{\mathrm{int}}(C)$ (respectively, $\overline{\mathrm{int}}(C')$), then Γ and Γ' are said to be *plane-isomorphic* if there is an isomorphism of Γ to Γ' such that a cycle in Γ is a face boundary of Γ iff the image of the cycle is a face boundary of Γ' and such that the outer cycle of Γ is mapped onto the outer cycle of Γ'.

Theorem 3.1. *If f is a homeomorphism of a simple closed curve C onto a simple closed curve C', then f can be extended into a homeomorphism of the whole plane.*

Proof: Without loss of generality we can assume that C' is a convex polygon. We shall first extend f to a homeomorphism of $\overline{\mathrm{int}}(C)$ to $\overline{\mathrm{int}}(C')$. Let B denote a countable dense set in $\mathrm{int}(C)$ (for example the points with rational coordinates). Since the points on C accessible from $\mathrm{int}(C)$ are dense on C, there exists a countable dense set A in C consisting of points accessible from $\mathrm{int}(C)$. Let $p_1, p_2, \ldots$ be a sequence of points in $A \cup B$ such that each point in $A \cup B$ occurs infinitely often in that sequence. Let Γ_0 denote any 2-connected graph consisting of C and some simple polygonal arcs in $\overline{\mathrm{int}}(C)$. Let Γ_0' be a graph consisting of C' and simple polygonal arcs in $\overline{\mathrm{int}}(C')$ such that Γ_0 and Γ_0' are plane-isomorphic (with isomorphism g_0) such that g_0 and f coincide on $C \cap V(\Gamma_0)$. We now extend f to $C \cup V(\Gamma_0)$ such that g_0 and f coincide on $V(\Gamma_0)$. We shall define a sequence of 2-connected graphs $\Gamma_0, \Gamma_1, \Gamma_2 \ldots$ and $\Gamma_0', \Gamma_1' \ldots$ such that, for each $n \geq 1$, Γ_n is an extension of a subdivision of Γ_{n-1}, Γ_n' is an extension of a subdivision of Γ_{n-1}', there is a plane isomorphism g_n of Γ_n onto Γ_n' coinciding with g_{n-1} on $V(\Gamma_{n-1})$, and Γ_n (respectively Γ_n') consists of C (respectively C') and simple polygonal arcs in $\overline{\mathrm{int}}(C)$ (respectively $\overline{\mathrm{int}}(C')$). Also, we shall assume that $\Gamma_n' \setminus C'$ is connected for each n. We then extend f to $C \cup V(\Gamma_n)$ such that f and g_n coincide on $V(\Gamma_n)$.

Suppose we have already defined $\Gamma_0, \Gamma_1, \ldots, \Gamma_{n-1}, \Gamma_0', \Gamma_1', \ldots, \Gamma_{n-1}'$, and $g_0, g_1, \ldots, g_{n-1}$. We shall define Γ_n, Γ_n' and g_n as follows. We consider the point p_n. If $p_n \in A$, then we let P be a simple polygonal arc from p_n to a point q_n of $\Gamma_{n-1} \setminus C$ such that $\Gamma_{n-1} \cap P = \{p_n, q_n\}$. We let Γ_n denote the graph $\Gamma_{n-1} \cup P$. P

is drawn in a face of Γ_{n-1} bounded by a cycle S, say. We add to Γ'_{n-1} a simple polygonal arc P' in the face bounded by $g_{n-1}(S)$ such that P' joins $f(p_n)$ with $g_{n-1}(q_n)$ (if q_n is a vertex of Γ_{n-1}) or a point on $g_{n-1}(a)$ (if a is an edge of Γ_{n-1} containing the point q_n). Then we put $\Gamma'_n = \Gamma'_{n-1} \cup P'$ and we define the plane-isomorphism g_n from Γ_n to Γ'_n in the obvious way. We extend f such that $f(q_n) = g_n(q_n)$.

If $p_n \in B$ we consider the largest square which has vertical and horizontal sides, which has p_n as midpoint and which is in $\overline{\text{int}}(C)$. In this square (whose sides we are not going to add to Γ_{n-1} as they may contain infinitely many points of C) we draw a new square with vertical and horizontal sides each of which has distance $< 1/n$ from the sides of the first square. Inside the new square we draw vertical and horizontal lines such that p_n is on both a vertical line and a horizontal line and such that all regions in the square have diameter $< 1/n$. We let H_n be the union of Γ_{n-1} and the new horizontal and vertical straight line segments possibly together with an additional polygonal arc in int(C) in order to make H_n 2-connected and $H_n \setminus C$ connected. By Lemma 2.7, H_n can be obtained from Γ_{n-1} by successively adding paths in faces. We add the corresponding paths to Γ'_{n-1} and obtain a graph H'_n which is plane-isomorphic to H_n. Then we add vertical and horizontal lines in $\overline{\text{int}}(C')$ to H'_n such that the resulting graph has no (bounded) region of diameter $\geqslant 1/2n$. If necessary, we displace some of the lines a little such that they intersect C' only in $f(A)$ and such that all bounded regions have diameter $< 1/n$ and such that each of the new lines has only finite intersection with H'_n. This extends H'_n into a graph we denote by Γ'_n. We add to H_n polygonal arcs such that we obtain a graph Γ_n plane-isomorphic to Γ'_n. Then we extend f such that it is defined on $C \cup V(\Gamma_n)$ and coincides with the plane-isomorphism g_n on $V(\Gamma_n)$. When we extend H'_n into Γ'_n and H_n into Γ_n we are adding many edges and it is perhaps difficult to visualize what is going on. However, Lemma 2.7 tells us that we can look at the extension of H'_n into Γ'_n as the result of a sequence of simple extensions each consisting of the addition of a path (which in this case is a straight line segment in a face). We then just perform successively the corresponding additions in H_n. Note that we have plenty of freedom for that since the current f is only defined on the current vertex set. The images of the points on the current edges have not been specified yet. In this way we extend f to a $1-1$ map defined on $F = C \cup V(\Gamma_0) \cup V(\Gamma_1) \cup \cdots$ and with image $C' \cup V(\Gamma'_0) \cup V(\Gamma'_1) \cup \cdots$. These sets are dense in $\overline{\text{int}}(C)$ and $\overline{\text{int}}(C')$, respectively. If p is a point in int(C) on which f is not yet defined, then we consider a sequence $q_1, q_2, \ldots$ converging to p and consisting of points in $V(\Gamma_0) \cup V(\Gamma_1) \cup \cdots$. We shall show that $f(q_1), f(q_2), \ldots$ converges and we let $f(p)$ be the limit. Let d be the distance from p to C and let p_n be a point of B of distance $< d/3$ from p. Then p is inside the largest square in $\overline{\text{int}}(C)$ having p_n as midpoint (and also inside what we called the new square if n is sufficiently large). By the construction of Γ_n and Γ'_n it follows that Γ_n has a cycle S such that $p \in \text{int}(S)$ and such that both S and $g_n(S)$ are in discs of radius $< 1/n$. Since f maps $F \cap \overline{\text{int}}(S)$ into $\overline{\text{int}}(g_n(S))$ and $F \cap \overline{\text{ext}}(S)$ into $\overline{\text{ext}}(g_n(S))$, it follows in particular, that the sequence $f(q_m), f(q_{m+1}), \ldots$ is in int$(g_n(S))$ for some m. Since n can be chosen arbitrarily large, $f(q_1), f(q_2), \ldots$ is a Cauchy sequence and hence convergent. It follows that f is well-defined. Moreover, using the above notation, f maps int(S) into int$(g_n(S))$. Hence f is continuous in int(C). Since $V(\Gamma'_0) \cup V(\Gamma'_1)$ is dense in int(C') the same argument shows that f maps int(C) onto int(C') that f is $1-1$ and that f^{-1} is continuous on int(C'). It only remains to be shown that f is continuous on C. (Then also f^{-1} is continuous since $\overline{\text{int}}(C)$ is compact). In order to prove this it is sufficient to

consider a sequence $q_1, q_2, \ldots$ of points in int(C) converging to q on C and then show that $f(q_1), f(q_2), \ldots$ converges to $f(q)$. Suppose therefore that this is not the case. Since $\overline{\mathrm{int}(C')}$ is compact we can assume (by considering an appropriate subsequence, if necessary) that $\lim_{n\to\infty} f(q_n) = q' \neq f(q)$. Since f^{-1} is continuous on int(C'), q' is on C'. Since A is dense in C, $f(A)$ is dense in C' and hence each of the two arcs on C' from q' to $f(q)$ contain a point $f(q_1)$ and $f(q_2)$, respectively, in $f(A)$. For some n, Γ_n has a path P from q_1 to q_2 having only q_1 and q_2 in common with C. By Lemma 2.13, P separates int(C) into two regions. These two regions are mapped on the two distinct regions of $\mathrm{int}(C') \setminus g_n(P)$. One of these contains almost all the $f(q_n)$ while the other has $f(q)$ on its boundary, but not the boundary common to both regions. Hence we cannot have $\lim_{n\to\infty} f(q_n) = q'$. This contradiction shows that f has the appropriate extension to int(C).

By similar arguments, f can be extended to ext(C). We consider a coordinate system in the plane. Without loss of generality we can assume that int(C) contains the origin and that both C and C' are in the interior of the quadrangle T with corners $(\pm 1, \pm 1)$. Let L_1, L_2, L_3 be the line segments (on lines through the origin) from $(1,1)$, $(-1,-1)$ and $(1,-1)$, respectively, to C. Let p_i be the end of L_i on C for $i = 1,2,3$. Let L'_1 and L'_2 be polygonal arcs from $f(p_1)$ to $(1,1)$ and from $f(p_2)$ to $(-1,-1)$, respectively, such that $L'_1 \cap L'_2 = \emptyset$ and L'_i has only its ends in common with C' and T for $i = 1,2$. It is easy to see that we can find a polygonal arc L'_3 from $f(p_3)$ to either $(1,-1)$ or $(-1,1)$ such that L'_3 is disjoint from $L'_1 \cup L'_2$ and has only its ends in common with C' and T. After a reflection of C' in the line through $(1,1)$ and $(-1,-1)$, if necessary, we can assume that L'_3 goes to $(1,-1)$. Now we can use the method of the first part of the proof to extend f to a homeomorphism of $\overline{\mathrm{int}(T)}$ such that f is the identity on T. Then f extends to a homeomorphism of the whole plane such that f is the identity on ext(T). □

If F is a closed set in the plane, then we say that point p in F is *curve-accessible* if, for each point q not in F, there is a simple arc from q to p having only p in common with F. The Jordan-Schönflies theorem implies that every point on a simple closed curve is curve-accessible. Hence we have the following extension of part of Theorem 2.12.

Theorem 3.2. *If F is a closed set in the plane with at least three curve-accessible points, then $R^2 \setminus F$ has at most two regions.*

Proof: If p_1, p_2, p_3 are curve-accessible on F and q_1, q_2, q_3 belong to distinct regions of $R^2 \setminus F$, then we get, as in the proof of Theorem 2.12, a plane graph isomorphic to $K_{3,3}$ with vertices $p_1, p_2, p_3, q_1, q_2, q_3$, a contradiction to Lemma 2.5. □

In Theorem 3.2, "three" cannot be replaced by "two." To see this, we let F be a collection of internally disjoint simple arcs between two fixed points.

Theorem 3.3. *Let Γ and Γ' be 2-connected plane graphs such that g is a homeomorphism and plane-isomorphism of Γ onto Γ'. Then g can be extended to a homeomorphism of the whole plane.*

Proof: The proof is by induction on the number of edges of Γ. If Γ is a cycle, then Theorem 3.3 reduces to Theorem 3.1. Otherwise it follows from Lemma 2.7 that Γ has a path P and a 2-connected subgraph Γ_1 containing the outer cycle of Γ

such that Γ is obtained from Γ_1 by adding P in $\overline{\text{int}}(C)$ where C bounds a face of Γ_1. Now we apply the induction hypothesis first to Γ_1 and then to the two cycles of $C \cup P$ containing P.

4. TRIANGULATING A SURFACE. Consider a finite collection of pairwise disjoint convex polygons (together with their interiors) in the plane such that all side lengths are 1. Form a topological space S as follows: Every side in a polygon is identified with precisely one side in another (or in the same) polygon. This also defines a graph G whose vertices are the corners and the edges the sides. Clearly S is compact. Now S is a surface iff S is connected (i.e., G is connected) and S is locally homeomorphic to a disc at every vertex v of G. If this is the case then we say that G is a *2-cell embedding* in S. If all polygons are triangles, then we say that G is a *triangulation* of S and that S is a *triangulated surface*. In case of a triangulation we shall assume that there are at least four triangles and that there are no multiple edges.

Theorem 4.1. *Every surface S is homeomorphic to a triangulated surface.*

Proof: Since the interior of a convex polygon can be triangulated it is sufficient to prove that S is homeomorphic to a surface with a 2-cell embedding. For each point p on S, let $D(p)$ be a disc in the plane which is homeomorphic to a neighbourhood of p on S. (Instead of specifying a homeomorphism we shall use the same notation for a point in $D(p)$ and the corresponding point on S.) In $D(p)$ we draw two quadrangles $Q_1(p)$ and $Q_2(p)$ such that $p \in \text{int}(Q_1(p)) \subset \text{int}(Q_2(p))$. Since S is compact, it has a finite number of points $p_1, p_2, \ldots, p_n$ such that $S = \bigcup_{i=1}^{n} \text{int}(Q_1(p_i))$. Viewed as subsets in the plane, $D(p_1), \ldots, D(p_n)$ can be assumed to be pairwise disjoint. In what follows we are going to keep $D(p_1), D(p_2), \ldots, D(p_n)$ fixed in the plane (keeping in mind, though, that they also correspond to subsets of S). However, we shall modify the homeomorphism between $D(p_i)$ and the corresponding set on S and consider new quadrangles $Q_1(p_i)$. More precisely, we shall show that $Q_1(p_1), \ldots, Q_1(p_n)$ can be chosen such that they form a 2-cell embedding of S.

Suppose, by induction on k, that they have been chosen such that any two of $Q_1(p_1), Q_1(p_2), \ldots, Q_1(p_{k-1})$ have only a finite number of points in common on S. We now focus on $Q_2(p_k)$. We define a *bad segment* as a segment P of some $Q_1(p_j)$ $(1 \leqslant j \leqslant k-1)$ which joins two points of $Q_2(p_k)$ and which has all other points in $\text{int}(Q_2(p_k))$. Let $Q_3(p_k)$ be a square between $Q_1(p_k)$ and $Q_2(p_k)$. We say that a bad segment inside $Q_2(p_k)$ is *very bad* if it intersects $Q_3(p_k)$. There may be infinitely many bad segments but only finitely many very bad ones. The very bad ones together with $Q_2(p_k)$ form a 2-connected graph Γ. We redraw Γ inside $Q_2(p_k)$ such that we get a graph Γ' which is plane-isomorphic to Γ and such that all edges of Γ' are simple polygonal arcs. This can be done using Lemma 2.7. Now we apply Theorem 3.3 to extend the plane-isomorphism from Γ to Γ' to a homeomorphism of $\overline{\text{int}}\, Q_2(p_k)$ keeping $Q_2(p_k)$ fixed. This transforms $Q_1(p_k)$ and $Q_3(p_k)$ into simple closed curves Q_1' and Q_3' such that $p_k \in \text{int}\, Q_1' \subseteq \text{int}\, Q_3'$. We consider a simple closed polygonal curve Q_3'' in $\text{int}\, Q_2(p_k)$ such that $Q_1' \subseteq \text{int}\, Q_3''$ and such that Q_3'' intersects no bad segments except the very bad ones (which are now simple polygonal arcs). (The existence of Q_3'' can be established as follows: For every point p on Q_3' we let $R(p)$ be a square with p as midpoint such that $R(p)$ does not intersect either Q_1' nor any bad segment which is not very bad. We consider a (minimal) finite covering of Q_3' by such squares. The union of those

squares is a 2-connected plane graph whose outer cycle can play the role of Q_3''). By redrawing $\Gamma' \cup Q_3''$ (which is a 2-connected graph) and using Theorem 3.3 once more we can assume that Q_3'' is in fact a quadrangle having Q_1' in its interior. If we let Q_3'' be the new choice of $Q_1(p_k)$, then any two of $Q_1(p_1), \dots, Q_1(p_k)$ have only finite intersection. The inductive hypothesis is proved for all k.

Thus we can assume that there are only finitely many very bad segments inside each $Q_2(p_k)$ and that those segments are simple polygonal arcs forming a 2-connected plane graph. The union $\bigcup_{i=1}^{n} Q_1(p_i)$ may be thought of as a graph Γ drawn on S. Each region of $S \setminus \Gamma$ is bounded by a cycle C in Γ. (We may think of C as a simple closed polygonal curve inside some $Q_2(p_i)$). Now we draw a convex polygon C' of side length 1 such that the corners of C' correspond to the vertices of C. The union of the polygons C' forms a surface S' with a 2-cell embedding Γ' which is isomorphic to Γ. An isomorphism of Γ to Γ' may be extended to a homeomorphism f of the point set of Γ on S onto the point set of Γ' on S'. In particular, the restriction of f to the above cycle C is a homeomorphism onto C'. By Theorem 3.1, f can be extended to a homeomorphism of $\overline{\mathrm{int}}(C)$ to $\overline{\mathrm{int}}(C')$. This defines a homeomorphism of S onto S'. □

5. THE CLASSIFICATION OF SURFACES. Consider now two disjoint triangles T_1, T_2 (such that all six sides have the same length) in a face F of a surface S with a 2-cell embedding G. We form a new surface S' by deleting from F the interior of T_1 and T_2 and identifying T_1 with T_2 such that the clockwise orientations around T_1 and T_2 disagree. (We recall that S consists of polygons and their interiors in the plane. So when we speak of clockwise orientation we are simply referring to the plane. We are not discussing orientability of surfaces.) If the orientations agree we obtain instead a surface S''. Finally, we let S''' denote the surface obtained by deleting the interior of T_1 and identifying "diametrically opposite" points on T_1. We say that S', S'', S''' are obtained from S by adding a *handle*, a *twisted handle*, and a *crosscap*, respectively. It is easy to extend G to a 2-cell embedding of S', S'' and S''', respectively. Also, it is an easy exercise to show that S', S'' and S''' are independent (up to homeomorphism) of where T_1 and T_2 are located since it is easy to continuously deform a pair of triangles into another pair of triangles inside a given triangle. In fact, they may belong to distinct faces, also, except that then (at this stage) we cannot distinguish between a handle and a twisted handle. When adding a crosscap it is sufficient that T_1 is a simple closed polygonal curve, which can be continuously deformed into a point (and hence to a triangle in a face).

We shall now consider all surfaces obtained from the sphere S_0 (which we here think of as a tetrahedron) by adding handles, twisted handles and crosscaps. If we add to S_0 h handles we obtain the surface S_h, and if we add to S_0 k crosscaps we obtain N_k. S_1, N_1, N_2 are the *torus*, the *projective plane* and the *Klein bottle*, respectively. N_2 is also S_0 plus a twisted handle. One way to see this is as follows: Let T_1 and T_2 be two disjoint tetrahedra (which are homeomorphic to S_0). Select a triangle in T_1 and T_2 and add in that triangle a twisted handle or two crosscaps. This transforms T_1 into T_1' and T_2 into T_2'. Now choose your favourite representation of the Klein bottle and your favourite triangulation G of it. Then for each $i = 1, 2$, draw G on T_i' such that the face boundaries are the same triangles in G in all three triangulations. Then the graph isomorphism of G on T_1' to G on T_2' can be extended to a homeomorphism of T_1' onto T_2'. Moreover, if we have already added a crosscap, then adding a handle amounts to the same, up to homeomorphism, as adding a twisted handle. (First observe that when we add a crosscap, it

does not matter where we add it; we get always the same surface up to homeomorphism. So we only need to verify the statement when we add a crosscap and then a handle or twisted handle inside the same triangle of the surface. This can be done by triangulating the two surfaces by the same graph G as above). So, the surfaces obtained from S_0 by adding handles, twisted handles and crosscaps are precisely the surfaces S_h $(h \geq 0)$ and N_k $(k \geq 1)$.

Theorem 5.1. *Let S be a surface and G a 2-cell embedding of S with n vertices, e edges and f faces. Then S is homeomorphic to either S_h or N_k where h and k are defined by the equations*

$$n - e + f = 2 - 2h = 2 - k.$$

Proof: We first show that $n - e + f \leqslant 2$. For this we successively delete edges from G until we get a minimal connected subgraph of G, that is, a spanning tree H of G. For each edge deletion the number of faces (which are now not necessarily 2-cells) is unchanged or decreased by 1. Since H has n vertices, $n - 1$ edges and only one face it follows that $n - e + f \leqslant 2$.

We next extend G to a triangulation of S as follows: For each face F of G which is a convex polygon with corners $v_1, v_2, \dots, v_q$, where $q \geqslant 4$ and their indices are expressed modulo q, we add new vertices $u, u_1, \dots, u_q$ in F and we add the edges $u_iv_i, u_iv_{i+1}, u_iu_{i+1}, u_iu$ for $i = 1, 2, \dots, q$. Let n', e', f' be the number of vertices edges and faces, respectively, of G'. Clearly, $n' - e' + f' = n - e + f$. Thus it is sufficient to prove the Theorem in the case where G is a triangulation which we now assume. Suppose (*reductio ad absurdum*) that S, G are a counterexample to Theorem 5.1 such that G is a triangulation with at least four vertices and

(1) $2 - n + e - f$ is minimum.

(2) n is minimum subject to (1), and

(3) the minimum valency m of G is minimum subject to (1), (2). (The *valency* of a vertex is the number of edges incident with it.)

Let v be a vertex of minimum valency. Let $v_1, v_2, \dots, v_m$ be the neighbours of v such that $vv_1v_2, vv_2v_3, \dots, vv_mv_1$ are the faces incident with v and the indices are expressed modulo m. Since v_1 and v_m are joined only by one edge, $m \geq 3$. If $m = 3$, then $G - v$ is a triangulation of S unless $n = 4$ in which case S is the tetrahedron. This contradicts (2) or the assumption that S, G are a counterexample to the Theorem. So $m \geqslant 4$.

If for some $i = 1, 2, \dots, m$, v_i is not joined to v_{i+2} by an edge, then we let G' be obtained from G by deleting the edge vv_{i+1} and adding the edge v_iv_{i+2} instead. Clearly, G' triangulates S, contradicting (3). So we can assume that G contains all edges v_iv_{i+2}, $i = 1, 2, \dots, m$, when v is a vertex of minimum valency.

Intuitively, we complete the proof by cutting the surface (using a pair of scissors, say) along the triangle T: vv_1v_3. This transforms T into either two triangles T_1 and T_2 or into a hexagon H (in case S has a Möbius strip that contains T). We get a new surface S' by adding two new triangles (and their interior) or a hexagon (and its interior which we triangulate) and identify their sides with T_1 and T_2 or with H, respectively. Then S' is a triangulated surface with smaller $2 - n + e - f$ than S. By the minimality of this parameter, S' is of the form S_g, or N_k. Then S is of that form, too.

Formally, we argue as follows. Recall that S is a triangulated surface, i.e., S is obtained by identifying sides of pairwise disjoint triangles in the plane. Let M denote the topological space which is formed by using the same triangles and the

same side identifications, except that those six sides that correspond to the edges vv_1, v_1v_3, v_3v are not identified with any other side. Let us call those six sides *boundary sides* of M. Let G' be the graph whose vertices are the corners of the triangles of M and whose edges are the sides of the triangles. It is easy to see that G' has precisely six vertices which are incident with boundary sides and that each of these six vertices is incident with precisely two boundary sides. Thus the boundary sides are a subgraph C of G' with vertices each of which has valency 2. There are only two such graphs (up to isomorphism): C is either a hexagon or two disjoint triangles. If C is two disjoint triangles, then we add to M two disjoint triangles (and their interior) in the plane and identify their sides with the edges of C such that we obtain a new surface S' which is triangulated by G'. If C is a hexagon, then we add to M a hexagon in the plane together with its interior (which we triangulate) and then we identify the sides of this hexagon with the edges of C. In this way M is extended to a surface S'' and G' is extended to a graph G'' which triangulates S''. Thus we have transformed G and S into a triangulation G' with n' vertices e' edges and f' faces of a surface S', or a triangulation G'' with n'' vertices e'' edges and f'' faces of a surface S''. In the former case we have

$$e' - n' + f' = e - n + f + 2.$$

In the latter case we have

$$e'' - n'' + f'' = e - n + f + 1.$$

By (1), S' or S'' is homeomorphic to a surface of the form $S_{h'}$ or $N_{k'}$. (Note that G' is obtained from G by "cutting" the triangle vv_1v_3. Then G' is connected because of the edge v_2v_m. Hence also the spaces M, S', S'' are connected.) If C consists of two triangles, then clearly S is obtained from S' by adding a handle or a twisted handle. If C is a hexagon, then in S'', C can be continuously deformed into a point, and hence S is obtained from S'' by adding a crosscap (see the discussion preceding Theorem 5.1). In the latter case (where C is a hexagon) S is homeomorphic to $N_{k'+1}$ or $N_{2h'+1}$ (by the discussion preceding Theorem 5.1). This contradicts the assumption that S and G are a counterexample to Theorem 5.1. Similarly, if C is two triangles, then S is homeomorphic to either $N_{k'+2}$ or $S_{h'+1}$ or $N_{2h'+2}$, and again we obtain a contradiction which finally proves the theorem. □

We have now completed the proof of the classification theorem without referring to orientability of surfaces or using Euler's formula (which consists of the equations of Theorem 5.1 and which is therefore now a corollary of Theorem 5.1). To complete the discussion we indicate a proof of the fact that all the surfaces $S_0, S_1, \ldots, N_1, N_2 \ldots$ are pairwise nonhomeomorphic. In this discussion, however, many details will be left for the reader.

First we observe that Euler's formula holds for all 2-cell embeddings since any such embedding can be extended to a triangulation. Now let us consider any connected graph G with n vertices and e edges drawn on S_h. Using Lemma 2.2 we assume that all edges are simple polygonal arcs. Let f be the number of faces for this drawing. If G' is a 2-cell embedding of S_h, then $G \sqcup G'$ is a 2-cell embedding satisfying Euler's formula and containing a subdivision of G. By successively deleting edges (and isolated vertices) from $G \sqcup G'$ until we get a subdivision of G we conclude that

$$n - e + f \geqslant 2 - 2h.$$

Since

$$3f \leqslant 2e$$

we conclude that

$$e \leqslant 3n - 6 + 6h$$

with equality if and only if G is a triangulation of S_h. Thus a triangulation of S_h has too many edges in order to be drawn on $S_{h'}$ when $h' < h$, and hence S_h and $S_{h'}$ are nonhomeomorphic for $h' < h$. More generally, this argument shows that $S_0, S_1, \dots, N_1, N_2, \dots$ are pairwise nonhomeomorphic except that S_h and N_{2h} might be homeomorphic. We sketch an argument which shows that they are not.

It is easy to describe a simple closed polygonal curve C in N_{2h} such that, when we traverse C, left and right interchange. Also it is easy (though a little tedious) to show that S_h has no such simple closed polygonal curve C'. (It is convenient to consider a 2-cell embedding G such that G contains no such C' and then extend the argument to an arbitrary C' in S.) So it suffices to show the following: If there exists a homeomorphism f: $N_{2h} \to S_h$, then there exists a homeomorphism f': $N_{2h} \to S_h$ such that $f'(C)$ is a simple closed polygonal curve. To see this we let G be a 2-cell embedding of N_{2h}. Then also $G \sqcup C$ may be regarded as a 2-cell embedding, and $G \sqcup C$ can be extended to a triangulation H of N_{2h}. We construct H such that it has no other triangles than the face boundaries. Then $\varphi(H)$ is a graph drawn on S_h and we apply Lemma 2.2 to redraw $\varphi(H)$ (resulting in a graph H') such that all edges are simple polygonal arcs. Since H' and H are isomorphic and H is a triangulation of N_{2h}, it follows from Euler's formula that H' is a triangulation of S_h. Hence the face boundaries of H' are the same as the face boundaries of H. So, any isomorphism $H \to H'$ can be extended into a homeomorphism φ': $N_{2h} \to S_h$ taking C into a simple closed polygonal curve.

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Are Mathematics and Poetry Fundamentally Similar?

JoAnne S. Growney

If you doubt their intrinsic similarity, consider the following quotations. In each of the following, the key word ("mathematics" or "poetry" or "mathematician" or "poet" or a variation of one of these terms) has been left out, although the name of the author may provide a give-away clue. Can you guess which art form is being described in each case? The missing words are supplied at the end of the quotations.

(1) ________ is the art of uniting pleasure with truth. —*Samuel Johnson*

(2) To think is thinkable—that is the ________'s aim. —*Cassius J. Keyser*

(3) All ________ [is] putting the infinite within the finite. —*Robert Browning*

(4) The moving power of ________ invention is not reasoning but imagination. —*A. DeMorgan*

(5) When you read and understand ________, comprehending its reach and formal meanings, then you master chaos a little. —*Stephen Spender*

(6) ________ practice absolute freedom. —*Henry Adams*

(7) I think that one possible definition of our modern culture is that it is one in which nine-tenths of our intellectuals can't read any ________. —*Randall Jarrell*

(8) Do not imagine that ________ is hard and crabbed, and repulsive to common sense. It is merely the etherealization of common sense. —*Lord Kelvin*

(9) The merit of ________, in its wildest forms, still consists in its truth; truth conveyed to the understanding, not directly by words, but circuitously by means of imaginative associations, which serve as conductors. —*T. B. Macaulay*

(10) It is a safe rule to apply that, when a ________ or philosophical author writes with a misty profundity, he is talking nonsense. —*A. N. Whitehead*

(11) ________ is a habit. —*C. Day-Lewis*

(12) ... in ________ you don't understand things, you just get used to them. —*John von Neumann*

(13) ________ are all who love—who feel great truths
And tell them. —*P. J. Bailey*
Festus

(14) The ________ is perfect only in so far as he is a perfect being, in so far as he perceives the beauty of truth; only then will his work be thorough, transparent, comprehensive, pure, clear, attractive, and even elegant. —*Goethe*

(15) ... [In these days] the function of ________ as a game ... [looms] larger than its function as a search for truth .... —*C. Day-Lewis*

(16) A thorough advocate in a just cause, a penetrating ________ facing the starry heavens, both alike bear the semblance of divinity. —*Goethe*

(17) ________ is getting something right in language. —*Howard Nemerov*

See pg. 133 for answers.

These quotations are taken from an article by Professor Growney entitled "Mathematics and Poetry: Isolated or Integrated" which appeared in the *Humanistic Mathematics Network Newsletter* #6 (May 1991), 60–69. To subscribe contact Alvin White, Harvey Mudd College.

A Pigeonhole Proof of Kaplansky's Theorem

Ira Rosenholtz

The purpose of this little note is to sketch a simple proof of the following result, which Kaplansky has referred to as his "infamous little exercise"*. (See **[1]**, **[2]**, **[3]**, **[4]**.)

Theorem (Kaplansky). *Suppose that an element in a ring with identity has two right inverses. Then it has infinitely many right inverses.*

The proof consists of the following two lemmas. It is analogous to solving linear differential equations and is a nice application of the pigeonhole principle.

Lemma 1 (The Homogeneous Solution). *If b has N right inverses with N at least* 2, *then the equation $bx = 0$ has at least $(N + 1)$ solutions.*

Proof of Lemma 1: Suppose b has distinct right inverses $a_1, a_2, \ldots, a_N$. Then $a_1 - a_1, a_2 - a_1, \ldots, a_N - a_1$ are N distinct solutions of $bx = 0$. We will show that the set $\{1 - a_1b, 1 - a_2b, \ldots, 1 - a_Nb\}$ contains at least one additional solution of $bx = 0$.

Clearly all of the elements of this set are solutions. If there were not a new solution in this set, then for each j there is a k so that $1 - a_jb = a_k - a_1$. However, $1 - a_jb$ cannot equal $a_1 - a_1 = 0$, because then a_j would be a left inverse for b, and in this case it is easy to see that b has only one right inverse, a contradiction. Thus, since there are N $(1 - a_jb)$'s (the pigeons) and only $(N - 1)$ acceptable $(a_k - a_1)$'s (the pigeon-holes), by the pigeon-hole principle we must have that for some $m \neq n$, $1 - a_mb = 1 - a_nb$. But then $a_mb = a_nb$, and multiplying this on the right by a_1, we get $a_m = a_n$, a contradiction.

Lemma 2 (The Non-Homogeneous Solution). *If b has N right inverses with N at least* 2, *then b has $(N + 1)$ right inverses.*

Proof of Lemma 2: By Lemma 1, $bx = 0$ has $(N + 1)$ distinct solutions $x_1, x_2, \ldots, x_{N+1}$. But then if a_1 is a right inverse of b, then $\{a_1 + x_1, a_1 + x_2, \ldots, a_1 + x_{N+1}\}$ is a set of $(N + 1)$ distinct right inverses of b.

*Personal communication with the author.

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Algebra is generous: she often gives more than is asked of her.

—*D'Alembert*

Answers to Mathematics and Poetry

The words missing are: (1) Poetry, (2) mathematician, (3) poetry, (4) mathematical, (5) a poem, (6) Mathematicians, (7) poetry, (8) mathematics, (9) poetry, (10) mathematician, (11) Poetry, (12) mathematics, (13) Poets, (14) mathematician, (15) poetry, (16) mathematician, (17) Poetry.

Some Aspects of Products of Derivatives

A. M. Bruckner, J. Mařík and C. E. Weil

1. INTRODUCTION. In 1921, Wilcosz [W] showed that the function $f(x) = \cos 1/x$ ($f(0) = 0$) is a derivative, but the function f^2 is not. (Saying f "is," rather than "has," a derivative means that there is a differentiable function F such that $F'(x) = f(x)$ for all x.) The Wilcosz example shows simultaneously that the class of derivatives is not closed under multiplication nor under outside composition with continuous functions. As the title suggests, this article deals primarily with the first consequence. However, concerning the second, it is natural to seek functions φ such that for each derivative f the composition $\varphi \circ f$ is again a derivative. It is obvious that linear functions φ have this property. However, it is not difficult to prove that there are no other possibilities; every such function φ is linear.

The Wilcosz example has other consequences as well. It is well known that a function f is continuous if and only if each of its associated sets, i.e., sets $\{x : f(x) > a\}$ and $\{x : f(x) < a\}$, where a is any real number, is open. One might be tempted to find a similar characterization for derivatives; in other words, to prove that a function is a derivative if and only if each of its associated sets has a certain property. The Wilcosz example can be used to show that there is no such theorem. Namely, if f is that function and if $F = f + 1$, then, since $F \geqslant 0$, F and F^2 have the same system of associated sets while F is a derivative but F^2 is not.

Incidentally, the theorem about the outside composition mentioned above yields another way, although a little less elementary, of showing that such a characterization of derivatives is not possible. Namely, if φ is any nonlinear, continuous, increasing function on $\mathbb{R}$ with range $\mathbb{R}$, then, for some derivative f, the composition $\varphi \circ f$ is not a derivative while, obviously, f and $\varphi \circ f$ have the same system of associated sets. A more complete treatment of this associated sets problem and a discussion of the topological character of the class of derivatives together with some applications can be found in [**B**, pp. 135–144].

The fact that the class of derivatives is not an algebra raises a number of interesting questions, some of which have been studied only in the past few years. The purpose of this article is to state these questions, to try to impart some of the flavor of the subject to the reader, and to indicate applications of some of the results. We shall try to present the material in a nontechnical, expository manner.

2. FOUR QUESTIONS. We shall denote by Δ the class of differentiable functions on $\mathbb{R}$ and by Δ' the class of derivatives. Thus, $f \in \Delta'$ if and only if there exists $F \in \Delta$ such that $F'(x) = f(x)$ (finite) for all $x \in \mathbb{R}$. The Wilcosz example immediately raises the following two questions.

Question 1. If f and g are in Δ', what else should be required of one or both of them to conclude that $fg \in \Delta'$?

Question 2. Given that the product of derivatives need not be a derivative, what functions f admit a representation of the form $f = f_1 f_2 \cdots f_n$ ($f_1, f_2, \dots, f_n$ all in Δ')?

These two questions lead to the next two.

Question 3. What other algebraic representations of functions by derivatives are of interest?

Question 4. What functions are in Alg Δ', the algebra generated by the derivatives?

These questions are the obvious ones to ask, but attempts to solve them have led to some surprisingly deep mathematics. The first one has the longest history; we discuss it in Section 4. The other three have been investigated only recently and we treat them in Sections 5, 6 and 7. The next section contains necessary information which may not be known to some readers.

3. SOME NEEDED FACTS. First, we recall that every continuous function is in Δ'. Of course not every function in Δ' is continuous, but every member of Δ' has the Intermediate Value (or Darboux) Property. It should be emphasized that a derivative can behave rather "unreasonably." For example, a derivative need not be locally summable (that is, locally Lebesgue integrable). We will soon see examples of functions $f \in \Delta'$ that are continuous on $(0, 1]$ such that $\int_0^1 |f| = \infty$.

When we deal with derivatives we often come across an essential, but not well-known concept, namely, approximate continuity which is defined next.

Let m be Lebesgue measure. Saying "a function f is approximately continuous at a point x" means that there is a Lebesgue measurable set E such that

$$\lim_{h \to 0} + m(E \cap (x - h, x + h))/2h = 1 \tag{1}$$

and that $\lim_{y \to x,\, y \in E} f(y) = f(x)$. So ordinary continuity is weakened by requiring that $f(y)$ converges to $f(x)$ only as y approaches x through a subset E; one that is "dense" enough at x so that, among other things, the limit is unique (does not depend on the choice of E). The set E is also dense enough to guarantee that the sum and the product of two functions approximately continuous at x are again approximately continuous at x. In what follows, "a function is approximately continuous" will mean that it is approximately continuous everywhere (i.e. at each point in $\mathbb{R}$). For the purpose of this article it is important to know that every bounded, approximately continuous function is in Δ'. (Every such function is the derivative of its indefinite Lebesgue integral.) Finally, we state the following two important facts. The second is somewhat deeper than the first.

Fact 1. If F is differentiable and monotone, then its derivative F' is locally summable. Consequently, if f is nonnegative and not locally summable, then $f \notin \Delta'$.

Fact 2. If $F \in \Delta$ and if F' is summable on $[a, b]$, then $\int_a^b F' = F(b) - F(a)$. Therefore, if a locally summable function is a derivative, it is the derivative of its indefinite integral.

4. MULTIPLIERS FOR Δ'. The Wilcosz example shows that the product of two derivatives f and g need not be a derivative. What happens if we require more of one of the factors, say f? Can we then conclude that $fg \in \Delta'$ for all $g \in \Delta'$? If so, we would call f a "multiplier" for the class of derivatives. What sort of regularity conditions would imply that a function is a multiplier for Δ'? It is not hard to see that continuity is not enough. So let us suppose more; for example, that the first factor, now denoted by F, is differentiable (i.e., $F \in \Delta$). Does this imply that $Fg \in \Delta'$ for all $g \in \Delta'$? If one believes that differentiability provides enough regularity, then, in view of Fact 2, one would perhaps try to prove that if $H(x) = \int_0^x Fg$, then $H'(x) = F(x)g(x)$ for all x. This may seem a plausible approach, but one immediately encounters difficulties involving the summability of the integrand (even g need not be summable). This difficulty, together with Fact 1, actually provides a clue toward obtaining a counter-example. We need only construct F and g so that the product Fg is nonnegative and not locally summable. No function H could meet the requirement that H' is nonnegative (everywhere) and not locally summable. Such combinations of functions F and g are easy to find using properties of functions of the form $x^n \sin x^{-m}$ and $x^n \cos x^{-m}$. For example, if

$$F(x) = x^2 \sin x^{-5} \qquad (F(0) = 0)$$

and

$$G(x) = \tfrac{1}{5}x^2 \cos x^{-5} - \tfrac{2}{5}\int_0^x t \cos t^{-5}\, dt \qquad (G(0) = 0),$$

then the function $g = G'$ fulfills the relations $g(x) = x^{-4} \sin x^{-5}$, $g(0) = 0$ and $(Fg)(x) = x^{-2} \sin^2 x^{-5}$ $((Fg)(0) = 0)$. This product is nonnegative and (as can be easily shown) not summable in any neighborhood of the origin. Thus Fg cannot be a derivative.

What happens if we remove the apparent problem in our example? That is, if we require Fg to be locally summable, can we conclude that $Fg \in \Delta'$? According to Fact 2, we must then try to prove that if $H(x) = \int_0^x Fg$, then $H'(x) = F(x)g(x)$ for all x. After some unsuccessful attempts we may arrive at the following example: Let $F(x) = x^2 \sin x^{-3}$, $G(x) = x^2 \cos x^{-3}$ $(F(0) = G(0) = 0)$. We verify easily that FG' and GF' are bounded and therefore summable on any bounded interval. Straightforward calculations show that

$$F(x)G'(x) - F'(x)G(x) = \begin{cases} 3, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

If either of the functions FG' or GF' were a derivative, then the other would be also since $FG' + GF' = (FG)' \in \Delta'$, and the same would be true of their difference. But it is not, since derivatives have the Darboux property.

So we are forced to assume even more about F. Suppose that F' is continuous. Let G be a primitive of g (i.e. $G' = g$). Then, obviously, $Fg = (FG)' - F'G$. The function $F'G$ is continuous and $(FG)' \in \Delta'$. Thus $Fg \in \Delta'$ and we have our first positive result!

(A_1) If $g \in \Delta'$ and F' is continuous, then $Fg \in \Delta'$.

More generally, one can prove

(A_2) If $g \in \Delta'$ and F' is locally summable, then $Fg \in \Delta'$.

Thus, such functions F are multipliers for Δ'. So we see that local summability is relevant—but for F' rather than for Fg.

Getting to the essence of (A_2), if F' is locally summable, then, as is easily proved, F is the difference of two continuous nondecreasing functions. We see that (A_2) follows from the next assertion:

(A'_2) If $g \in \Delta'$ and if F is continuous and nondecreasing, then $Fg \in \Delta'$.

(If $G' = g$ and if $H(x) = F(x)G(x) - \int_0^x G\,dF$, then $H(x+h) - H(x) = F(x+h)(G(x+h) - G(x)) - \int_x^{x+h}(G - G(x))\,dF$ which easily implies that $H' = Fg$.)

Using the product formula we obtain from (A_2) a companion theorem:

(A_3) If $F \in \Delta$, $g \in \Delta'$ and if g is locally summable, then $Fg \in \Delta'$.

The assertion (A_3), however, is not of the same type as (A_1), (A_2), and (A'_2). In (A_1), (A_2) and (A'_2) we impose conditions only on F whereas in (A_3) we require also local integrability of g.

It is natural to ask whether we can improve (A_3) by weakening the requirement that $F \in \Delta$ to simply that F be continuous. The following example shows that we cannot.

Let

$$F(x) = \sqrt{x}\cos\frac{1}{x}, \qquad g(x) = \frac{1}{\sqrt{x}}\cos\frac{1}{x} \qquad (x > 0),$$

$$F(x) = g(x) = 0 \qquad (x \leqslant 0).$$

Then F is continuous and one can calculate that g is a locally summable derivative. Yet

$$(Fg)(x) = \begin{cases} \cos^2\dfrac{1}{x} & (x > 0) \\ 0 & (x \leqslant 0), \end{cases}$$

a function which, according to Wilcosz, is not a derivative.

If, however, we require g to be nonnegative (which, by Fact 1, implies that g is locally summable), then we can conclude that $Fg \in \Delta'$:

(A_4) If $g \in \Delta'$, $g \geqslant 0$ and if F is continuous, then $Fg \in \Delta'$.

It is easy to see that the zero function in (A_4) can be replaced by any nonpositive derivative. In this way we obtain the following generalization of (A_4):

(A_5) If $g, h \in \Delta'$, $g \geqslant h$, $h \leqslant 0$ and if F is continuous, then $Fg \in \Delta'$.

It is also easy to see that the following three properties of a function $g \in \Delta'$ are equivalent:

(i) There is an $h \in \Delta'$ such that $h \leqslant 0$ and $h \leqslant g$.
(ii) There are $h_1, h_2 \in \Delta'$ such that $h_1 \geqslant 0$, $h_2 \geqslant 0$ and $g = h_1 - h_2$.
(iii) There is an $h \in \Delta'$ such that $|g| \leqslant h$.

These conditions suggest obvious modifications of (A_5).

The preceding results may be formulated also in another way, if we speak about multipliers for subclasses of Δ'. A function f is said to be a multiplier for such a subclass S, in symbols $f \in M(S)$, if and only if $fg \in \Delta'$ for all $g \in S$. Using this terminology we get the following:

(A'_2) A continuous, nondecreasing function is a multiplier for Δ'.
(A_3) A differentiable function is a multiplier for locally summable derivatives.
(A_4) A continuous function is a multiplier for nonnegative derivatives.

We have also seen that a continuous function need not be a multiplier for locally summable derivatives (such a derivative need not be the difference of two nonnegative derivatives).

For certain subclasses of Δ' the multipliers have been completely characterized. For example, let S_0 be the set of all locally bounded derivatives. Then $M(S_0)$ is the set L of Lebesgue functions, i.e., functions f such that

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|\, dt = 0$$

for each $x \in \mathbb{R}$. It is easy to prove that each element of L is locally summable and approximately continuous, that $L \subset \Delta'$ and that each bounded approximately continuous function is in L. Surprisingly, the "dual" statement $M(L) = S_0$ is also valid.

It can be proved that $M(\Delta')$ is the class of all derivatives F such that

$$\limsup_{n \to \infty} \left(\operatorname{var}\left(F, \left[x + \frac{1}{n}, x + \frac{2}{n} \right] \right) + \operatorname{var}\left(F, \left[x - \frac{2}{n}, x - \frac{1}{n} \right] \right) \right) < \infty \quad \text{for each } x \in \mathbb{R}. \tag{2}$$

The multipliers for the class of all summable derivatives can be characterized in a similar way.

Our results and our examples give a sense of the delicacy of determining conditions on two functions $F, g \in \Delta'$ such that $Fg \in \Delta'$. We are looking for some regularity conditions that, when imposed on F, would imply that $Fg \in \Delta'$ for each $g \in \Delta'$, or for each $g \in S$, where S is a given class of derivatives. However, such conditions have sometimes surprisingly little to do with continuity or differentiability of F. It is easy to construct a discontinuous derivative F fulfilling (2); thus continuity is not a necessary condition for being a member of $M(\Delta')$. On the other hand, we have seen that differentiability is not sufficient.

A different notion of multipliers has also been studied. A function f is sometimes called a multiplier for S if and only if $fg \in S$ for each $g \in S$. In this setting, the multipliers of locally bounded derivatives consist of the locally bounded approximately continuous functions.

It is obvious that these two definitions of multipliers yield the same result, if $S = \Delta'$; an analogous assertion holds also, if S is the class of locally summable derivatives.

The interested reader may wish to consult Fleissner [**F**]. This survey article was current at the time it was written.

Some of the results we have mentioned can be found in [**Mi** 1–4]; the proof of the relation $M(S_0) = L$ and the characterization of $M(S)$ for some other classes S have yet to be published.

5. REPRESENTATIONS AS PRODUCTS OF DERIVATIVES. Since the product of two or more derivatives need not be a derivative, it is natural to ask what functions admit such a representation. Now any $f \in \Delta'$ is in B_1, the first class of Baire (that is, it is the pointwise limit of a sequence of continuous functions) and it has the Darboux property as has already been mentioned. Since B_1 is an algebra, any product of derivatives must also be in B_1. What can we say about the Darboux property for the product? The fact that the product of two functions with the Darboux property need not have that property suggests that the product of two derivatives need not have the Darboux property. On the other hand, in spite of the

fact that the *quotient* of two functions with the Darboux property need not have that property, the quotient of two derivatives will have the Darboux property (if the denominator is never zero) [**Hr**]. This suggests that products of derivatives may have the Darboux property.

Let us first try to settle this question by considering the simplest sort of function without the Darboux property. Let

$$h(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$$

That is, h is the characteristic function of the origin, $\chi_{\{0\}}$. If we wish to express h in the form $h = fg$ ($f, g \in \Delta'$), then f must be zero whenever g is not (except at $x = 0$). It is not difficult to construct two differentiable functions F and G, both of whose graphs are trapped between the curves $y = x$ and $y = x^2 + x$ such that for every $x \neq 0$ there is an interval containing x on which F or G is constant. Then $F'(x)G'(x) = 0$, if $x \neq 0$. Clearly $F'(0) = G'(0) = 1$. This provides the desired construction, $h = F'G'$.

Another (more "arithmetical") such representation is the following: Let

$$f(x) = \max\left(\pi \sin \frac{1}{x}, 0\right), \qquad g(x) = \max\left(-\pi \sin \frac{1}{x}, 0\right) \qquad (x \neq 0),$$
$$f(0) = g(0) = 1.$$

It is not difficult to prove that $f, g \in \Delta'$ and that $fg = \chi_{\{0\}}$. So the product of two derivatives need not have the Darboux property.

Using refined versions of either of these two arguments, one can actually prove that if K is any closed set, then χ_K is the product of two derivatives.

What other simple non-Darboux functions are the product of two derivatives? What about χ_U, where U is an open set, say $U = (0, \infty)$?

Suppose $h = \chi_U$ and $h = fg$, $f, g \in \Delta'$. Then f and g have the same sign on $(0, \infty)$. Let $x > 0$. Since f and g are in Δ', both are summable on $[0, x]$ according to Fact 1. It follows from the Cauchy-Schwarz inequality that

$$x^2 = \left(\int_0^x \sqrt{fg}\right)^2 \leqslant \left(\int_0^x f\right)\left(\int_0^x g\right) = (F(x) - F(0)) \cdot (G(x) - G(0)),$$

where $F' = f$ and $G' = g$. Hence $f(0)g(0) = F'(0) \cdot G'(0) \geqslant 1$, a contradiction.

This shows not only that h is not the product of two derivatives, but also that if h were redefined at 0 to be such a product, it would have to satisfy $h(0) \geqslant 1$. Similar arguments show that h is not the product of any number of derivatives.

We have arrived at the following comparison: $\chi_{[0,\infty)}$ can be expressed as the product of two derivatives but $\chi_{[0,\infty)}$ cannot be expressed as the product of any number of derivatives. Yet these two functions, in addition to differing at only one point, are closely related by various identities; for example, $\chi_{[0,\infty)}(x) + \chi_{[0,\infty)}(-x) = 1$ for each x.

The mentioned results concerning characteristic functions are special cases of Corollary 3.7, page 33 of [**BMW**]. Also see [**Mi5**]. A more general (but still not-too-technical) special case is the following:

Theorem. *Let $u > 0$ on $[0, \infty)$, let u be continuous on $(0, \infty)$ and constant on $(-\infty, 0]$. There exist nonnegative numbers $q_2 \geqslant q_3 \geqslant q_4 \geqslant \cdots$ such that if $u(0) \geqslant q_k$, then u can be expressed as the product of k derivatives but if $u(0) < q_k$, no such representation is possible.*

Explicit values of the numbers q_k are given in [**MW**].

As an illustration of this theorem let us consider a function u with the following properties: Let $0 < a < b$, $\alpha, \beta > 0$, $\alpha + \beta = 1$. Let u be continuous on $(0, \infty)$, constant on $(-\infty, 0]$ and let

(i) $a \leqslant u \leqslant b$ on $(0, \infty)$,

$$\text{(ii)} \qquad \lim_{h \to 0^+} \frac{m\{x \in (0,h) : u(x) = a\}}{h} = \alpha,$$

$$\text{(iii)} \qquad \lim_{h \to 0^+} \frac{m\{x \in (0,h) : u(x) = b\}}{h} = \beta.$$

(It is not difficult to construct such a function u.) Then one can calculate (using Prop. 5.3 and Remark 1 on page 367 of [**MW**]) that $q_n = (\alpha a^{1/n} + \beta b^{1/n})^n$. An elementary application of L'Hôpital's Rule yields the result $q_n \to a^\alpha b^\beta$. (See also Prop. 6.6 of [**MW**].) For example, if $a = 1$, $b = 4$, $\alpha = \beta = 1/2$, then $q_2 = 9/4$ and $q_n \to 2$.

In this example, if $u = 5/2$ on $(-\infty, 0]$, then u is a derivative; if $u = 9/4$ on $(-\infty, 0]$, then u is not a derivative but can be expressed as the product of two derivatives, and as $u(0)$ decreases, the number of factors in a representation of u as a product of derivatives increases. When $u(0) \leqslant 2$, no such representation exists.

Let P be the set of all functions that can be expressed as the product of (finitely many) derivatives. How big is P in B_1? Let us equip B_1 with the topology of uniform convergence. Our function u with $u(0) = 2$ is not in P, but, obviously, is in its closure. Hence (as we could expect) P is not closed.

We have indicated that the characteristic function of a nonempty open set $G \neq \mathbb{R}$ is not in P. Similarly, the following can be proved: If $c \in \mathbb{R}$, $\varepsilon \in [0, \infty)$ and if f is a function such that $\varepsilon < \liminf f(x)$ $(x \to c+)$, then $f \notin P$. It is easy to see that such an f is not even in the closure of P. Using the fact that each Baire one function has points of continuity we now see that P is nowhere dense in B_1.

On the other hand, P contains some rather complicated functions. For example, every Baire 1 function that is zero almost everywhere (a.e.) is the product of two derivatives [**BMW**]. This fact provides a very simple solution to a problem which at one time baffled some of the leading mathematicians of the day. Let us discuss this problem briefly and then show how our theorem on products of derivatives provides a simple solution.

Over one hundred years ago DuBois-Reymond held the view that a differentiable function must be monotone on some interval. Dini, on the other hand, believed the existence of nowhere monotone differentiable functions highly probable. (See [**Ho**], page 412.) In 1887, Koepcke provided a construction of such a function [**K**]. In discussing Koepcke's work, Denjoy wrote in 1915 [**D1**], "In 1887, Koepcke gave in Math. Annalen an example of a function possessing at each point (or so he thought) a derivative which vanished and took both signs in every interval contained in its domain of definition. This geometer returned to this subject on several occasions, correcting each time the errors contained in the previous proofs." This question of differentiable nowhere monotone functions has also provoked many other works.

The Koepcke constructions Denjoy referred to were quite complicated. They were later simplified by Pereno and other mathematicians. Denjoy then gave four separate constructions of his own, which were also quite complicated.

Hobson modified Pereno's modification of Koepcke's construction in the second edition of his book [**Ho**]. This edition was published in 1921, about forty years after

Koepcke's first correction, thirty years after Pereno's modification and fifteen years after Denjoy's several developments. It required ten pages!

Today a number of faster proofs of the existence and constructions of differentiable nowhere monotone functions exist. Here is a quick one based on the result we mentioned; namely that each Baire 1 function that equals 0 a.e. is the product of two derivatives.

Let

$$h(x) = \begin{cases} \dfrac{1}{q} & \text{if } x = \dfrac{p}{q},\ q \text{ even} \\ -\dfrac{1}{q} & \text{if } x = \dfrac{p}{q},\ q \text{ odd}, \\ 0 & \text{elsewhere}, \end{cases}$$

where p and q are relatively prime integers and $q > 0$. Then h is continuous except on a denumerable set, and therefore in Baire class 1 [**N**]. Clearly h is zero a.e. According to the result alluded to, there exist $f, g \in \Delta'$ such that $h = fg$. If f takes both signs on each interval, then a primitive of f is the desired function. If not, then there is an interval I on which f is unsigned. But since h takes both signs on dense subsets of I, so does g and then a primitive of g is the desired function.

6. OTHER REPRESENTATIONS BY DERIVATIVES. We have seen that the characteristic function of a proper nonempty open subset of $\mathbb{R}$ cannot be expressed as the product of any number of derivatives. If we allow addition as well as multiplication, then such a function can be expressed in terms of derivatives as we shall now see. Recall that if $F(x) = x^2 \sin x^{-3}$ and $G(x) = x^2 \cos x^{-3}$ for $x > 0$ and $F(x) = G(x) = 0$ for $x \leqslant 0$, then $FG' - F'G = 3\chi_{(0,\infty)}$. Of course $(FG)' = FG' + F'G$. Thus $2FG' - (FG)' = 3\chi_{(0,\infty)}$; that is, there are functions $F, G, H \in \Delta$ such that $\chi_{(0,\infty)} = FG' + H'$. It will not surprise the reader to learn that the characteristic function of any open set can be written in the same fashion. It follows that the characteristic function of any closed set can also be thusly written.

Another representation of $\chi_{(0,\infty)}$ may interest the reader. In the previous section we have encountered bounded derivatives f and g with $fg = \chi_{\{0\}}$. Let us define functions f_1, g_1 setting $f_1 = g_1 = 1$ on $(-\infty, 0)$ and $f_1 = f$, $g_1 = g$ on $[0, \infty)$. It is easy to see that f_1 and g_1 are bounded derivatives and that $f_1 g_1 = \chi_{(-\infty,0]}$. Hence $\chi_{(0,\infty)} = 1 - f_1 g_1$. Our previous representation $\chi_{(0,\infty)} = FG' + H'$ is, in some sense, better; we multiply the derivative G' by a "more reasonable" function. However, the function G' is obviously unbounded. It is worth mentioning that the unboundedness of G' was not caused by our awkwardness. No matter how we represent the function $\chi_{(0,\infty)}$ in the form $FG' + H'$ with $F, G, H \in \Delta$, G' must be unbounded (in fact, not even locally integrable); because if it were bounded, we would have (see (A_3) in section 4) $FG' \in \Delta'$ and hence $\chi_{(0,\infty)} \in \Delta'$ which is impossible.

The association of open sets with functions admitting this type of a representation is intrinsic as the following theorem from [**ABBM**] shows.

Theorem. *Let* $\Phi\colon \mathbb{R} \to \mathbb{R}$. *The following two conditions are equivalent*:

1. *There are* $F, G, H \in \Delta$ *such that* $\Phi = FG' + H'$.
2. *There is an open set* U, *a function* $K \in \Delta$, *and a function* L *differentiable on* U *such that* $\Phi = L'$ *on* U *and* $\Phi = K'$ *on* $\mathbb{R} \setminus U$.

It follows from (A_2) (section 4) that if $F \in \Delta$ has a summable derivative, then $FG' + H'$ is a derivative, which, along with condition 2 shows that functions of the form $FG' + H'$ $(F, G, H \in \Delta)$ are very close to being derivatives. This fact can be exploited to show that some desirable properties are possessed by certain classes of functions that arise naturally in differentiation theory. For example, every approximately continuous function satisfies conditions 1 and 2. The same is true of functions in O'Malley's class B_1^* (see **[O]**) which are also called generalized continuous functions. A function f is in B_1^* if to every nonempty closed set E there corresponds an open interval I intersecting E such that $f|I \cap E$ is continuous. It is well-known (see **[N]**) that a function f is in B_1 if and only if every nonempty closed set E contains a point x such that $f|E$ is continuous at x. So $B_1^* \subset B_1$. Actually the class B_1^* is much smaller than the class B_1. To see this, let us denote by V the system of all functions Φ with the property 1 (or 2). It is obvious that $V \supset \Delta'$ and (because B_1 is an algebra) that $V \subset B_1$. We can easily construct a derivative with a dense set of points of discontinuity and we have a function that is in V, but not in B_1^*. On the other hand, every function in V is a derivative on some interval. It follows that no increasing function with a dense set of points of discontinuity is in V. Thus we see that the inclusions $B_1^* \subset V \subset B_1$ are proper in a rather strong sense.

We know that there are functions $F, G \in \Delta$ such that $FG' \notin \Delta'$; we have, of course, $FG' \in V$. Moreover, it follows from 2 that V contains some functions that do not have the Darboux property. So V is also "much bigger" than Δ'.

Of particular interest is the fact that the so-called approximate derivatives are in V.

The approximate derivative is the most thoroughly studied generalized derivative. It serves as an excellent substitute for the ordinary derivative when the latter is not known to exist. To say that f is approximately differentiable at x with approximate derivative $f'_{ap}(x)$ means that there is a set E satisfying the same conditions as in the definition of approximate continuity such that

$$\lim_{y \to x,\, y \in E} \frac{f(y) - f(x)}{y - x} = f'_{ap}(x).$$

The reason that f'_{ap} is such a good substitute for the ordinary derivative is that it shares all the known desirable properties of ordinary derivatives. This fact was established, in pieces, by various authors **[D2]**, **[C]**, **[Mc]**, **[We1]**, **[We2]**, **[P1]**. Moreover, one has, for example, the result that any monotonicity theorem valid for differentiable functions has a complete analogue for approximately differentiable functions **[OW]**.

Much of this good behavior of f and f'_{ap} can be understood by the fact that f'_{ap} satisfies conditions 1 and 2 above. For example, one sees immediately that $f \in B_1^*$ and that f is differentiable on a dense open set.

When dealing with a class S of functions, one often wonders whether the members of S must remain in S when "perturbed" algebraically or topologically; that is, is S closed under the perturbations under consideration? For many classes the answer to specific questions of this type is often an unqualified "yes." For classes whose definitions involve the notion of derivative, the answer is usually "only in exceptional cases". The class Δ' is sensitive to algebraic and topological perturbations. We have seen, for example, that multiplication of a derivative by even a differentiable function can result in a function that is not a derivative. We

have also seen that compositions of functions in Δ' with homeomorphisms may result in functions that are not derivatives. We mentioned in Section 1 that if $\varphi \circ f \in \Delta'$ for every $f \in \Delta'$, then φ is linear. As a further example, if $f \in \Delta'$, and $\varphi \circ f \in \Delta'$ for *some* strictly convex φ, then f is approximately continuous. (Thus, the reciprocal of a positive derivative is usually not a derivative.) For inner compositions we mention that if $f \in \Delta'$ and $f \circ h$ is a derivative for every homeomorphism h, then f is continuous. (These results can all be found in [**B**]).

Recent results involving the representation of functions by derivatives provide illustrations of a similar phenomenon. The general idea can be roughly described in the following way. If a well-behaved function is expressed algebraically in terms of several derivatives, then these derivatives are themselves well-behaved. (This statement is, of course, a vague one and shouldn't be taken too literally.) We present some illustrations. But first we remark that within the class of bounded derivatives, the class of approximately continuous (a.c.) functions is "small"; more precisely, it is a nowhere dense subset when the bounded derivatives are equipped with the sup norm.

We have seen that the product of several derivatives may be rather badly behaved. The Baire one functions that vanish almost everywhere can serve as an illustration. (We mentioned in Section 5 that every such function f is the product of two derivatives. If, moreover, $f \geqslant 0$, then both factors can be taken to be nonnegative.)

What happens if the product is well-behaved? It is clear that the approximate continuity alone would not help much; the product of two very wild functions can be identically zero. We have, however, the following result [**MW**]: If the product is a.c. and positive, then each factor is a.c. This result actually holds "pointwise": If $f_k \in \Delta'$ for all $k = 1, \dots, n$, if the function $f = \prod_{k=1}^{n} f_k$ is a.c. at x_0 and if $f > 0$, then each f_k is a.c. at x_0.

It is natural to ask various analogous questions. For example: What can we say about derivatives f and g, if we know that the sum of their squares is well behaved? One possible answer is contained in the following theorem: Let $f, g, h \in \Delta'$ and let ε be a positive number such that (everywhere) $f^2 + g^2 = h^2 \geqslant \varepsilon$. Then both ratios f/h, g/h are a.c. If, in particular, h is a.c., then also f and g are a.c.

In a similar way it can be proved that derivatives f and g are in L (= Lebesgue functions) if and only if $(f^2 + g^2)^{1/2} \in L$. Or, equivalently: Let $h \in L$. Then the set of all pairs (f, g) of derivatives such that $f^2 + g^2 = h^2$ is identical with the set of all pairs (f, g) of Lebesgue functions fulfilling the equation.

Instead of squares we may, of course, investigate also other powers; the corresponding results are sometimes even better. For example, the following theorem holds: Let $f, g, h \in \Delta'$, $h \geqslant 0$ and let $f^4 + g^4 = h^2$. Then $f, g, h \in L$ (in particular, f, g, and h are all a.c.).

7. THE ALGEBRA GENERATED BY Δ'. We have already seen that many types of Baire 1 functions can be represented algebraically by derivatives. This leads naturally to Question 4: What functions are in Alg Δ', the algebra generated by the derivatives? Since $\Delta' \subset B_1$ and B_1 is an algebra, it is clear that Alg $\Delta' \subset B_1$. It is also not difficult to verify that the class B_1^* mentioned in section 6 is uniformly dense in B_1. One need only observe (see [**ABBM**], Lemma 5 and Proposition 3) that a Baire 1 function with isolated range is in B_1^*. Since each $f \in B_1^*$ admits the representation $f = gh' + k'$ $(g, h, k \in \Delta)$, it is clear that Alg Δ' is uniformly dense

in B_1. This suggests that perhaps Alg $\Delta' = B_1$. On the other hand, there is a good deal of evidence that might cause one to believe that B_1 is much larger than Alg Δ'. For one thing, Baire 1 functions can exhibit a great deal more pathology than can any derivative. For another, Δ' is closed with respect to uniform convergence from which it follows without much difficulty that Δ' is nowhere dense in B_1 (in the topology of uniform convergence). Finally, if it is true that Alg $\Delta' = B_1$, then there is an integer N such that each $f \in B_1$ can be represented algebraically in terms of no more than N derivatives. To see this one need only observe that if this were not the case one could construct $f \in B_1$ with the property that on the interval $[n, n+1]$ at least n derivatives are needed to represent f algebraically. Then no algebraic representation of f in terms of finitely many derivatives would be possible. If one believes that Alg $\Delta' \neq B_1$, one may attempt to prove this by showing that for each n there exists $f \in B_1$ which cannot be expressed algebraically in terms of fewer than n derivatives.

During the beginning of this decade, one of the authors used this approach (unsuccessfully) while another obtained several classes of functions whose members admitted a representation of the form $f = g'h' + k'$ $(g, h, k \in \Delta)$. For example, each function of bounded variation admits such a representation (but not necessarily representations as products of derivatives or representations of the form $gh' + k'$ $(g, h, k \in \Delta)$). In fact, no matter what approach was used, no examples of Baire 1 functions which didn't admit such a representation were forthcoming. Eventually, this led to the conjecture that *every* Baire 1 function admits such a representation. Various attempts to prove this conjecture seemed promising—but none worked. The problem was a very elusive one.

Finally, in 1982, David Preiss [**P2**] succeeded in proving the conjecture. In fact he was able to impose additional conditions on the derivatives appearing in the representation. We state his result as a Theorem.

Theorem (Preiss [**P2**]). *Let $f \in B_1$. There exist functions g, h and k such that $f = g'h' + k'$, g' is bounded and k is a Lebesgue function. If f is bounded, one can choose g', h' and k' all bounded.*

The representation in Preiss' theorem may be compared with the representation $\Phi = FG' + H'$ discussed in the previous section. Functions admitting the latter representation have many desirable properties. Yet replacing the function $F \in \Delta$ by a function $f \in \Delta'$ may result in a function with no specific properties (beyond the obvious one of membership in the algebra B_1). This contrast may be viewed as another indication of the unstable nature of derivatives.

We close by returning to Question 2. Preiss' remarkable theorem provides an indication of the difficulty inherent in attempting to answer this question. We have seen that the class of functions whose members are representable as the product of two or more derivatives is quite restricted. Yet because of Preiss' theorem we see that each $f \in B_1$ differs from a product of two derivatives by a derivative!

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Boolean Circulants, Groups, and Relation Algebras

Chris Brink and Jan Pretorius

1. INTRODUCTION. Recall that a circulant matrix is determined by its top row: each successive row is determined from the preceding one by shifting entries one position to the right, with wraparound. Since the publication of Davis [**4**] there has been increasing interest in circulants. Some examples from this *Monthly* alone are Wong [**12**], Ungar [**11**] and Clark [**3**]. A *Boolean* circulant matrix is a circulant with entries from a Boolean algebra–usually (and here as well) just from the simple Boolean algebra $\mathbf{2} = \{0, 1\}$. Davis [**4**] does not explicitly mention Boolean circulants, but they receive some attention in Kim [**8**], a book on Boolean matrix theory (from which further references can be obtained). One aspect of Boolean matrices not treated in Kim's book, however, is that n-square Boolean matrices form a model of Tarski's concept of a *relation algebra*. These algebras date back to Tarski [**10**]; they were introduced in an attempt to do for the algebra of binary relations what Boolean algebras do for the calculus of sets. A recent survey paper is Jónsson [**7**]; detailed references can be found in Henkin, Monk and Tarski [**6**].

Our purpose here is to embed some results on Boolean circulants into the context of relation algebras, and then to generalise them. Interestingly enough, the route lies through group theory.

2. CIRCULANTS. Let $\mathscr{B}_n$ be the algebra which has as base set all n-square Boolean matrices and is endowed with the componentwise Boolean operations of complementation $'$, meet $\cdot$ and join $+$ (under which it forms a Boolean algebra), as well as the matrix operations of transposition and multiplication ; , and the identity matrix I.

Definition 1. An element $[c_{i,j}]$ of $\mathscr{B}_n$ is a *circulant* iff $c_{0,k} = c_{j,m}$ whenever $j + k \equiv m \pmod n$, where $0 < j, k, m \leqslant n - 1$.

This is just the formal version of the characterization of circulants as matrices produced by the top-row vector. As in Clark [**3**] we indicate the circulant corresponding to a Boolean vector $(c_1, c_2, \dots, c_n)$ by $Circ(c_1, c_2, \dots, c_n)$. Let $\mathscr{C}_n$ be the set of all n-square circulants. The first thing to know is whether $\mathscr{C}_n$ is closed under all the operations of $\mathscr{B}_n$. For this purpose we introduce the special circulant P, defined by

$$P := Circ(0, 1, 0, 0, \dots, 0) \tag{1}$$

(where the vector is understood to be of length n). Then it is easy to check that

$$\begin{aligned} P^2 &= Circ(0, 0, 1, 0, \dots, 0) \\ P^3 &= Circ(0, 0, 0, 1, \dots, 0) \\ &\vdots \\ P^{n-1} &= Circ(0, 0, 0, 0, \dots, 1) \end{aligned}$$

where the exponents indicate repeated matrix product. Also, we define:

$$P^0 := P^n = Circ(1, 0, 0, 0, \ldots, 0) = I. \tag{2}$$

These n circulant matrices are closed under the matrix operations of transposition and multiplication. In fact, for $0 \leqslant j, k < n - 1$ and $j + k \equiv m \pmod{n}$:

$$(P^k)^{\smile} = P^{n-k}$$
$$(P^j);(P^k) = P^m.$$

Which is just another way of saying:

Theorem 1. $\{P^0, P^1, \ldots, P^{n-1}\}$ *endowed with matrix operations is isomorphic to* Z_n, *the (cyclic) group of integers* $\bar{k}$ *modulo* n.

(Note: we use overlining to indicate integers modulo n.) The usefulness of the P^k's is that any circulant C can be written canonically as a Boolean sum of some of them:

$$C = \pi_0 I + \pi_1 P^1 + \pi_2 P^2 + \cdots + \pi_{n-1} P^{n-1} \tag{3}$$

where each $\pi_i \in \{0, 1\}$ is an indicator telling us whether or not 'P^i' is part of the expression. To see that this is so it is sufficient to keep in mind that C corresponds to its top-row Boolean vector; that any such Boolean vector is a Boolean sum of the atomic vectors corresponding to the matrix powers of P, and that accordingly the top row of C is precisely the vector $(\pi_0, \pi_1, \ldots, \pi_n)$. Hence the Boolean operations of complementation, meet and join on circulants correspond to these same operations on the top-row Boolean vectors, and so circulants are closed under Boolean operations. In fact, we can say more. Both transposition and multiplication of Boolean matrices distribute over Boolean sums:

$$([a_{i,j}] + [b_{i,j}])^{\smile} = [a_{ij}]^{\smile} + [b_{i,j}]^{\smile}$$
$$([a_{i,j}] + [b_{i,j}]);[c_{i,j}] = [a_{i,j}];[c_{i,j}] + [b_{i,j}];[c_{i,j}].$$

Hence converses and products of circulants may be calculated from their canonical forms (3) by using distribution and Theorem 1. Thus, for example

$$\begin{aligned}[Circ(0,1,1,0,1,0)]^{\smile} &= (P^1 + P^2 + P^4)^{\smile} \\ &= (P^1)^{\smile} + (P^2)^{\smile} + (P^4)^{\smile} \\ &= P^5 + P^4 + P^2 \\ &= Circ(0,0,1,0,1,1).\end{aligned}$$

And

$$\begin{aligned}Circ(0,1,1,0,1,0);Circ(0,0,0,1,0,1) &= (P^1 + P^2 + P^4);(P^3 + P^5) \\ &= P^1;P^3 + P^2;P^3 + P^4;P^3 \\ &\quad + P^1;P^5 + P^2;P^5 + P^4;P^5 \\ &= P^4 + P^5 + P^1 + P^0 + P^1 + P^3 \\ &= I + P^1 + P^3 + P^4 + P^5 \\ &= Circ(1,1,0,1,1,1).\end{aligned}$$

We may conclude that circulants are closed also under matrix operations. And so we get:

Theorem 2. *The algebra of circulants $\mathscr{C}_n$ is a subalgebra of the algebra $\mathscr{B}_n$, under Boolean and matrix operations.*

Note that $\mathscr{C}_n$, being more specialised, has properties not enjoyed by $\mathscr{B}_n$. For example, unlike $\mathscr{B}_n$, matrix multiplication in $\mathscr{C}_n$ is *commutative*. This is evident from Theorem 1: multiplication of atomic circulants is commutative because of the isomorphism with $\mathbf{Z}_n$, where addition is commutative. And multiplication of non-atomic circulants inherits this commutativity via the canonical forms. So we may call $\mathscr{C}_n$ an *Abelian* subalgebra of $\mathscr{B}_n$. In fact

Theorem 3 (Butler and Krabill [**1**]). *$\mathscr{C}_n$ is a maximal Abelian subalgebra of $\mathscr{C}_n$.*

We shall generalize this result shortly to the context of relation algebras (see Theorem 7).

A close scrutiny of just how circulants inherit commutativity from the P^k's allows us to flesh out Theorem 1 to an isomorphism theorem for $\mathscr{C}_n$. By (3), any circulant C in canonical form corresponds to a subset of $\{P^0, P^1, \dots, P^n\}$, hence, looking just at the exponents, to a subset of $\mathbf{Z}_n$. Multiplication of two circulants A and B then involves multiplication of each P^i in (the canonical form of) A with each P^j in (the canonical form of) B. But, by Theorem 1, this amounts to adding each $\bar{j}$ in the subset of $\mathbf{Z}_n$ corresponding to A to each $\bar{k}$ in the subset of $\mathbf{Z}_n$ corresponding to B. And the same for converses: the converse of a circulant A is obtained by finding the converse of each P^i in its canonical form, which corresponds to finding the negative modulo n of each $\bar{j}$ in that subset of $\mathbf{Z}_n$ which goes with A. And, of course, the identity matrix I goes with the singleton set $\{\bar{0}\}$ contained in $\mathbf{Z}_n$.

A little abstractness at this point will pay off. We define, for any group $\mathscr{G}$, a Boolean algebra with operators over the power set of $\mathscr{G}$ as follows:

Definition 2. For any group $\mathscr{G} = (G, \times, ^{-1}, e)$, its *power algebra* is $\mathscr{P}(\mathscr{G}) = (\mathscr{P}(G), \cup, ^{-}, \otimes, ^{-1}, \{e\})$, where

(i) the power set $\mathscr{P}(G)$ is the set of all subsets of G;

(ii) $\cup$ and $^{-}$ are the set-theoretic operations of union and complementation, respectively, and

(iii) $^{-1}$ and $\otimes$ are the power operations of the operations in G—i.e. for any $A, B \subseteq G$ we have $A^{-1} = \{a^{-1} | a \in A\}$ and $A \otimes B = \{A \times b | a \in A \text{ and } b \in B\}$. (In what follows we indicate both multiplications simply by juxtaposition.)

(Note: in some older textbooks in group theory, such as Macdonald [**9**], any subset of a group is called a *complex*. Accordingly, the power algebra construction is sometimes also referred to as the *complex algebra* construction—for example in Grätzer [**5**].)

$\mathscr{P}(\mathbf{Z}_n)$, the set of all subsets of $\mathbf{Z}_n$, is an example of such a power algebra. The (power) sum of two subsets of $\mathbf{Z}_n$ is the set of all sums of their elements, and the (power) negative of a set is the set of all negatives of its elements. And, as we have just seen, these power operations correspond to matrix operations on circulants.

This observation effectively establishes

Theorem 4. *The algebra of circulants* $\mathscr{C}_n = (C_n, +, ', ;, ^{\smile}, I)$ *is isomorphic to the power algebra* $(\mathscr{P}(\mathbf{Z}_n), \cup, ^{-}, \oplus, -, \{\bar{0}\})$.

Together with Theorem 2 this shows that the power algebra of the group of integers modulo n can be embedded in the algebra of n-square Boolean matrices. (This, too, we shall generalise to the context of relation algebras.) Alternatively, Theorem 4 may be viewed as reducing circulants and their operations to simple manipulations of integers modulo n. For example, since a subset A of a cyclic group is a subgroup iff $AA = A$, we get as a simple corollary to Theorem 4:

Theorem 5 (Butler and Schwarz [2]). *A circulant is idempotent (under matrix multiplication) iff the subset of* $\mathbf{Z}_n$ *to which it correspond is a subgroup of* $\mathbf{Z}_n$.

3. RELATION ALGEBRAS. n-Square Boolean matrices, we have already remarked, form a model of the concept of relation algebra, for which we adopt the definition of Jónsson [7].

Definition 3. A relation algebra is an algebra $\mathscr{A} = (\mathscr{A}_0, ;, ^{\smile}, e)$ such that

(i) $\mathscr{A}_0 = (A, +, 0, \cdot, 1, ')$ is a Boolean algebra.

(ii) $(A, ;, e, ^{\smile})$ is an involuted monoid. That is, for all $a, b, c \in A$:

$$a;(b;c) = (a;b);c$$
$$a;e = a = e;a$$
$$(a;b)^{\smile} = b^{\smile};a^{\smile}$$
$$(a^{\smile})^{\smile} = a.$$

(iii) The operations ; and $^{\smile}$ are, respectively, left-distributive and right-distributive over $+$. That is, for all $a, b, c \in A$:

$$a;(b+c) = a;b + a;c$$
$$(a+b)^{\smile} = a^{\smile} + b^{\smile}.$$

(iv) For all $a, b \in A$: $a^{\smile};(a;b)' \leqslant b'$.

The standard model of a relation algebra is the set $\mathscr{P}(U^2)$ of all binary relations over some set U, endowed with the relational operations of relative product and converse, and the identity relation. [*Note*: The *relative product* of two relations R and S is defined by $R;S = \{(x, y) | (\exists z)[(x, z) \in R \text{ and } (z, y) \in S]\}$. The *converse* of R is $R^{\smile} = \{(y, x) | (x, y) \in R\}$. And of course the *identity relation* is $I = \{(x, y) | x, y \in U \text{ and } x = y\}$.] Following Jónsson [7] we shall use the notation '$\mathscr{R}(\mathscr{U})$' for the standard model. If U is finite, with say n elements, then $\mathscr{R}(\mathscr{U})$ is easily seen to be isomorphic to $\mathscr{B}_n$, the algebra on n-square Boolean matrices. Namely, any relation $R \subseteq U^2$ corresponds to the matrix $[r_{i,j}]$ which has entry '1' in position (i, j) iff $(u_i, u_j) \in R$, where $U = \{u_1, u_2, \ldots, u_n\}$. Boolean operations in $\mathscr{R}(\mathscr{U})$ correspond to Boolean operations in $\mathscr{B}_n$; relative product corresponds to matrix multiplication, converses correspond to transposes and the identity relation corresponds to the identity matrix.

So relation algebras generalise n-square Boolean matrices. What generalises circulants? By Theorem 3, we expect the answer to be: the power algebra of an Abelian group. And indeed, as is well known to relation algebra theorists (and can

easily be checked), the power algebra of any group forms a relation algebra. As promised, we strengthen the analogy by proving two results. The first (Theorem 6) shows how $\mathscr{P}(\mathscr{G})$ can be embedded into $\mathscr{R}(\mathscr{G})$, which generalizes the embedding of $\mathscr{P}(\mathbf{Z}_n)$ into Boolean matrices implicitly given by Theorems 2 and 4. The second result (Theorem 7) shows that if $\mathscr{G}$ is Abelian then $\mathscr{P}(\mathscr{G})$ is a maximal Abelian subalgebra of $\mathscr{R}(\mathscr{G})$, which generalizes Theorem 3.

Theorem 6. *For any group $\mathscr{G} = (G, \times, ^{-1}, e)$, the mapping r defined by*

$$r(A) = \{(x, y) | x, y \in G \text{ and } x^{-1}y \in A\}, \quad \text{for every } A \subseteq G \tag{4}$$

embeds $\mathscr{P}(\mathscr{G})$ into $\mathscr{R}(\mathscr{G})$. (It maps every subset of G onto a relation over G.)

Proof: It is easy to see that $r(\{e\}) = I$, and that $r(A^{-1}) = r(A)^{\smile}$ for any $A \subseteq G$. Moreover, if $(x, y) \in r(AB)$ for any $A, B \subseteq G$ then there is some $a, b \in G$ such that $x^{-1}y = ab$. So $x^{-1}(yb^{-1}) \in A$ and $(yb^{-1})^{-1}y \in B$, hence we have found some $z \in G$ (namely yb^{-1}) such that $x^{-1}z \in A$ and $z^{-1}y \in B$, which means that $(x, z) \in r(A)$ and $(z, y) \in r(B)$, which means that $(x, y) \in r(A); r(B)$. Conversely, if $(x, y) \in R(A); r(B)$ then, for some $z \in G$, $x^{-1}z \in A$ and $z^{-1}y \in B$; hence $x^{-1}y = (x^{-1}z)(z^{-1}y) \in AB$ and so $(x, y) \in r(AB)$. Thus r is a homomorphism. To see that it is also an injection suppose $r(A) = r(B)$ for some $A, B \subseteq G$. Let $a \in A$ arbitrarily, then $(e, a) \in r(A) = r(B)$; hence $e^{-1}a = a \in B$. So $A \subseteq B$, and similarly $B \subseteq A$. □

It is worth noting just how this theorem cashes out in the case where $\mathscr{G} = \mathbf{Z}_n$. To synchronise the numbering, think of the elements of $\mathbf{Z}_n$ as being $z_1 = \overline{0}$, $z_2 = \overline{1}, \ldots, z_n = \overline{n-1}$. Then to any subset $A = \{z_{i_1}, z_{i_2}, \ldots, z_{i_k}\}$ of $\mathbf{Z}_n$ corresponds the relation $\{(z_{i_p}, z_{i_q}) | \overline{n - (i_p - 1) + i_q - 1} \in A\}$, which simplifies to $\{(z_{i_p}, z_{i_q}) | \overline{n - i_p + i_q} \in A\}$. And the Boolean matrix which corresponds to this relation has a '1' in position (i_p, i_q) iff $\overline{n - i_p + i_q} \in A$ (where $1 \leqslant i_p, i_q \leqslant n$). So, scaling Theorem 6 down to $\mathbf{Z}_n$ we get a mapping r which takes any set A of integers modulo n to that Boolean matrix which has a '1' in position (i, j) iff $\overline{n - i + j} \in A (1 \leqslant i, j \leqslant n)$. Definition 1 shows that $r(A)$ is indeed a circulant. And it is not difficult to check that r is inverse to the mapping which maps a circulant in canonical form (3) onto the subset of $\mathbf{Z}_n$ given by the relevant powers of P.

From now on let $\mathscr{G}$ be Abelian. Then those relations $R \subseteq G^2$ which make up the image of $\mathscr{P}(\mathscr{G})$ under r are, in a sense, *generalized circulants*. Fortunately, we can characterise them very simply.

Lemma 1. *A relation $R \subseteq G^2$ is the image $r(A)$ of some set $A \subseteq G$ iff it satisfies the condition*

$$(x, y) \in R \text{ iff } (e, x^{-1}y) \in R, \quad \text{for every } x, y \in G. \tag{5}$$

Proof: Suppose first that $R = r(A)$ for some $A \subseteq G$. Then by, definition of $r, (x, y) \in R$ iff $x^{-1}y \in A$, iff $e^{-1}(x^{-1}y) \in A$ iff $(e, x^{-1}y) \in r(A) = R$ (for any $x, y \in G$). This proves that R satisfies (5). For the converse we consider any relation $R \subseteq G$ and assume that it satisfies (5). Let $s(R) = \{a \in G | \exists x, \ y \in G$ such that $(x, y) \in R$ and $a = x^{-1}y\}$. Then $s : \mathscr{R}(\mathscr{G}) \to \mathscr{P}(\mathscr{G})$ turns any relation over G into a subset of G. We now show that $R = r(s(R))$. Left to right is easy: if

$(u, v) \in R$ then $u^{-1}v \in s(R)$ hence $(u, v) \in r(s(R))$. Right to left is a bit more subtle. Let $(u, v) \in r(s(R))$, then $u^{-1}v \in s(R)$, hence $u^{-1}v = x^{-1}y$ for some $x, y \in G$ such that $(x, y) \in R$. But then $(e, x^{-1}y) \in R$ by (5), hence $(e, u^{-1}v) \in R$ and so by (5) again $(u, v) \in R$. □

Note how (5) corresponds to the defining condition of a circulant in Definition 1. It says that whenever $xa = y$ we have $(e, a) \in R$ iff $(x, y) \in R$, for any $x, y, a \in G$. Moreover, in the case where we are dealing with a singleton subset $\{a\}$ of G, the image relation $r(\{a\})$ is neatly characterised by

$$r(\{a\}) = \{(x, xa) | x \in G\}. \tag{6}$$

Such relations correspond to the atomic circulants P^k $(0 \leqslant k \leqslant n-1)$. It is easy to check that, as with circulants, any image relation $r(A)$ is the union of all such relations $r(\{a\})$, $a \in A$, so that actually we could also define:

$$r(A) = \bigcup \{r(\{a\}) | a \in A\}. \tag{7}$$

This generalizes the canonical form (5) of circulants. Note further that the condition '$x^{-1}y$' used in Theorem 6 to associate a relation with a subset A of a group is a familiar one from group theory: it is the textbook case of associating an equivalence relation with a subgroup. More generally, then, it associates with any subset of a group a generalized circulant; in a sense these relations therefore emerge as a generalisation of equivalence relations.

Theorem 7. *For any Abelian group $\mathscr{G}$, $\mathscr{P}(\mathscr{G})$ is (up to isomorphism) a maximal Abelian subalgebra of $\mathscr{R}(\mathscr{G})$.*

Proof: We have already ascertained that $\mathscr{P}(\mathscr{G})$ is (up to isomorphism) a subalgebra of $\mathscr{R}(\mathscr{G})$, and evidently commutativity of products in $\mathscr{G}$ implies commutativity of complex products in $\mathscr{P}(\mathscr{G})$, so it only remains to check maximality. We do so by showing that any relation $R \in \mathscr{R}(\mathscr{G})$ which commutes with every element of $\mathscr{P}(\mathscr{G})$ must already be an element of $\mathscr{P}(\mathscr{G})$. To show this we invoke Lemma 1. Assume that $R \in \mathscr{R}(\mathscr{G})$ commutes with every element of $\mathscr{P}(\mathscr{G})$. We now show that condition (5) is satisfied; this will complete the proof.

First suppose $(x, y) \in R$. By hypothesis R commutes with $r(\{x\}) = \{(u, ux) | u \in G\}$ (by (6)). Since $(e, x) \in r(\{x\})$ and $(x, y) \in R$ we have $(e, y) \in r(\{x\}); R = R; r(\{x\})$. Hence there exists some $w \in G$ such that $(e, w) \in R$ and $(w, y) \in r(\{x\})$. Then by (6) $y = wx$, so $w = x^{-1}y$, so $(e, x^{-1}, y) \in R$. For the converse, suppose $(e, x^{-1}y) \in R$. Since $(x, e) \in r(\{x^{-1}\})$ we get $(x, x^{-1}y) \in r(\{x^{-1}\}); R = R; r(\{x^{-1}\})$. Hence there exists some w such that $(x, w) \in R$ and $(w, x^{-1}y) \in r(\{x^{-1}\})$. But then by (6) $x^{-1}y = wx^{-1}$, so $x^{-1}y = x^{-1}w$ by commutativity, so $y = w$ and hence $(x, y) \in R$. □

To conclude, here is a question: can we characterise the concept of a circulant element of $\mathscr{R}(\mathscr{G})$ using only Boolean- and relation-algebraic operations? If so, that characterisation could be used in any relation algebra $\mathscr{R}(\mathscr{U})$, and this would yield the general concept of a *circulant relation*. This would be interesting because, for one thing, it would allow also *infinite* circulants—a concept not covered by our present definition.

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In science the credit goes to the man who convinces the world, not to the man to whom the idea first occurs.

—*Sir William Osler*

Construction of Self-Dual Graphs

Brigitte Servatius and Peter R. Christopher

1. INTRODUCTION. Given a planar graph G, we introduce two concepts.

The Geometric Dual of G: Let the plane graph $G = (V, E, F)$ be a planar representation of G, with vertex set V, edge set E and face set F. The *geometric dual* $G^* = (V^*, E^*, F^*)$ is obtained from G as follows: within each face f of G, choose a vertex f^* of G^*; for each edge e separating faces f_i and f_j of G, let e^* be an edge of G^* joining vertices f_i^* and f_j^*. There is a natural one-to-one correspondence between V and F^*, E and E^*, F and V^*. FIGURE 1 shows a graph and its geometric dual. More familiar examples are provided by planar representations of the platonic solids: the dual of the cube and dodecahedron are the octahedron and icosahedron, respectively; the dual of the tetrahedron is itself.

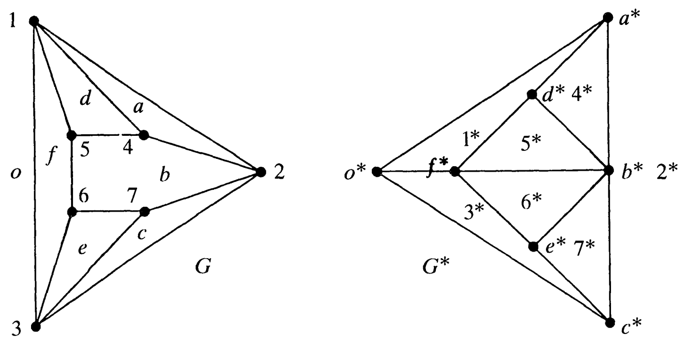

FIG. 1. A graph and its dual.

The Rigidity of G: Since rigidity theory is a relatively new field, we want to provide some intuition first, a good reference is [**3**].

Let $G = (V, E)$ be a planar graph without loops or multiple edges. G can be drawn in the plane such that all edges are straight lines (see Lovasz [**4**]). So we could build a planar model of G by replacing the edges by rigid rods and the vertices by flexible joints. (Cardboard strips and nails work well. For more ideas on construction materials see Baglivo and Graver [**1**]). We can describe a mechanical motion of such a plane structure by giving the position of each vertex as a differentiable function of time such that the distance of two vertices which are joined by an edge is constant. This yields a system of quadratic equations from which we may obtain, through differentiation, a system of linear equations whose solutions are called *infinitesimal motions* of the structure. G is called *rigid* if the infinitesimal motions form a 3-dimensional linear subspace of $\mathbb{R}^{2|V|}$ namely the space generated by horizontal and vertical translation and rotation about the origin. This definition of rigidity is due to Laman [**2**]. The rigidity of a structure depends on the coordinates of the vertices in the plane.

The vertices of a plane structure are in *generic* [3] position if their coordinates are algebraically independent over the rational field. This highly nonmechanical assumption means that the linear dependence of the system of infinitesimal motions depends only on the underlying graph, and consequently rigidity depends on the graph only. A graph G is called *generically rigid* if there is a generic embedding of G in the plane which is rigid. 2-dimensional generic rigidity is characterized in the following

Theorem 1 (Laman's Theorem, 1970). *Let $G = (V, E)$ be a graph. G is generically rigid, if there is a subset F of E such that*

$$|F| = 2|V| + 3, \quad \textit{and} \tag{1}$$

$$|F'| \leqslant 2|\sigma(F')| - 3 \tag{2}$$

holds for all nonempty subsets F' of F, where $\sigma(X)$ denotes the set of endpoints of the edge set X.

We may now *define* G to be rigid if the conditions in Laman's theorem are satisfied.

Equation (1) ensures that G has enough edges to be rigid. The inequalities in (2) ensure that no subset of vertices is overbraced by the edges satisfying (1).

The purpose of this note is to develop a somewhat surprising relationship between these two seemingly unrelated concepts.

2. SELF-DUAL GRAPHS. A graph G is said to be *self-dual* if there is an embedding in the plane such that G is isomorphic to G^*. While the ancient subject of duality has been well-studied, curiously little attention has been given to self-dual graphs. Examples in the literature are sparse. With the procedures that follow we are able to construct self-dual graphs from arbitrary plane graphs.

Adhesion: Let G be a plane graph and v a vertex of G on the face f. Let v^* and f^* be the face and vertex of G^* corresponding to v and f. We form a graph $G \overset{v,f}{\circ} G^*$, the adhesion of G and G^*, by identifying the vertices v and f^*.

FIGURE 2 illustrates the construction, G and G^* are as in FIGURE 1.

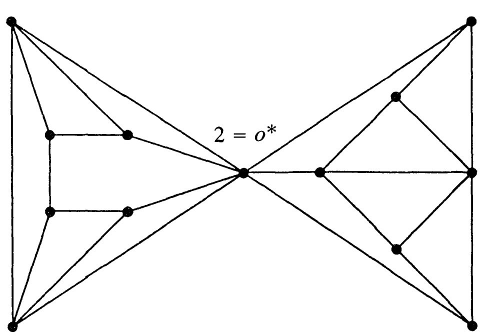

FIG. 2. $G \overset{2,o}{\circ} G^*$.

Theorem 2. *The adhesion of a plane graph G and its dual is a self-dual graph.*

Proof: Let v, v^*, f, f^* be as defined above. Embed G^* with outer face v^* within the face f of G and identify vertices v and f^*. Observe that the union of f and v^*

is a face of the constructed graph. Label it $(f.v^*)$ and label the vertex of attachment $(f.v^*)^*$. All other faces and vertices of $G \overset{v,f}{\circ} G^*$ inherit their labels from G, G^*, respectively. The desired isomorphism between $(G \overset{v,f}{\circ} G^*)^*$ and $G \overset{v,f}{\circ} G^*$ is obtained by mapping vertices of $(G \overset{v,f}{\circ} G^*)^*$ to vertices with the same label in $G \overset{v,f}{\circ} G^*$. $\square$

Adhesion produces self-dual graphs with cut vertices. Furthermore the notion of self-duality here is dependent on the embedding. Since three-connected graphs possess unique embeddings on the sphere, (see Welsh **[6]**), self-duality of such graphs becomes a property independent of the embedding. We are therefore interested in generating 3-connected self-dual graphs. Our next construction produces 3-connected self-dual graphs from sufficiently connected plane graphs.

Explosion: Let G be a plane graph with a face f whose boundary is a cycle. If f^* is the vertex in G^* corresponding to f, form a graph $G \overset{f}{\square} G^*$, the *explosion of* G *in* G^*, as follows: Label the vertices of f consecutively $1, \dots, n$ and let e_i denote the edge connecting i and $i + 1 \pmod n$. In G^*, label the edge corresponding to e_i by e_i^*. Replace the vertex f^* with the graph G by choosing as a new endpoint of each edge e_i^* a vertex labeled j in G such that $i + j \pmod n$ is a constant k.

Theorem 3. *The explosion of a plane graph G in its dual is a self-dual graph.*

Proof: Embed G^* with f^* on its outer face within the face f of G and perform the construction, thereby subdividing the face f. Label these subdivisions as inherited from G^*, and all other faces of the constructed graph as inherited from G. To show that mapping vertices of $(G \overset{f}{\square} G^*)^*$ to vertices with the same label in $G \overset{f}{\square} G^*$ produces an isomorphism, we have to examine if the subgraphs corresponding to G and G^* in $(G \overset{f}{\square} G^*)^*$ are properly connected. Let us consider the face of $G \overset{f}{\square} G^*$ that contains the edges e_{i-1}^* and e_i^* of G^* and the edge e_k of G. It corresponds to a vertex of G labeled i after taking the dual, and an edge labeled e_k^* crosses e_k. Hence, our connection rule $i + j = k \pmod n$ is satisfied. $\square$

The simple examples provided in FIGURE 3 show that explosion reduces to adhesion if the face of G chosen for the construction is a loop, and that explosion, unlike adhesion, is not a symmetric operation. f was not specified since all choices of f are equivalent in these examples.

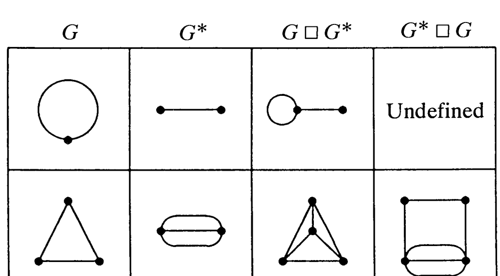

FIG. 3. Exploding small graphs.

FIGURE 4 shows $G \overset{o}{\square} G^*$, where o is determined by the vertex set $\{1, 2, 3\}$ and $k = 2$. The dualization process is indicated by dotted lines.

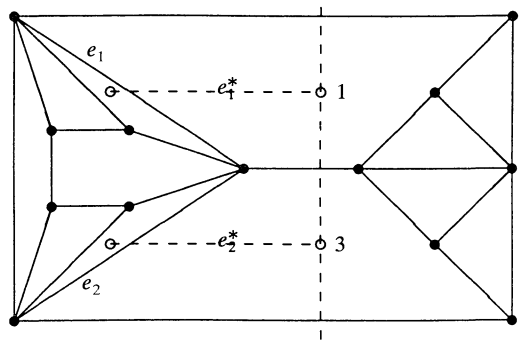

FIG. 4. $G \overset{\circ}{\square} G^*$.

Clearly, if G and G^* are both 3-connected, then so is $G \overset{f}{\square} G^*$.

3. RIGIDITY AND DUALITY. A plane graph G on n vertices with $2n - 2$ edges such that (2) holds for every *proper* subset of E is called a C-graph. Such graphs were first examined by Sugihara **[5]**. A C-graph is rigid, in fact overbraced, and the edges are distributed so that the removal of any edge from G leaves a rigid graph. Observe also, that any graph on n vertices and more than $2n - 3$ edges must, as an immediate consequence of Laman's Theorem, contain a C-graph. We shall refer to the following

Lemma 1. *If $G = (V, E, F)$ is a C-graph, then $|V| = |F|$.*

Proof: Since the Euler characteristic of the sphere is 2, we have $|V| - |E| + |F| = 2$. Moreover $|E| = 2|V| - 2$ holds in a C-graph, and the result follows. $\square$

G and G^* in FIGURE 1 are examples of C-graphs, motivating the following:

Theorem 4. *If G is a C-graph, then its geometric dual, G^*, is also a C-graph.*

Proof: The Lemma implies that G^* has the same number n of vertices and faces as G. If G^* is not a C-graph, we know from the observations above that it must properly contain a C-graph C on $k < n$ vertices and $2k - 2$ edges. The geometric dual of C is a contraction of G obtained by contracting $2(n - k)$ edges of G. Each connected component of the subgraph induced by these $2(n - k)$ edges is contracted to a single vertex. Since these edge sets are proper subsets of E, they satisfy (2). Let $x_1, \dots, x_r$ be the cardinalities of the edge sets of the connected components. The contracted graph has at most $n + r - \sum \frac{1}{2}(x_i + 3) \leqslant k - 1$ vertices, contradicting the fact that C has k faces. $\square$

Wheels are examples of C-graphs which have the additional property of being self-dual. FIGURE 1 illustrates that not all C-graphs have the property of being isomorphic to their geometric duals.

4. MINIMAL SELF-DUAL GRAPHS. Let $G = (V, E, F)$ be a self-dual graph. $|V|$ necessarily equals $|F|$ and the same calculation as in the proof of the Lemma gives $|E| = 2|V| - 2$. By Laman's theorem it is therefore either a C-graph or contains a C-graph H as proper subgraph and G as well as G^* can be contracted to H^*. Moreover, if G is a C-graph it does not contain any self-dual proper subgraphs, since, by (2) any subgraph on $k < |V|$ vertices contains at most $2k - 3$ edges.

A self dual graph is called *minimal* if it does not contain any proper self dual subgraph. Self-dual C-graphs are minimal. The next theorem shows that minimal self dual graphs are not necessarily C-graphs.

Theorem 5. *If G is a C-graph which is not self-dual, then adhesion and explosion be performed on G such that the resulting self-dual graph is minimal.*

Proof: **Adhesion:** $G \overset{v,f}{\circ} G^*$ contains exactly 2 C-graphs, therefore any self-dual subgraph must properly contain at least G or G^* and, since the cut vertex of $G \overset{v,f}{\circ} G^*$ is a fixed point of any automorphism, must equal $G \overset{v,f}{\circ} G^*$.

Explosion: Choose a triangle t bounding a face for the construction, observing that G contains at least 4 such triangles. The deletion of t^* from G^* leaves a rigid graph G', since the removal of an edge with endpoint t^* leaves a rigid graph where t^* is of valence 2, and, by Laman's theorem, the removal of a vertex of valence 2 never alters the rigidity properties of a graph. Since $G \overset{f}{\square} G^*$ is rigid and has only one edge more than required by Laman's Theorem, it contains exactly one C-graph, namely G. Any self-dual subgraph S has to properly contain G and a subgraph isomorphic to G'. S must be rigid, because it contains exactly one C-graph, hence all edges of t^* belong to S. So there is a set of 3 independent edges in S whose removal disconnects S such that one of the resulting components is isomorphic to G'. Since at least 4 independent edges are necessary to separate a C-graph, S must equal $G \overset{f}{\square} G^*$. $\square$

Observe that we have shown that the adhesion of G and G^* in the last theorem yields a minimal self-dual graph for any choice of vertex and face incident with it. We chose the weaker statement of the theorem, since we were unable to find a simple proof of an equally general result for explosion.

The following questions remain unanswered: Does one obtain all self-dual graphs from C-graphs through a sequence of adhesions and explosions? Can we construct all *minimal* self-dual graphs? Is there a relationship between self-dual graphs embedded on surfaces other than the plane, or equivalently the sphere, and rigidity in higher dimensions?

ACKNOWLEDGMENTS. The idea for this paper originated in an NSF sponsored research experience program for undergraduates at WPI in the summer of 1988, grant No. DMS-8804212, with Patricia Berkebile, Ari Juels, and Amy Oudaise as student participants.

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Teacher's Gift

Confined you are, have always been,
by bonds unfelt: by bars unseen.
But not so I, I soar on wings
of thought. And thinking, dream these things:
two worlds made one yet ever two
apart; a labyrinth traced clear through
from end to end; a tone more pure
than Circe's voice; a keep secure
from even time's travail; a bright-
ness that confers the pain of sight
so keen it pierces to the heart.
To this and more I am conveyed.
Come, break those chains. Take up the blade
by Euclid forged, and polished since
by ev'ry soul who saw its glint
in reason's fire, and passed from hand
to hand down all the age of man
until at last here now we two.
Hold out your hand. I give it you.
Your fetters can't withstand its aim.
Here. Mathematics is its name.

—*Dan Kalman*

On Functions of Bounded Variation in Higher Dimensions

Pawel Gora and Abraham Boyarsky

Among the most important properties of a function of bounded variation f in one dimension are the boundedness of f and the fact that the support of f, $\operatorname{supp} f = \{x\colon f(x) \neq 0\}$, is a union of intervals and, therefore, has interior [**1**]. It was with surprise that the authors learned that in higher dimensions, functions of bounded variation may have neither of these properties.

The modern definition of variation is given in a distributional sense and can be found in [**2**]:

$$V(f) = \int_{R^N} \|Df\| = \sup\left\{\int_{R^N} f \operatorname{div}(g)\, d\lambda_N\colon g = (g_1, \dots, g_N) \in C_0^1(R^N, R^N)\right.$$
$$\left.\text{and } |g(x)| \leqslant 1 \text{ for } x \in R^N\right\}, \quad (1)$$

where $f \in L_1(R^N)$ has bounded support, Df denotes the gradient of f in the distributional sense, $C_0^1(R^N, R^N)$ is the space of continuously differentiable functions from R^N into R^N which vanish at ∞, and λ_N is Lebesgue measure on R^N. For example, if $f = \chi_A$ is the characteristic function of a set A having piecewise C^2 boundary, ∂A, then [**2**]: $V(f) = \lambda_{N-1}(\partial A)$. In two dimensions, (1) reduces to the Tonnelli definition of variation [**2**]:

$$V(f) = \max\left\{\int_R V_x f\, dy, \int_R V_y f\, dx\right\},$$

where V_x denotes one-dimensional variation in the x-direction and analogously for V_y [**2**].

We shall now construct a function on the unit square $S \subset R^2$ which is of bounded variation and whose support has no interior. The example is based on Ex. 1.10 in [**2**].

Let $\{x_i\}$ be the sequence of all rational points in S and let

$$E = \bigcup_{i=0}^{\infty} B(x_i, \varepsilon/2^i),$$

where $B(x, \delta)$ is a ball centered at x with radius δ. Then

$$\lambda_2(E) \leqslant \varepsilon^2 \sum_{i=0}^{\infty} \pi/2^{2i} = 4\pi\varepsilon^2/3.$$

We choose ε small enough so that $\lambda_2(E) < 1/2$.

Let $F = S - E$ and consider $f = \chi_F$. Since $V(\chi_F)$ is the perimeter of F,

$$V(f) \leqslant 2\pi\varepsilon \sum_{i=0}^{\infty} 1/2^i = 4\pi\varepsilon.$$

Hence f is of bounded variation, but $\operatorname{supp} f = \{x: \chi_F > 0\}$ has no interior.

Now let $f: S \to R$ be given by $f(x, y) = 1/(\sqrt{x} + \sqrt{y})$. Clearly f is unbounded on S. Since

$$\int_0^1 V_x f\, dy = \int_0^1 \left(1/\sqrt{y} - 1/\left(1 + \sqrt{y}\right)\right) dy < \infty$$

it follows by symmetry that f is of bounded variation.

Note that f^2 is not of bounded variation. Hence, unlike the one dimensional case, the product of two functions of bounded variation is not necessarily of bounded variation.

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PROBLEMS AND SOLUTIONS

Edited by:
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PROBLEMS

10193. *Proposed by Solomon Golomb, University of Southern California, Los Angeles, CA.*

Determine all pairs of integers n, k such that

$$\binom{n}{k} = \binom{n+1}{k-1}, \qquad n > k > 1.$$

10194. *Proposed by Jiro Fukuta, Gifu-ken, Japan.*

(a) For any four-digit number x in base 12, *excluding the eleven numbers with all digits equal*, form the number $A = a_1a_2a_3a_4$ obtained by arranging the four digits in descending order of magnitude. Next form the number $B = a_3a_4a_1a_2$ obtained by exchanging the first two with the last two digits. Put $K(x) = A - B$ and $K^{i+1}(x) = K(K^i(x))$ for $i = 1, 2, \dots$. Prove that $K^i(x) = 4378$ if $i \geqslant 5$.

(b) Generalize to the base $3 \cdot 2^n (n = 0, 1, 2, \dots)$.

10195. *Proposed by Andrew Granville, University of Georgia, Athens, GA.*

For $m \geqslant k \geqslant 1$, define numbers $b(k,m)$ by

$$b(1,m) = 1 \text{ for } m \geqslant 1,$$

$$b(k+1,m) = \sum_{j=k}^{m-1} b(k,j)\left(\frac{1}{j} + \frac{1}{m-j}\right) \quad \text{for } m \geqslant k+1 \geqslant 2.$$

For example, $b(2,m)$ is twice the $(m-1)$th partial sum of the harmonic series (for $m \geqslant 2$).

(a) Prove that

$$\sum_{k=1}^{m} \frac{(-1)^k}{k!} b(k,m) = 0 \quad \text{for } m \geqslant 2.$$

(b) Prove that $(m-1)!\, b(k,m) = k!\ \sigma(m,k)$, where $\sigma(m,k)$ is the unsigned Stirling number of the first kind.

10196. *Proposed by Barry Hayes, Stanford University, Stanford, CA, and David S. Pearson, Cornell University, Ithaca, NY.*

Let M_n be the set of n-bit binary strings containing no pairs of consecutive ones. For example,

$$M_3 = \{(0,0,0),(0,0,1),(0,1,0),(1,0,0),(1,0,1)\}.$$

Find the probability p_n that if $(\delta_1, \delta_2, \ldots, \delta_n)$ and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ are in M_n, then

$$(\max\{\delta_1, \varepsilon_1\}, \max\{\delta_2, \varepsilon_2\}, \ldots, \max\{\delta_n, \varepsilon_n\})$$

is in M_n.

10197. *Proposed by Uri Peled, University of Illinois at Chicago, Chicago, IL.*

Light bulbs $L_1, L_2, \ldots, L_n$ are controlled by switches $S_1, S_2, \ldots, S_n$. Switch S_i changes the on/off status of light L_i and possibly the status of some other lights. Assume that if S_i changes the status of light L_j, then S_j changes the status of light L_i. Initially all the lights are off. Prove that it is possible to operate the switches in such a way that all the lights are on.

10198. *Proposed by David M. Bloom, Brooklyn College of CUNY, Brooklyn, NY.*

Suppose f is a continuous map of $[0,1]$ onto a circle. Prove that there exist two closed subintervals of $[0,1]$ intersecting in at most one point whose images under f are complementary semicircles (i.e., semicircles intersecting only at their endpoints).

10199. *Proposed by Richard Stanley, Massachusetts Institute of Technology, Cambridge, MA.*

Given a finite partially ordered set P, let $f(P)$ denote the number of ways to partition the elements of P into pairwise disjoint nonempty saturated chains.

(a) Prove that if P_n is the product of two n-element chains, i.e., if $P_n = \{(i,j)\colon 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n\}$, with $(i,j) \leqslant (i',j')$ if and only if $i \leqslant i'$ and $j \leqslant j'$, then $f(P_n) = \prod_{j=1}^{n} F_{2j}^2$, where F_k is the kth Fibonacci number.

(b) If every element of P covers at most two elements and is covered by at most two elements, prove that $f(P)$ factors into Fibonacci and Lucas numbers.

10200. *Proposed by Daniel Goffinet, St. Etienne, France.*

(a) Prove that a (square) matrix over a field F is singular if and only if it is a product of nilpotent matrices.

(b) If $F = \mathbb{C}$, prove that the number of nilpotent factors can be bounded independently of the size of the matrix.

10201. *Proposed by Gunnar Blom, University of Lund and Lund Institute of Technology, Lund, Sweden.*

An urn contains one white ball and one black ball. Draw a ball at random. With probability $1/2$ return it to the urn; otherwise (again with probability $1/2$) put a ball of the opposite color in the urn. Perform n such drawings in succession. Find the mean and variance of the number X_n of white balls appearing in the n drawings. Find the limiting distribution of $n^{-1/2}(X_n - E(X_n))$.

NOTES

(10194) A similarity with problem E2222 [1970, 307; 1971, 197] has been observed. (10195) The "unsigned Stirling number of the first kind", $\sigma(m,k)$, is defined as the number of permutations of m symbols which have exactly k cycles. Riordan, "An Introduction to Combinatorial Analysis", may be used as a reference for known recurrences and generating functions of these numbers. The quantities $b(k,m)$ are related to the "generalized harmonic numbers" considered by Yuri Matiyasevich in the January 1992 issue of this *Monthly*, pp. 74–75. (10196) Some sample values of p_n are, $p_2 = 7/9$ and $p_3 = 19/25$. (10199) A chain $x_1 < x_2 < \cdots < x_k$ is *saturated* if x_i covers x_{i-1} in P for $i = 2, 3, \ldots, k$. Hence, if P_n is an n-element chain, then $f(P) = 2^{n-1}$. The Fibonacci numbers are defined by $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n > 2$; the Lucas numbers are defined by $L_1 = 1$, $L_2 = 3$, $L_n = L_{n-1} + L_{n-2}$ for $n > 2$. (10200) Discussion of algorithms for performing this factorization, or bounds on the number of factors for fields other than $\mathbb{C}$, would be welcome supplements to solutions of the problems stated here.

SOLUTIONS

Writing Integers with Exactly Three Fours

E 3363 [1990, 63]. *Proposed by N. J. Fine, Deerfield Beach, FL.*

A printer is given three copies of the numeral 4 and an unlimited supply of each of the six symbols

$$+, \cdot, -, \div, \sqrt{}, \lfloor\ \rfloor,$$

as well as an unlimited supply of left and right parentheses. (Here $\lfloor t \rfloor$ denotes the

greatest integer not exceeding the real number t.) Show that he can compose an expression for each positive integer N. For example,

$$19 = \left\lfloor \sqrt{4 \div (\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{4}}}}}} - \lfloor\sqrt{\sqrt{4}}\rfloor)} \right\rfloor.$$

Solution by Marcin E. Kuczma, University of Warsaw, Warsaw, Poland. The example suggests the general method. Given nonnegative integers m, n, let $F(m,n) = (4/(4^{1/2^m} - 1))^{1/2^n}$. By writing 1 as $\lfloor\sqrt{\sqrt{4}}\rfloor$, an expression for $F(m,n)$ can be composed. Since $t < 4^t - 1 < 4t$ for $t \in (0,1]$, we obtain $2^m < 4/(4^{1/2^m} - 1) < 2^{m+2}$. Hence $m2^{-n} < \log_2 F(m,n) < (m+2)2^{-n}$, from which it follows that the range of F is dense in $(1,\infty)$. This implies that $\lfloor F(m,n)\rfloor = N$ infinitely often for any positive integer N. (Any integer exceeding 1 can be used in place of 4.)

Editorial comment. Related material can be found in J. H. Conway and M. J. T. Guy, "π in four 4's," Eureka 25 (1962), 18–19, and in M. Bicknell and V. E. Hoggatt, "64 ways to write 64 using four 4's," *Recreational Mathematics Magazine* 14 (1964), 13–15.

Solved also by F. Brulois, S. Gaignoux (France), R. J. Hendel, M. Hildebrand, E. Levine, H. Lipman, O. P. Lossers (The Netherlands), R. Martin, L. E. Mattics, J. G. Merickel, T. S. Norfolk, R. E. Prather, R. Stong, J. S. Sumner, A. Zulauf (New Zealand), and the proposer.

A Consequence of the Continuity of the Composition

E 3379 [1990, 342]. *Proposed by Hugh Thurston, University of British Columbia, Vancouver, BC, Canada.*

Suppose g and f are real-valued functions defined on subsets of $\mathbb{R}$ such that:
(i) the domain of g is an interval, I.
(ii) g is continuous on I.
(iii) the domain of f contains the range of g.
Is it true that if $f \circ g$ is continuous on I, then f is continuous on the range of g?

Solution by Dale Varberg, St. Paul, MN. The answer is "Yes". Suppose that f is discontinuous at some point $g(z)$ in the range of g. Then there exists $\varepsilon > 0$ and a monotone sequence $\{g(x_n)\}$ of distinct values such that $g(x_n) \to g(z)$, but $|f(g(x_n)) - f(g(z))| \geqslant \varepsilon$. Let J be the closed interval with endpoints x_1 and z. By the Intermediate Value Theorem, there are points x'_n in J such that $g(x'_n) = g(x_n)$. The sequence $\{x'_n\}$ is bounded, so it has a convergent subsequence $\{x'_{n_k}\}$, converging to some value z'. This implies $g(x'_{n_k}) \to g(z')$, but also we have $g(x'_{n_k}) = g(x_{n_k}) \to g(z)$, so $g(z') = g(z)$. We now have $|f(g(x'_{n_k})) - f(g(z'))| = |f(g(x_{n_k})) - f(g(z))| \geqslant \varepsilon$. However, since $x'_{n_k} \to z'$, this contradicts the continuity of $f \circ g$.

Editorial comment. As several solvers pointed out, there are two possible interpretations of the conclusion that "f is continuous on the range of g." The interpretation intended by the proposer, the interpretation assumed by most solvers, and the interpretation made in the solution given above is that the function obtained by restricting f to the range of g is continuous. An alternative but very reasonable interpretation is that if Y is the range of g, then the given function f is continuous at each point of Y. With this alternative interpretation, the conclusion can fail at boundary points of Y contained in Y. For example, if

$Y = (0, 1]$ and $f(x) = \lceil x \rceil$, then $(f \circ g)(x) = 1$ for all x in I but f is discontinuous at 1. (An instance of this would be $g(x) = 1/(1 + x^2)$, $I = (-\infty, \infty)$.)

Three solvers obtained generalizations of the following form: If X, Y, Z are topological spaces, g maps X continuously onto Y, f maps Y into Z, and $f \circ g$ is continuous, then under certain conditions f must be continuous. The generalizers and their conditions are as follows: 1) Frédéric Brulois: Y is a subset of $\mathbb{R}$, and X is connected and locally connected. 2) Sam B. Nadler, Jr.: Y is a subset of $\mathbb{R}$, and every two points of X are contained in a compact connected subspace of X. 3) Peter Wakker: X is connected, Y is arcwise connected, and $Z = R$. Wakker's results appear in a forthcoming paper in the *Journal of Mathematical Analysis and Applications*.

Solved also by D. W. Bailey, L. Blaine, F. Brulois, H. Chen (student), A. del Rio (Spain), B. Elkins (alternative interpretation only), W. Hensgen (Germany), E. A. Herman, C. Hill, Y. Ionin, I. E. Leonard & J. E. Lewis (Canada), J. G. Merickel, M. D. Meyerson, S. B. Nadler, Jr., A Pedersen (Denmark), E. R. Pujals & J. P. Bes (Argentina), B. Richmond, A. Riese, K. Schilling, I. Szalkai, P. Wakker (The Netherlands), Western Maryland College Problems Group, and the proposer. Partially solved by W. J. Bühler (along with alternative interpretation) and E. Swenson. Two incorrect solutions were received.

How to Produce a "Harmonic Convergence"

E 3381 [1990, 342]. *Proposed by David Gurarie, Case Western Reserve University, Cleveland, OH.*

Suppose a is a fixed real number greater than 1. For positive integral j, let $\log^{(j)}$ denote the jth iterate of the logarithmic function to the base a. For example,

$$\log^{(2)} x = \log_a(\log_a x).$$

If $1 \leqslant k < a$, put $d_k = 0$. If $k \geqslant a$, define d_k by

$$\log^{(d_k)} k \geqslant 1, \quad \log^{(d_k+1)} k < 1.$$

Does

$$\sum_{k=1}^{\infty} \left\{ k \prod_{j=1}^{d_k} \log^{(j)} k \right\}^{-1}$$

converge?

Composite solution by Martin Goldstern, Vienna, Austria, and Reiner Martin (student), University of California, Los Angeles. When $1 < a \leqslant e^{1/e}$, the numbers d_k do not exist for all k, so the problem is not well defined. For $e^{1/e} < a < e$, the sequence converges. For $e \leqslant a$, the sequence diverges. We use log for $\log_a$.

Let $f(x) = \ln x/x$. The equation $\log x = x$ or $f(x) = \ln a$ has a solution if and only if $a \leqslant e^{1/e}$, since differentiation yields a global maximum of $1/e$ for f at $x = e$ and $\lim_{x \to \infty} f(x) = 0+$. Furthermore, there is a unique solution y with $y \geqslant e$. Since $k \geqslant y$ implies $\log^{(n)} k \geqslant \log^{(n)} y = y > 1$ for all n, the value d_k does not exist for these k when $a \leqslant e^{1/e}$.

For $a > e^{1/e}$, the equation $\ln x/x = \ln a$ has no solution and so $\log x = \ln x/\ln a < x$ for all positive x. Moreover, the definition of d_k is valid for all real $k \geqslant 1$, and the convergence of the sum is equivalent to the convergence of the integral

$$\int_1^{\infty} \frac{1}{t \prod_{j=1}^{d_t} \log^{(j)} t} \, dt.$$

Define the sequence N_n recursively by $N_0 = 1$ and $N_{n+1} = a^{N_n}$. If $a > e^{1/e}$, then $N_n \to \infty$. This follows from the fact that for any $k, \log^{(n)} k$ is eventually less than 1, since otherwise log(base a) has a fixed point, which happens only for $a \leqslant e^{1/e}$. Now $d_k = m$ where $N_m \leqslant k \leqslant N_{m+1}$.

Thus the integral equals the sum over $n \geqslant 0$ of

$$A_n = \int_{N_n}^{N_{n+1}} \frac{1}{t \prod_{j=1}^{d_t} \log^{(j)} t} \, dt.$$

If we replace t by a^t in this integral, we obtain $A_n = (\ln a) A_{n-1}$, so $\Sigma A_n = A_0 \Sigma_{n=0}^{\infty} (\ln a)^n$, which is well-known to converge if and only if $\ln a < 1$, i.e. $a < e$.

Editorial comment. E. M. Reingold noted that this problem has already been solved in the literature. Essentially the same argument appears in "Some equivalences between Shannon entropy and Kolmogorov complexity," by S. K. Leung-Yan-Cheong and T. M. Cover, *IEEE Trans. Info. Th.* 24 (1978), 331–338 (see inequality B26). Also of interest in this regard are Appendix A of J. Rissanen's "A universal prior for integers and estimation by minimum description length," *Ann. Stat.* 11 (1983), 416–431, and Lemma 21 in R. Beigel's "Unbounded searching algorithms," *SIAM J. Comput.* 19 (1990), 522–537.

The convergence/divergence question depends critically on the precise style in which the logarithm is used. Define $L(n) = \lfloor \lg n \rfloor + \lfloor \lg \lg n \rfloor + \lfloor \lg \lg n \rfloor + \cdots$, with the sum stopping when the values cease to be positive. Then $\Sigma_{n \geqslant 1} 2^{-L(n)}$ diverges, albeit extremely slowly. See J. L. Bentley and A. C.-C. Yao's "An almost optimal algorithm for unbounded searching," *Info. Proc. Lett.* 5 (1976), 82–87. (The closely related results attributed in this paper to Chung and Graham are incorrect.) Also see D. E. Knuth's "Supernatural numbers" in *The Mathematical Gardner*, edited by D. A. Klarner (Wadsworth, 1981), 310–325. X. Shen and E. M. Reingold, in their work on unbounded searching, have found extensions to this sum that are much more slowly converging/diverging; see "More nearly optimal algorithms for unbounded searching." *SIAM J. Comput.* 20 (1991), 156–208.

Finally, P. Erdös supplied a reference to an article by R. P. Agnew in this *Monthly* 54 (1947), 273, in which this problem is solved for the case $a = e$.

Solved also by H. Chen, M. Getz, C. Hill, J. G. Merickel (student), T. S. Norfolk, A. Pedersen (Denmark), J. H. Steelman, and D. B. Tyler. Four incorrect solutions asserting divergence for all $a > 1$ were received.

Identifying a Factorial

6633 [1990, 433]. *Proposed by Horacio Porta, University of Illinois at Urbana-Champaign.*

Suppose we are given a large positive number N and the further information that $N = k!$ for some positive integer k. Show that we can determine k in at most $C \log \log \log N$ steps.

Solution by Harold G. Diamond, University of Illinois, Urbana, IL. Let $y = \log N$. We shall determine the integer k satisfying $\log \Gamma(k+1) = y$ by the following sequence of three steps. (A) Approximate $k+1$ from below with the aid of Stirling's formula. (B) Find further approximations to $k+1$ by Newton's method. (C) Quit when $k+1$ is determined to within 1. We show that this can be done in $O(\log \log \log N)$ steps. To achieve reasonable constants in our estimates we assume that $N \geqslant (50!)$. Note that $\log \log \log 50! = 1.609$.

For (A) we use Stirling's estimate in the form

$$0 \leqslant \log \Gamma(x) - \left\{\left(x - \frac{1}{2}\right)\log x - x + \log\sqrt{2\pi}\right\} \leqslant \frac{1}{12x},$$

valid for $x \geqslant 1$. We start the iteration at

$$x_0 = \frac{y}{\log y}\left(1 + \frac{1 + \log\log y}{\log y}\right).$$

The following lemma will be used to show that $\log \Gamma(x_0) < y$, and that $\log \Gamma(x_0)$ is close to y.

Lemma. *Let* $g(x) = x\log x - x$ *and* $L = \log y$. *Then*

$$0 \leqslant \frac{g(x_0)}{y} - \{1 - L^{-2}\log L(1 + \log L)\} \leqslant L^{-3}(1 + \log L)^2.$$

Proof: We have

$$\frac{g(x_0)}{y} = \left\{1 + \frac{1 + \log L}{L}\right\}\left\{1 - \frac{1 + \log L}{L} + L^{-1}\log\left(1 + \frac{1 + \log L}{L}\right)\right\}.$$

Since $L^{-1}(1 + \log L) > 0$ the result follows from

$$\varepsilon - \frac{\varepsilon^2}{2} \leqslant \log(1 + \varepsilon) \leqslant \varepsilon.$$

(We omit some details; there is nothing difficult in principle.)

In particular, for $N \geqslant 50!$ we have $y > 148.4777$ and $\log y > 5$. Thus $g(x_0) < y$ and $x_0 > 45$. By our Stirling estimate

$$\log \Gamma(x_0) < g(x_0) + \log\sqrt{2\pi} - \frac{1}{2}\log x_0 + \frac{1}{12 \cdot 45} < y$$

and x_0 is a lower bound for $k + 1$.

For (B) apply Newton's method to the equation

$$f(x) = \log \Gamma(x) - y = 0.$$

This gives the recurrence

$$x_{n+1} = x_n + \{y - \log \Gamma(x_n)\}/\{(\Gamma'/\Gamma)(x_n)\}.$$

By Taylor's formula

$$0 = f(k+1) = f(x_n) + (k + 1 - x_n)f'(x_n) + \frac{(k + 1 - x_n)^2}{2}f''(x_n^*)$$

where x_n^* is between x_n and $k + 1$. Together these two equations yield

$$x_{n+1} - (k+1) = \frac{(k + 1 - x_n)^2}{2}\,\frac{f''(x_n^*)}{f'(x_n)}.$$

Note that $x_{n+1} > k + 1$ since f' and f'' are strictly positive.

From the gamma function identity

$$(\Gamma'/\Gamma)'(x) = \sum_{v=0}^{\infty}(v + x)^{-2}$$

and the convexity inequality $u^{-2} < \int_{u-1/2}^{u+1/2} t^{-2}\,dt$ we obtain

$$(\Gamma'/\Gamma)'(x) < \int_{x-1/2}^{\infty} t^{-2}\,dt = \left(x - \frac{1}{2}\right)^{-1}, x > \frac{1}{2}.$$

From this estimate it follows that

$$(\Gamma'/\Gamma)(x) - \log\left(x - \frac{1}{2}\right) = -\int_x^{\infty} \left\{(\Gamma'/\Gamma)'(t) - \frac{1}{t - \frac{1}{2}}\right\} dt > 0$$

since

$$(\Gamma'/\Gamma)(x) - \log x \to 0 \quad \text{as } x \to \infty.$$

The above elementary estimates on the first and second logarithmic derivatives of the gamma function yield

$$x_{n+1} - k - 1 < \frac{(x_n - k - 1)^2}{2(x_n^* - \frac{1}{2})\log(x_n - \frac{1}{2})}.$$

For $n \geqslant 1$ we have $x_n, x_n^* > k + 1$, so

$$\left(x_n^* - \frac{1}{2}\right)\log\left(x_n - \frac{1}{2}\right) > \left(k + \frac{1}{2}\right)\log\left(k + \frac{1}{2}\right).$$

By the definition of y and the Stirling upper bound,

$$\begin{aligned} y = \log \Gamma(k+1) &< \left(k + \frac{1}{2}\right)\log(k+1) - k \\ &< \left(x_n^* - \frac{1}{2}\right)\log\left(x_n - \frac{1}{2}\right), \end{aligned}$$

so for $n \geqslant 1$ we have

$$x_{n+1} - k - 1 < \frac{(x_n - k - 1)^2}{2y}.$$

For (C) it is enough to find an integer $n \ll \log\log\log N$ such that $x_n < k + 2$, since $k + 1 < x_n$ for $n \geqslant 1$. The Stirling lower bound and the lower bound of the lemma give

$$\begin{aligned} x_1 - x_0 = \frac{y - \log\Gamma(x_0)}{(\Gamma'/\Gamma)(x_0)} &\leqslant \frac{y - g(x_0) + \frac{1}{2}\log x_0 - \log\sqrt{2\pi}}{\log(x_0 - \frac{1}{2})} \\ &\leqslant \frac{y \log L(1 + \log L)}{L^2(L - \log L)} + \frac{1}{2}. \end{aligned}$$

For $L \geqslant 5$ the factor of $(\log L)(1 + \log L)/(L - \log L)$ in the first term is bounded by 1.239 (it is decreasing). Hence

$$x_1 - x_0 < 1.24yL^{-2} + \frac{1}{2} < 1.33yL^{-2}$$

and $(x_0 < k + 1 < x_1)$

$$x_1 - k - 1 < 2yL^{-2}.$$

From this inequality and the last inequality of (B) we have

$$x_n - k - 1 < 2y/L^{2^n}, \quad n \geqslant 1.$$

If we take $n \geqslant (\log\log y)/\log 2$ then $x_n - k - 1 < 1$ and we are done. Thus it suffices to use

$$\lceil (\log\log\log n)/\log 2 \rceil$$

steps (after the choice of x_0) to solve $k! = N$.

Editorial comment. For a more precise discussion of what is meant by a "step" see the study of computational complexity in Chapter 6 of J. and P. Borwein, *Pi and the AGM*, John Wiley and Sons, New York, 1987.

No other solutions were received.

The Area of a Pedal of a Pedal Triangle

E 3392 [1990, 528]. *Proposed by Antal Bege, Miercurea-Ciuc, Romania.*

Given an acute-angled triangle ABC with orthocenter H, let A_1, B_1, C_1 be the feet of the altitudes from A, B, C, respectively, and let A_2, B_2, C_2 be the feet of the perpendiculars from H onto B_1C_1, C_1A_1, A_1B_1, respectively. Prove that

$$\text{area}(\Delta ABC) \geqslant 16\text{area}(\Delta A_2B_2C_2)$$

and determine when equality holds.

Solution I by Ilias Kastanas, California State University, Los Angeles, CA. By using the fact that $A_2C_1B_2H$ is a cyclic quadrilateral and applying the reflection property of the orthic triangle, we have $\angle A_2B_2C_1 = \angle A_2HC_1 = \angle B_1C_1A = \angle B_2C_1B$. Thus A_2B_2 is parallel to AB, and similarly for B_2C_2 and C_2A_2, so the sides of $A_2B_2C_2$ are parallel to those of ABC.

Let K, K_1, K_2 be the circumcircles of $ABC, A_1B_1C_1, A_2B_2C_2$, of respective radii R, R_1, R_2. Then K_1 is the Euler circle of ABC, that is, the circle passing through the midpoints of the sides, and so $R_1 = R/2$. Now K_2 is the incircle of $A_1B_1C_1$; hence R_2 is at most the radius of the Euler circle of $A_1B_1C_1$, which in turn is $R_1/2 = R/4$. Thus $R_2 \leqslant R/4$, and by the similarity of the triangles, area$(ABC) \geqslant 16$area$(A_2B_2C_2)$.

Equality holds when the Euler circle of $A_1B_1C_1$ coincides with its incircle. For this to happen, $A_1B_1C_1$ must be equilateral, and hence ABC must be equilateral as well.

Solution II and generalization by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada. More generally, the pedal triangle of a triangle ABC with respect to a point P is the triangle whose vertices A_1, B_1, C_1 are the feet of the perpendiculars from P onto the sides of ABC. For P lying within or on ABC, it is known [**2**, p. 139] that

$$[A_1B_1C_1] = [ABC](1 - OP^2/R^2)/4 \leqslant [ABC]/4,$$

where [ ] denotes area and O, R are, respectively, the circumcenter and circumradius of ABC. There is equality if and only if P coincides with O (and this requires that ABC be non-obtuse). Then if $A_2B_2C_2$ is the pedal triangle of $A_1B_1C_1$ with

respect to P,

$$[A_2B_2C_2] \leqslant [A_1B_1C_1]/4 \leqslant [ABC]/16.$$

For equality in both places here, P must be the circumcenter of both ABC and $A_1B_1C_1$. This requires that ABC is equilateral.

Solution III and generalization independently by Arvind Subramanian (student), D. G. Ruparel College, Bombay, India, and O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. We use the following lemma [**3**, p. 342] (notation $[\cdot]$ for area as above), which is not difficult to prove: "If D, E, F are points on the sides BC, CA, AB of a triangle ABC such that AD, BE, CF are concurrent at an interior point of triangle ABC, then $[ABC] \geqslant 4[DEF]$, with equality if and only if AD, BE, CF meet at the centroid of ABC."

In the acute-angled triangle ABC, the altitudes AA_1, BB_1, CC_1 meet at the interior point H, which is the center of the inscribed circle of $A_1B_1C_1$. Hence A_2, B_2, C_2 are the points of contact of this circle with the sides of $A_1B_1C_1$. By [**2**, p. 184] A_1A_2, B_1B_2, C_1C_2 meet at a point inside $A_1B_1C_1$, the so-called Gergonne point of triangle $A_1B_1C_1$. Hence, by the lemma,

$$[ABC] \geqslant 4[A_1B_1C_1] \geqslant 16[A_2B_2C_2].$$

Equality holds if and only if H is the centroid of ABC; i.e., if and only if ABC is equilateral.

Editorial comment. Many solvers used analytic and other means to establish the relation $[A_2B_2C_2] = 4(\cos A \cos B \cos C)^2[ABC]$. This is a consequence of $R_2 = 2R \cos A \cos B \cos C$ (where R_2 is the circumradius of $[A_2B_2C_2]$ ([**2**, p. 191]) and the similarity of $A_2B_2C_2$ and ABC. The required inequality then follows from the easy inequality $\cos A \cos B \cos C \leqslant 1/8$.

Walther Janous suggests §1.9 of [1] as a good reference for properties of iterated pedal triangles. He also points out the related inequality $[ABC]^5 \geqslant R^8(27/4)^2[A_2B_2C_2]$ that can be obtained from inequalities found in [**3**, p. 271].

1. H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, New Mathematical Library, vol. 19 Math. Assoc. Amer., 1967.
2. R. A. Johnson, *Modern Geometry* Houghton Mifflin, 1929.
3. D. S. Mitrinović, J. E. Pečarić, and V. Volenec, *Recent Advances in Geometric Inequalities* Kluwer, 1989.

Also solved by C. Athanasiadis, J. Anglesio (France), L. Bilir & S. Demir (Turkey), R. J. Chapman (United Kingdom), J. Fukuta (Japan), J. Garfunkel, H. Guggenheimer, J. Heuver (Canada), W. Janous (Austria), H. Kappus (Switzerland), L. Kuipers (Switzerland), E. Lee, G. J. Masjuán (Chile), G. Nagy (Hungary), I. Sadoveanu, I. A. Sakmar (Turkey), V. Schindler (Germany), R. A. Simon (Chile), R. S. Tiberio, M. Vowe (Switzerland), R. L. Young, Central Michigan University Problem Group, and the proposer.

Polynomials in Computer-Aided Geometric Design

E 3400 [1990, 612]. *Proposed by Burt J. Totaro, Mathematical Sciences Research Institute, Berkeley, CA.*

Let S be the boundary of the unit square $[0, 1] \times [0, 1]$ in $\mathbb{R}^2$. Suppose f is a continuous real-valued function on S such that $f(x, 0)$ and $f(x, 1)$ are polynomial functions of x on $[0, 1]$ and such that $f(0, y)$ and $f(1, y)$ are polynomial functions

of y on $[0,1]$. Prove that f is the restriction to S of a polynomial function of x and y.

Solution by Michael Golomb, Purdue University, West Lafayette, IN. One such function f is given by $g(x, y)$, where

$$\begin{aligned} g(x,y) = {} & xf(1,y) + (1-x)f(0,y) + yf(x,1) + (1-y)f(x,0) \\ & + [f(0,0) - f(1,0)]x + [f(0,0) - f(0,1)]y \\ & + [f(0,1) + f(1,0) - f(0,0) - f(1,1)]xy - f(0,0). \end{aligned}$$

Here $f(0,0)$, $f(0,1)$, $f(1,0)$, $f(1,1)$ denote the common values of the bounding polynomials at the corners. If $h(x, y)$ is an arbitrary polynomial solution, then $h(x,y) - g(x, y)$ is a polynomial that vanishes on $x = 0$, $x = 1$, $y = 0$, and $y = 1$; hence it must have the factor $x(1 - x)y(1 - y)$. Thus all polynomial solutions have the form

$$h(x,y) = g(x,y) + x(1-x)y(1-y)p(x,y),$$

where p is an arbitrary polynomial.

It should be noted that the given polynomials on the sides of the square are defined on the extensions of those sides, and g extrapolates these one-variable polynomials to the plane. The result generalizes as follows: given n straight lines $\{l_i\}$ in the plane no three of which are concurrent and n polynomials $\{p_i\}$ such that p_i has the same value as p_j at $l_i \cap l_j$, there is a polynomial in x, y whose restriction to l_i is p_i for each i.

Editorial comment. Generalizations to higher dimensions were given by Frédéric Brulois, J. G. Mauldon, and José Heber Nieto. Michael Kallay and Eugene Lee (independently) remarked that this problem is well known in Computer-Aided Design and appears in textbooks on the subject, such as I. D. Faux and M. J. Pratt, *Computational Geometry for Design and Manufacture* (Ellis-Horwood, 1979). See also W. J. Gordon, "Spline-blended interpolation through curve networks," *J. Math. Mech.* 18 (1969) 931–952.

Solved by 48 readers and the proposer.

An Old Chestnut

E 3401 [1990, 612]. *Proposed by James A. Davis, Michael Kerckhove, and J. Van Bowen, University of Richmond, VA.*

Suppose n points are independently chosen at random on the perimeter of a circle. What is the probability that all points lie in some semicircle?

Solution by Ellen Hertz, National Highway Traffic Safety Administration, Washington, DC. Denote the points by $X_1, X_2, \ldots, X_n$ and let A_i be the semicircular arc described by travelling π radians counterclockwise from X_i. If E_i is the event that no X_j lies in A_i, $i \neq j$, then $P(E_i) = (1/2)^{n-1}$ since A_i is half of the circle. Since the events E_i are disjoint,

$$P(\cup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i) = \frac{n}{2^{n-1}}.$$

Editorial comment. This problem was proposed independently (and even a bit earlier) by Gunnar Blom of the University of Lund and Lund Institute of Technology, Lund, Sweden. Blom's solution and twenty-nine of the other solutions were equivalent to the solution given above.

Actually this problem is rather old. It can be derived from the work of C. Jordan (Questions de probabilités, *Bull. de la Soc. Math. de France*, 1 (1872–1873) 256–258) who considered the following generalization: Let the circumference of a circle with unit perimeter be divided into n parts by n points chosen at random. Denote by $\nu_n(x)$, $0 < x < 1$, the number of subintervals of length larger than x. Then

$$P(\nu_n(x) = k) = \binom{n}{k} \sum_{j=k}^{\lfloor 1/x \rfloor} (-1)^{j-k} \binom{n-k}{j-k} (1-jx)^{n-1}.$$

The present problem is the case $x = 1/2$ and $k = 1$ (or equivalently $k = 0$). The case $k = 0$, x arbitrary, was the subject of S30 [1980, 403; 1982, 332]. Some other previous appearances were in the papers listed in the references below and in the following books: H. A. David, *Order Statistics*; Arthur Engel, *Wahrscheinlichkeitsrechnung und Statistik*, Volume 2; William Feller, *An Introduction to Probability Theory and Its Applications*, Volume 2; M. G. Kendall and P. A. P. Moran, *Geometric Probability*; P. Hall, *The Theory of Coverage Processes*; H. Solomon, *Geometric Probability*; W. A. Whitworth, *Choice and Chance*.

J. G. Wendel in "A problem in geometric probability," *Math. Scand.*, 11 (1962) 109–111, has given the following generalization to k-space: Let n points be scattered at random on the surface of the unit sphere in k-space. Let $p_{k,n}$ be the probability that all the points lie on some hemisphere. Then

$$p_{k,n} = \frac{1}{2^{n-1}} \sum_{j=0}^{k-1} \binom{n-1}{j}.$$

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Solved also by the proposers and seventy other readers. One incorrect solution was received. Many solvers submitted multiple solutions.

Series Involving the Central Binomial Coefficient

6638 [1990, 622]. *Proposed by Stan Philipp, Pennsylvania State University, Altoona, PA.*

Let

$$\alpha_k = (-1)^k \binom{-1/2}{k} = 4^{-k} \binom{2k}{k} \quad (k = 0, 1, 2, \dots).$$

(i) Prove that

$$\sum_{k=0}^{\infty} \frac{\alpha_k}{2k+1} = \frac{\pi}{2}, \qquad \sum_{k=0}^{\infty} \frac{\alpha_k}{2k-2n+1} = 0 \quad \text{for } n = 1, 2, 3, \dots .$$

(ii) Prove that

$$\sum_{k=0}^{\infty} \alpha_k^2 \left(\frac{1}{2k+2n+2} + \frac{1}{2k-2n-1} \right) = 0 \quad \text{for } n = 0, 1, 2, \dots .$$

Editorial remark. In the published version of the problem the second minus sign in the second denominator of (ii) was erroneously replaced by a plus sign. All of the solvers of (ii) corrected this misprint.

Solution I by Mourad E. H. Ismail, University of South Florida, Tampa, FL. We use the shifted factorial notation $(a)_k = \Gamma(a+k)/\Gamma(a)$, so $\alpha_k = (1/2)_k/k!$. Clearly

$$\sum_{k=0}^{\infty} \frac{(1-2n)\alpha_k}{2k-2n+1} = {}_2F_1\left(\begin{matrix} 1/2, -n+1/2 \\ -n+3/2 \end{matrix} ; 1 \right) = \frac{\Gamma(-n+3/2)\Gamma(1/2)}{\Gamma(-n+1)\Gamma(1)},$$

by Gauss's theorem. The above identity gives the required sum for any $n \neq 1/2, 3/2, \dots$. In particular both parts in (i) follow, since $\Gamma(1/2) = \sqrt{\pi} = 2\Gamma(3/2)$ and $1/\Gamma(-z)$ vanishes for $z = 0, 1, \dots$. Next note that

$$(2n+1)(2n+2)\left[\frac{1}{2k+2n+2} + \frac{1}{2k-2n-1} \right]$$

$$= -\frac{(n+1)_k(-n-1/2)_k(5/4)_k}{(n+2)_k(-n+1/2)_k(1/4)_k}$$

identifies the sum in (ii) as

$$\frac{-1}{(2n+1)(2n+2)}\, {}_5F_4\left(\begin{matrix} 1/2, & 5/4, & 1/2, & -n-1/2, & n+1 \\ & 1/4, & 1, & n+2, & -n+1/2 \end{matrix} ; 1 \right).$$

This can be summed using the ${}_5F_4$ summation theorem (see e.g. Appendix III in L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, 1966), namely

$${}_5F_4\left(\begin{matrix} a, & 1+a/2, & b, & c, & d \\ a/2, & 1+a-b, & 1+a-c, & 1+a-d \end{matrix} ; 1 \right)$$

$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}.$$

The only restrictions in the above steps are that neither n nor $-n+1/2$ are negative integers. The desired sum is $[\Gamma(n+1)\Gamma(-n-1/2)]/[2\Gamma(n+3/2)\Gamma(-n)]$, which vanishes for $n = 0, 1, 2 \dots$.

Solution II by Rolf Richberg, Institut für Reine und Angewandte Mathematik der RWTH, Aachen, Germany. (The editors omit the easier part (i) to save space.) The well-known infinite series for the complete elliptic integral of the first kind $K(t)$ is

$$K(t) = \int_0^1 (1-s^2)^{-1/2}(1-s^2t^2)^{-1/2}\, ds = \frac{\pi}{2} \sum_{k=0}^{\infty} \alpha_k^2 t^{2k}, \qquad |t| < 1.$$

This is a straightforward consequence of the binomial expansion. Define

$$f(z) = \sum_{k=0}^{\infty} \frac{\alpha_k^2}{k+z}$$

for $z \in \mathbb{C}$ with $z \notin \{0, -1, -2, \dots\}$. For $x > 0$ a term by term integration yields

$$\begin{aligned} f(x) &= 2\int_0^1 \sum_{k=0}^{\infty} \alpha_k^2 t^{2k+2x-1}\,dt \\ &= \frac{4}{\pi}\int_0^1\int_0^1 t^{2x-1}(1-s^2t^2)^{-1/2}(1-s^2)^{-1/2}\,ds\,dt. \end{aligned}$$

Now for $0 < x < \frac{1}{2}$ the above formula applies to $\frac{1}{2} - x$ as well as x. Upon adding these two formulae and performing a change of variable in the resulting double integral we obtain

$$\begin{aligned} f(x) + f\left(\frac{1}{2} - x\right) &= \frac{4}{\pi}\int_0^1\int_0^1 (t^{2x-1} + t^{-2x})(1-s^2t^2)^{-1/2}(1-s^2)^{-1/2}\,ds\,dt \\ &= \frac{4}{\pi}\iint_{0\leqslant t\leqslant s\leqslant 1} (t^{2x-1}s^{-2x} + t^{-2x}s^{2x-1}) \\ &\qquad \times(1-t^2)^{-1/2}(1-s^2)^{-1/2}\,dt\,ds. \end{aligned}$$

Considerations of symmetry, together with separation of variables, change of variable, the Gamma formula for the Beta function, and the reflection formula for the Gamma function yield

$$\begin{aligned} f(x) + f\left(\frac{1}{2} - x\right) &= \frac{4}{\pi}\int_0^1\int_0^1 t^{2x-1}s^{-2x}(1-t^2)^{-1/2}(1-s^2)^{-1/2}\,dt\,ds \\ &= \frac{1}{\pi}\frac{\Gamma(\frac{1}{2})\Gamma(x)\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-x)}{\Gamma(\frac{1}{2}+x)\Gamma(1-x)} \\ &= \tan(\pi x)\left(\frac{\Gamma(x)}{\Gamma(\frac{1}{2}+x)}\right)^2, \qquad 0 < x < \frac{1}{2}. \end{aligned}$$

By analytic continuation we obtain

$$f(z) + f\left(\frac{1}{2} - z\right) = (\tan \pi z)\left(\frac{\Gamma(z)}{\Gamma(\frac{1}{2}+z)}\right)^2$$

for $z \in \mathbb{C}$, and $z \notin \{0, -1, -2, \dots\} \cup \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\}$. In particular, this gives (the corrected version of) (ii) when z is a positive integer. It also yields

$$\sum_{k=0}^{\infty} \frac{\alpha_k^2}{4k+1} = \frac{1}{16\pi^2}\left(\Gamma\left(\frac{1}{4}\right)\right)^4$$

upon putting $z = \frac{1}{4}$.

Editorial comment. R. J. Chapman's proof of (ii) was similar to Richberg's. The proposer began with the well-known differential equation

$$tK(t) = \frac{d}{dt}\left[t(1-t^2)K'(t)\right] =: \frac{d}{dt}[H(t)], \qquad -1 < t < 1,$$

and then performed a careful integration by parts on

$$\frac{\pi}{4} f\left(\frac{x+1}{2}\right) = \int_0^1 t^{x-1} H'(t)\, dt.$$

The other solvers, like Ismail, used the theory of the generalized hypergeometric functions ${}_AF_B$. For a wealth of infinite series formulas involving the central binomial coefficient see D. H. Lehmer, this MONTHLY 92 (1985), 449–457. Further examples and references are available in J. and P. Borwein, *Pi and the AGM*, Wiley-Interscience, New York, 1987.

Solved also by P. J. Bushell (U.K.), R. J. Chapman (U. K.), Carl Libis ((i) only), O. P. Lossers (The Netherlands), James A. Wilson, and the proposer.

Triangles With Sides of Integer Length Whose Area is an Integer Multiple of the Perimeter

E3408 [1990, 848]. *Proposed by Juan V. Savall and Jesús Ferrer, Oliva, Valencia, Spain.*

For each positive integer k let $f(k)$ denote the number of triangles with sides of integer length whose area is k times the perimeter. It is well-known (cf. E2420 [1973, 691; 1974, 662]) that $f(1) = 5$. Obtain an upper bound for $f(k)$ in terms of k.

The analogous problem for *right* triangles appeared as Problem 1447 in Crux Mathematicorum, 15 (1989) 148.

Joint Solution by the proposers and the editors. For a given positive integer k we shall give an algorithm for determining all triangles with sides of integer length whose area is k times the perimeter. This algorithm gives the crude upper bound $f(k) < 8k^2 \log(13k)$.

Specifically, we shall show that $f(k)$ is equal to the number of pairs r, s of positive integers such that

(i) $r \leqslant s$,

(ii) $4k^2 < rs \leqslant 12k^2$,

(iii) $4k^2(r+s)/(rs - 4k^2) \geqslant s$,

(iv) $4k^2(r+s)/(rs - 4k^2)$ is a positive integer.

The actual triangles can be determined by putting $t = 4k^2(r+s)/(rs - 4k^2)$ and taking

$$a = r + s, \qquad b = r + t, \qquad c = s + t.$$

The bound on $f(k)$ given above will be obtained by showing that the number of pairs r, s satisfying the two conditions (i) and (ii) is less than $8k^2 \log(13k)$.

Suppose a, b, c are integers with $a \leqslant b \leqslant c$ such that the area of the triangle with sides a, b, c is k times its perimeter. By Heron's formula

$$(a + b - c)(c + a - b)(b + c - a) = 16k^2(a + b + c). \tag{1}$$

Now $a + b - c, c + a - b, b + c - a$, and $a + b + c$ have the same parity; if they were odd, we would have a contradiction to (1). Hence there are integers r, s, t with $r \leqslant s \leqslant t$ such that

$$a + b - c = 2r, \qquad c + a - b = 2s, \qquad b + c - a = 2t,$$

and consequently $a + b + c = 2(r + s + t)$. Note that also $a = r + s$, $b = r + t$, $c = s + t$. In terms of r, s, t equation (1) becomes $rst = 4k^2(r + s + t)$, and so

finding all triples a, b, c with $a \leqslant b \leqslant c$ satisfying (1) is equivalent to finding all triples r, s, t such that

$$rst = 4k^2(r + s + t), \qquad r \leqslant s \leqslant t. \tag{2}$$

If (2) holds, then $t < r + s + t \leqslant 3t$ and so

$$4k^2 t < rst \leqslant 12k^2 t.$$

Thus two positive integers r, s can be the first two members of a triple satisfying (2) if and only if $r \leqslant s$, $4k^2 < rs \leqslant 12k^2$, and the number t determined by the equation $rst = 4k^2(r + s + t)$ is an integer not less than s. In other words, two positive integers r, s can be the first two members of a triple satisfying (2) if and only if (i), (ii), (iii), and (iv) all hold.

To obtain an upper bound for $f(k)$ we estimate the number of pairs r, s satisfying (i) and (ii). Clearly $r^2 \leqslant rs \leqslant 12k^2$, so that $r \leqslant 2\sqrt{3}\,k$. For a given value of r such that $1 \leqslant r \leqslant 2k$, any value of s satisfying (ii) is automatically greater than r, so that (i), is also satisfied. Hence for given r in the interval $[1, 2k]$ the number of values of s satisfying (i) and (ii) is the number of multiples of r in the interval $(4k^2, 12k^2]$, namely $\lfloor 12k^2/r \rfloor - \lfloor 4k^2/r \rfloor$. For a given value of r such that $2k < r \leqslant 2\sqrt{3}\,k$, the number of values of s satisfying (i) and (ii) is the number of multiples of r in the interval $[r^2, 12k^2]$, namely $\lfloor 12k^2/r \rfloor - r + 1$. Hence the number of pairs r, s satisfying (i) and (ii) is equal to

$$\begin{aligned}
&\sum_{r=1}^{2k} \left(\lfloor 12k^2/r \rfloor - \lfloor 4k^2/r \rfloor\right) + \sum_{r=2k+1}^{\lfloor 2\sqrt{3}\,k \rfloor} \left(\lfloor 12k^2/r \rfloor - r + 1\right) \\
&\leqslant \sum_{r=1}^{2k} (8k^2/r + 1) + \sum_{r=2k+1}^{\lfloor 2\sqrt{3}\,k \rfloor} (12k^2/r - r + 1) \\
&\leqslant 8k^2\{1 + \log(2k)\} + 2k + 12k^2 \int_{2k}^{2\sqrt{3}\,k} du/u - 2k \\
&= 8k^2 \log(2k) + (6 \log 3 + 8)k^2 \\
&< 8k^2 \log(13k).
\end{aligned} \tag{3}$$

The inequality (3) gives the announced estimate $f(k) < 8k^2 \log(13k)$.

Note that if $4k^2 < rs \leqslant 8k^2$, the condition (iii) is automatically satisfied. Thus the use of (iii) along with (i) and (ii) can only improve the constant and does not eliminate the logarithmic factor. For the number of pairs of positive integers r, s satisfying $r \leqslant s$ and $4k^2 < rs \leqslant 8k^2$ is easily seen to be $4k^2 \log(2k) + O(k^2)$, by a slight modification of the above argument.

On the other hand it seems difficult to use condition (iv) quantitatively.

Kevin Ford has kindly provided us with the following brief table of values, calculated by the use of the above algorithm:

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$f(k)$	5	18	45	45	52	139	80	89	184	145	103	312	96	225	379

The values of f grow irregularly, but it would not be unreasonable to conjecture that $f(k) = O(k^2)$.

No other solutions were received.

Large Abelian Subgroups

6641 [1990, 857]. *Proposed by Theodore M. Alper, Stanford University, Stanford, CA.*

(a) For every positive integer n, is there an integer N_n such that every finite group of order at least N_n has an abelian subgroup of order at least n?

(b) For every positive integer n, does every infinite group have an abelian subgroup (possibly infinite) of order at least n?

Solution by Reiner Martin (student), University of California at Los Angeles. The answers to (a) and (b) are yes and no respectively.

For (a) let $N_n = (n!)^{n^2}$. Given a group G of order $|G| \geqslant N_n$, a prime $p > n$ must divide $|G|$, or p^{n^2} must divide $|G|$ for a prime $p \leqslant n$. In the first case, by Cauchy's Theorem, there is an element of order p, which generates a cyclic subgroup of order $p > n$. In the second case, let H be a Sylow p-subgroup of G. So $|H| = p^a$ with $a \geqslant n^2$. Now H has an abelian subgroup with $p^n > n$ elements (see H. J. Zassenhaus, *The Theory of Groups* (2nd ed., Chelsea, 1958) p. 145, or B. Huppert, *Endliche Gruppen I* (Springer, 1967) Satz III.7.3, or J. D. Dixon, *Problems in Group Theory* (Dover, 1973) Problem 8.28, or W. Burnside, *Proc. London Math. Soc.* (2) 11 (1913) 225–245, particularly 225–227).

For (b) we use the fact that the free group of exponent 665 on two generators is infinite, but each of its abelian subgroups has order dividing 655. (See Theorems 1.5 and 3.3 in Chapter VI of S. I. Adian, *The Burnside Problem and Identities in Groups*, J. Lennox and J. Wiegold, translators, vol. 95, Ergebnisse der Math., Springer-Verlag, 1979.)

Editorial comment. For (b) a number of solvers quoted a result (ascribed variously to Rips, Ol'shanskii, or both) that there are infinite groups all of whose non-trivial proper subgroups have prime order p. See A. Yu. Ol'shanskii, Groups of bounded period with subgroups of prime order (Russian), *Algebra i Logika* 21 (1982), 553–618 or *Algebra and Logic* 21 (1982), 369–418 (English translation).

Solved also by O. P. Lossers (The Netherlands), Victor V. Pambuccian, Derek J. S. Robinson, Richard Stong, Douglas B. Tyler, Gary L. Walls, Z. Z. Uoiea, and the proposer.

Collaborating editors: *Paul T. Bateman, Bruce C. Berndt, Duane M. Broline, Barry W. Brunson, Frank S. Cater, Gulbank D. Chakerian, Michael A. Filaseta, Ira M. Gessel, Richard A. Gibbs, Douglas A. Hensley, John R. Isbell, Murray Klamkin, Daniel J. Kleitman, Fred Kochman, Frederick W. Luttmann, Marvin Marcus, Frank B. Miles, Richard Pfiefer, Stephen L. Portnoy, J. O. Shallit, John Henry Steelman, Daniel Ullman, and Edward T. H. Wang.*

UNSOLVED PROBLEMS

In this department the MONTHLY presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial or related results. Typescripts should be sent to Richard Guy, Department of Mathematics & Statistics, The University of Calgary, Alberta, Canada T2N 1N4.

Trapped Reflections?

John E. Connett

Let S be a container, such as a bottle or vase, which is coated inside with a perfectly reflective mirror surface. Assume you shine a beam of light into the mouth of S (Figure 1). Can it be shown that, regardless of the shape of S, some of the light rays in the beam will eventually be reflected back out again through the mouth of S?

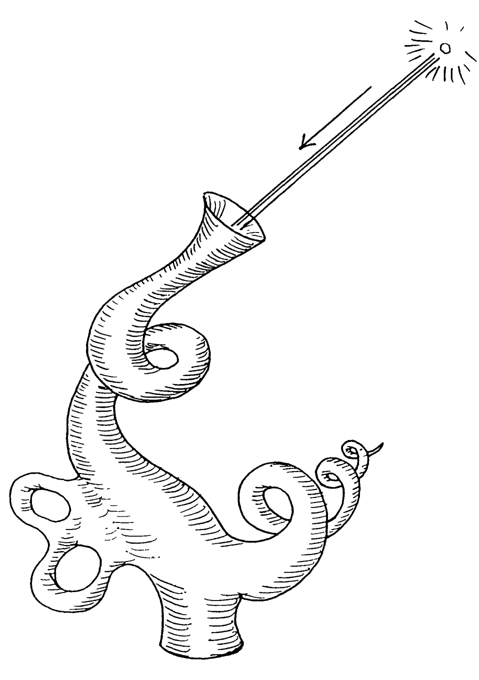

FIG. 1. Light beam entering a strangely shaped vase.

A counterexample to the question could be of practical value as a kind of battery or energy-thermos to store light rays, or at least delay their conversion into heat energy.

The two-dimensional version of the problem is easy to state more precisely. Let S be a piecewise-smooth curve in R with endpoints A and B. Assume S intersects the line segment AB only in points A and B. Thus $S \cup AB$ is a simple closed curve and divides the plane into components C (bounded) and D (unbounded). Assume that S acts as a perfect mirror, and that L is a beam of light (i.e., a pencil of parallel lines) which intersects line segment AB and which continues inside component C indefinitely, reflecting off the inner wall of S in accordance with the usual angle-of-incidence-equals-angle-of-reflection law. Will some elements of the beam L eventually intersect AB again, regardless of the shape of S?

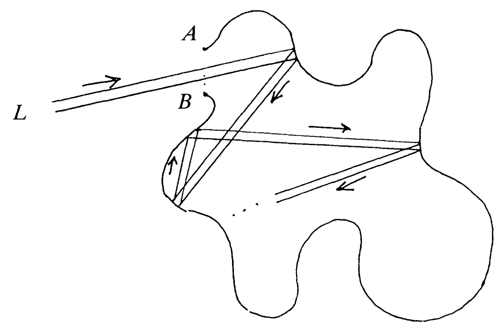

FIG. 2. Light beam bouncing around the interior of a simple curve S.

The somewhat related question of whether every bounded polygonal region in the plane could be illuminated from some point was posed by Klee (1969). This question appears to still be open. Rauch (1978), using a special property of the ellipse, showed that there are closed piecewise-smooth curves in the plane such that some subregions of the bounded component of the complement exist which cannot be illuminated either directly or by internal reflection from other subregions. Guy and Klee (1971) give an example of a region, bounded by a smooth closed curve, which cannot be illuminated from any point. The strongest *positive* result in this area was obtained by Boldrighini, Keane, and Marchetti (1978). They showed that, for a simple closed planar polygon all of whose angles are rational multiples of π, almost any internally reflected light-ray path will form a dense subset of the interior of the polygon. Generalizations of this result were obtained by Kerckhoff, Masur and Smillie (1986).

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LETTERS

In connection with the interesting paper by Foster and Richards on the "Gibbs Phenomena for Piecewise Linear Approximation" we would like to call attention of your readers to our paper "Smooth Polynomial Approximations of Piecewise-Differentiable Functions" in *Applied Mathematics Letters* 2, no. 4, 1989, pp. 377–379 which shows that piecewise-differentiable functions can be approximated continuously and accurately by the decomposition method without the Gibbs phenomenon.

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I have read with pleasure the pleading [**3**] for the Carathéodory definition of derivatives. This definition has been already used in [**2**], [**4**], [**5**], unfortunately without mentioning Carathéodory. I had not been aware of the fact that Carathéodory introduced this definition in his book [**1**]. It was a "quotient-free" proof of the chain rule in the book of Rothe [**6**, p. 49], which I had read as a student, which led me to the definition in question.

The proof of the product rule in [**3**] can be made a little bit easier by using $f(x) = f(a) + \phi(x)(x - a)$ instead of the difference $f(x) - f(a)$ in the following calculation (see [**4**], [**5**]):

$$\begin{aligned}(fg)(x) &= f(x)g(x) = f(x)(g(a) + \psi(x)(x-a)) \\ &= (f(a) + \phi(x)(x-a))g(a) + f(x)\psi(x)(x-a) \\ &= (fg)(a) + (\phi(x)g(a) + f(x)\psi(x))(x-a).\end{aligned}$$

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After presenting his solution to Advanced Problem 6613 [AAM, 98 (1991)], Professor Boyd remarks that for a trigonometric polynomial P of degree n, the inequality $\|P'\|_p \leqslant n\|P\|_p$, $0 < p < 1$, is yet to be proved. I wish to point out that the said inequality has in fact been established by V. V. Arestov [*Izv. Akad. Nauk SSSR*, *Ser. Mat.* 45 (1981), 3–22]. Further, a simpler proof of Arestov's theorem was later provided by M. V. Golitschek and G. G. Lorentz [*Rocky Mountain J. Math.*, 19 (1989), 145–156].

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REVIEWS

Stories About Maxima and Minima. By V. M. Tikhomirov, American Mathematical Society and the MAA, 1990, xi + 187 pp. Translated from the Russian by Abe Shenitzer.

Abe Shenitzer

> Nothing takes place in the world whose meaning is not that of some maximum or minimum.
>
> L. Euler

Euler's pronouncement is not the first of its kind. An early statement of the idea that nature is guided by extremal principles is attributed to Heron of Alexandria (first century A.D.). He presumably made it in connection with his discovery of the law of reflection of light from a flat surface. The same belief guided Fermat's derivation of Snel's law of refraction of light moving in an inhomogeneous medium. The most remarkable "technical" version of this philosophical tenet is Hamilton's principle of least action.

In connection with Snel's law of refraction we note that, while Fermat relied on an extremal principle, Huygens derived this law by positing a "wave mechanism" for the propagation of light. The axiomatic approach of Fermat and the model approach of Huygens continue to be fruitful to the present day.

The variety of extremal problems is staggering. They range from riddles to nature's way of doing things. They arise in economics, technology, the natural sciences, and in all areas of mathematics. There is an extremum problem "way back" in Euclid's *Elements* and extremal problems continue to inspire mathematical research in our own time.

On the technical side, Fermat's derivation of Snel's law leads to the simplest extremal problem in which one needs to minimize a function of a single variable with no constraints. Here the work of Fermat, Leibniz, and Newton provided the necessary condition for an extremum known as Fermat's principle: a "candidate" for an extremum must be a root of the derivative of the function to be extremized. The next step in the evolution of the subject of extremal problems was initiated by Johann Bernoulli's brachystochrone problem. Here one had to minimize what we now call a functional rather than a function and the candidates for an extremum were functions (rather than numbers) constrained by simple equalities. The analogue of Fermat's necessary condition for such problems was discovered by Euler, and is now known as the Euler differential equation. Euler called the subject he brought into being the calculus of variations.

After the leap from functions of one variable to functionals the subject of extremal problems returned to the case of functions of finitely many variables. Here the remarkable discovery was due to the young Lagrange who "put forward a principle for the solution of finite-dimensional problems with [constraints in the form of] equalities."

In the 20th century, the needs of economics and technology gave rise to new classes of extremal problems, such as convex problems and problems of optimal

control, that called for modifications of the existing methods of solution of extremal problems. Nevertheless, "the general conception of Lagrange remains valid for problems of the calculus of variations as well as for problems of optimal control."

This brief sketch gives the reader an idea of the technical and intellectual richness of this slim volume "intended primarily for high school students." Every issue I mentioned is discussed with rare skill in the book. Every problem is solved twice: once as originally solved, and a second time using the Lagrange principle. The best summary of the book's technical content is supplied by the author at the end of the penultimate (fourteenth) "story":

> Time to stop. I have kept my promise and solved all problems in the first part twice. Let's take a break from formulas and have a chat.
>
> Now what would I tell "the first high school student in the street" [a quote from an epigraph in the story] about the theory of extremal problems? Surely, something along these lines: In school they taught you about functions of one variable. They told you about Fermat's method of solution of extremum problems for such functions. But, in fact, there are very many problems that come down to the minimization of functions of many variables and even of functions of functions (say, curves), as in the case of the brachystochrone problem. They have been investigated in a chapter of mathematics called the calculus of variations. The notion of a derivative—the fundamental notion of "school" analysis—was generalized in functional (infinite-dimensional) analysis, a subject that arose in the beginning of this century. Infinite-dimensional analysis makes possible a unified view of the problem of minimization of a function of one and many variables and of problems of the calculus of variations.
>
> In this, most general, situation Fermat's theorem remains fully valid for problems without constraints: at an extremum the derivative must be zero. In the case of problems of the calculus of variations the decoded version of Fermat's theorem is a differential equation known as Euler's equation.
>
> The number of problems without constraints is relatively small. A large part of problems with constraints can be formalized as problems with constraints in the form of equalities.
>
> Lagrange put forward a principle for the solution of finite-dimensional problems with equalities. Its essence consists in the formation of the Lagrange function (that is the sum of the function to be minimized and the functions that determine the equalities multiplied by undetermined coefficients) and in treating it as if there were no constraints. (Here it would perhaps be best to quote the words of Lagrange in the epigraph in the twelfth story.) The general conception of Lagrange remains valid for problems of the calculus of variations as well as for problems of optimal control—a new chapter of theory of extremal problems.
>
> And if my new student acquaintance showed further interest then I would tell him/her what appears in the second part of this book (pp. 79–185).

While I don't expect too many high school students to take advantage of Tikhomirov's book, it is my ardent hope that it will do much to raise the technical and intellectual standards of many high school teachers, and the intellectual standards of many post-secondary school teachers.

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TELEGRAPHIC REVIEWS

Edited by
Lynn Arthur Steen
with the assistance of
the Mathematics Departments of Carleton, Macalester, and St. Olaf Colleges

Telegraphic Reviews are designed to alert readers in a timely manner to new books and computer software appropriate to mathematics teaching and research. Special codes classify reviews by subject area and appropriate use:

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Books and software submitted for review should be sent to *Reviews Editor, American Mathematical Monthly, St. Olaf College, Northfield, Minnesota 55057.*

General, S, P, L***.** *Mathematica in Action.* Stan Wagon. WH Freeman, 1991, xiv + 419 pp, (P). [ISBN: 0-7167-2229-1] A Michelin guide to *Mathematica* for the mathematical tourist beginning for beginners with prime numbers (e.g., public key encryption); visiting cycloids, surfaces, Julia sets, turtle geometry, three-dimensional shapes; and concluding with more advanced algorithms of number theory. Little programming required—most examples are just one-liners. Can be used to explore *Mathematica* or to explore mathematics: a superb resource for a senior seminar. LAS

General, P, L. *Miscellanea Mathematica.* Eds: Peter Hilton, Friedrich Hirzebruch, Reinhold Remmert. Springer-Verlag, 1991, xiii + 326 pp, $24.95. [ISBN: 0-387-54174-8] A *very* miscellaneous collection of papers by a score of world-famous mathematicians in honor of publisher Heinz Götze. Topics range from historical reminiscences to mathematical exposition; languages include English, French, and German; authors include Atiyah, Cartan, Eckmann, Faddeev, Hironaka, Serre, and Weil. LAS

Education. *Mathematical Thinking Activities for Student Groups.* J. Weston Walch. J. Weston Walch Publishers, 1991, xxxvii + 24 pp, $13.95 (P). Twenty-four activity cards for middle school students, each presenting a problem, hints, and possible solution strategies. Teachers' commentary lists objectives, prerequisites, and solutions for each problem. MW

Education, S(15-17). *Reaching Higher.* NCTM, 1990, vi + 30 pp, $39.50 (P), plus videotape. [ISBN: 0-87353-304-6] Three video taped lessons at primary, middle, and upper elementary levels illustrate a problem solving approach to instruction. Lessons model ways to engage students, including hands-on activities, questioning strategies, and grouping techniques. Printed materials allow classroom replication of video-taped lessons, and provide ideas for extension and follow-up activities. Tape and book both reproducible. Valuable for preservice and in-service elementary teachers. MW

Education, P*, L.** *National Curricula in Mathematics.* Geoffrey Howson. Mathematical Assoc (259 London Road, Leicester LE2 3BE), 1991, vii + 238 pp, £5.50 (P); £15.50. [ISBN: 0-906-588-219; 0-906-588-243] A comparative study of school mathematics curricula in nations of the European Community (EC) plus Hungary and Japan. Reveals wide variety in structure of school systems, in aims and objectives of mathematics education, in order and priority of mathematics topics, and most notably, in approach to differences among pupils. Written in part as a rebuttal to the newly established (and politically motivated) mathematics curriculum in England and Wales. "In some ways national curriculum are like New Year's resolutions—they are rarely kept for long but ...they tell us something about their makers, their aims, aspirations, ...and shortcomings." LAS

Education, P. *Mathematics in Preschool: An Aid for the Preschool Educator.* L.S. Metlina. Soviet Stud. in Math. Educ., V. 5. Transl: Joan Teller. NCTM, 1991, xiii + 371 pp, $25 (P). [ISBN: 0-87353-333-X] Curricula for the four-year Soviet preschool (ages 3-6) intended to raise "the required standard of mathematical concepts for pupils finishing preschool" to match the "changed content" of school instruc-

tion. Focuses on perception, measurement, number, and shape; based on lessons ("the basic type of work in forming mathematical conceptions") involving demonstration, explanation, performance, and practice. Originally published in the Soviet Union in 1977. LAS

Education, P, L. *Mathematicians and Education Reform, 1989-1990.* Eds: Naomi D. Fisher, Harvey B. Keynes, Philip D. Wagreich. CBMS Issues in Math. Educ., V. 2. AMS and MAA, 1991, vii + 176 pp, $40 (P). [ISBN: 0-8218-3502-5] Eleven papers selected from meetings of MER, the Mathematicians and Education Reform network, intended to illustrate the range of mathematicians' interest in education. Includes project reports as well as position papers. LAS

Education, P. *Teaching Mathematics: The Resource Implications.* Mathematical Assoc (259 London Road, Leicester LE2 3BE), 1991, 24 pp, (P). [ISBN: 0-906-588-219] Short list of resources needed to implement Britain's Cockcroft Report recommendations. Calls for enhanced learning environment, broad in-service programs, increased time for planning and teaching, fewer non-teaching demands on teachers' time, and smaller class sizes. Not much different from conditions in the U.S.! MW

Logic, T*(13-16: 1, 2), S, C. *The Language of First-Order Logic Including the Program Tarski's World.* Jon Barwise, John Etchemendy. Center for the Study of Language and Information (Stanford U., Stanford, CA 94305), 1990, xiii + 259 pp, $27.50 (P); Macintosh disk included. [ISBN: 0-937073-59-8] An innovative approach to teaching logic with a text whose exercises are intended to be worked on by small groups of students using the Macintosh program Tarski's World. Begins with propositional logic; covers quantifiers, first order set theory, Horn sentences, and various advanced topics. Tarski's World provides a "world window" with models against which formal sentences in a "sentence window" are evaluated. LAS

Discrete Mathematics, T(13-14: 1, 2). *Discrete Mathematics for Computer Scientists.* J.K. Truss. Intern. Comput. Sci. Ser. Addison-Wesley, 1991, xviii + 565 pp, $49.50 (P). [ISBN: 0-201-17564-9] Covers number systems, sets, relations, functions, algebra, combinatorics, and graphs with additional chapters on formal machines, complexity, and coding theory. Special emphasis on propositional and predicate logic. Over 700 exercises with selected solutions. DH

Discrete Mathematics, T(14-16: 1), S, L*.** *Applications of Discrete Mathematics.* Eds: John G. Michaels, Kenneth H. Rosen. McGraw-Hill, 1991, x + 515 pp, $17.95. [ISBN: 0-07-041823-3] Would be valuable used as a supplement to an introductory discrete mathematics course or as a complete second course. Contains twenty-four separate topics divided into three sections: discrete structures and computing, combinatorics, and graph theory. Each topic has a range of problems, complete solutions, suggested computer projects, and a suggested reading list. DH

Number Theory, S*(16-18), P, L*. *The Little Book of Big Primes.* Paulo Ribenboim. Springer-Verlag, 1991, xvii + 237 pp, $29.50 (P). [ISBN: 0-387-97508-X] A chatty abridged version of the author's *Book of Prime Number Records* (*First Edition*, TR, December 1988; Extended Review, August-September 1989; *Second Edition*, TR, February 1990) which tells the tale without including highly complex proofs. Recognizing primes, distribution of primes, special types of primes, heuristic and probabilistic results. Excellent bibliography, useful appendices and several indices. Brings methods and results of modern number theory within the group of any mathematician. (It is not, however, as hyped on the back cover, "thoroughly accessible to everyone.") LAS

Linear Algebra, T(14-15: 1). *Matrix Methods: An Introduction, Second Edition.* Richard Bronson. Academic Pr, 1991, xiii + 503 pp, $49.95. [ISBN: 0-12-135251-X] An introduction to matrix techniques emphasizing methodology rather than theory. This edition incorporates many shifts of emphasis that reflect changes in computational practice. Also, many new (mostly routine) exercises have been added. (*First Edition*, TR, April 1970.) AO

Linear Algebra, T(14-15: 1). *Matrices and Vector Spaces.* William C. Brown. Pure & Appl. Math., V. 145. Marcel Dekker, 1991, viii + 309 pp, $49.75. [ISBN: 0-8247-8419-7] "Written ... for those students with a serious interest in mathematics, and for those college instructors who want a challenging book for their students." Covers sytems of linear equations, vector spaces, determinants, and inner product spaces. AO

Algebra, T*(17-18: 1, 2), S, P, L. *A First Course in Noncommutative Rings.* T.Y. Lam. Grad. Texts in Math., V. 131. Springer-Verlag, 1991, xv + 397 pp, $49. [ISBN: 0-387-97523-3] Divided into twenty-five sections. Topics include the Wedderburn-Artin theory of semisimple rings, Jacobson's theory of the radical, representation theory of groups and algebras, prime and semiprime rings, primitive and semiprimitive rings, division rings, ordered rings, local and semilocal rings, perfect and semiperfect rings. Exercises at the end of each section. A conversational style that provides motivation and conveys a sense of perspective to the subject. LCL

Calculus, T*(14: 1, 2), S*, L*. *Second Year Calculus: From Celestial Mechanics to Special Relativity.* David M. Bressoud. Undergrad. Texts in Math. Springer-Verlag, 1991, xi + 386 pp, $29.95 (P). [ISBN: 0-387-97606-X] An inno-

vative honors-level text that relates the mathematics of three and four-dimensional vector calculus to the history of physics that motivated so much of its development. Opens with Newton's proofs of Kepler's laws (in modern form); concludes with derivations of Maxwell's equations and $E = mc^2$, using the full force of multivariable calculus expressed in the notation both of coordinates and of differential forms. A beautiful retelling of one of mathematics' greatest triumphs, eminently suited to use either as a text or as "bedtime reading," in the words of the book jacket. LAS

Complex Analysis, P. *Several Complex Variables and Complex Geometry.* Eds: Eric Bedford, *et al.* Proc. of Symposia in Pure Math., V. 52, Parts 1-3. AMS, 1991, $219 set [ISBN: 0-8218-1488-5]. *Part 1,* xv + 262 pp, [ISBN: 0-8218-1489-3]; *Part 2,* xv + 625 pp, [ISBN: 0-8218-1490-7]; *Part 3,* xv + 368 pp. [ISBN: 0-8218-1491-5] Proceedings from the Thirty-seventh Annual Summer Research Institute of the AMS held at the University of California at Santa Cruz, July 1989. A large collection of lectures intended to describe the current state of the field. MLR

Differential Equations, P. *Differential Equations and Mathematical Physics.* Ed: Christer Bennewitz. Math. in Sci. & Eng., V. 186. Academic Pr, 1992, xxx + 365 pp, $49.95. [ISBN:0-12-089040-2] Contains the plenary lectures given during a conference at the University of Alabama at Birmingham, March 1990. Mostly research articles, but features brief historical surveys on inverse scattering (P. Deift) and boundary control (W. Littman). SK

Numerical Analysis, P. *Multigrid Methods III.* Eds: W. Hackbusch, U. Trottenberg. ISNM, V. 98. Birkhäuser, 1991, ix + 394 pp, $98. [ISBN: 0-8176-2632-8] Proceedings of the Third European Conference on Multigrid Methods held in Bonn, October 1990. Contains seven invited papers and twenty-one contributed papers. MLR

Numerical Analysis, L. *Lectures on Numerical Mathematics.* Heinz Rutishauser. Transl: Walter Gautschi. Birkhäuser, 1990, xv + 546 pp, $49.50. [ISBN: 0-8176-3491-6] A posthumously published collection of Rutishauser's lecture notes for courses on numerical analysis at the E.T.H. in Zurich. The German edition was published in two volumes in 1976. This translation includes commentary describing subsequent developments. AO

Functional Analysis, P. *Lecture Notes in Mathematics-1477: Strong Limit Theorems in Noncommutative L_2-Spaces.* Ryszard Jajte. Springer-Verlag, 1991, x + 113 pp, $16 (P). [ISBN: 0-387-54214-0] The noncommutative versions of pointwise convergence theorems in L_2-spaces in the context of von Neumann algebras. A continuation of the 1975 volume *LNM-1110: Strong Limit Theorems in Non-Commutative Probability* (TR, August-September 1986). Contains several pages of open problems. DH

Analysis, S*(14-15), P. *An Introduction to the Laplace Transform and the z-Transform.* A.C. Grove. Prentice Hall, 1991, vii + 128 pp, (P). [ISBN: 0-13-488933-9] Intended to aid students' understanding of Laplace transforms and z-transforms. Mathematical theory is kept to a minimum. Good use of examples; with exercises. DH

Analysis, P. *Structural Properties of Polylogarithms.* Ed: Leonard Lewin. Math. Surveys & Mono., V. 37. AMS, 1991, xviii + 412 pp, $128. [ISBN: 0-8218-1634-9] Studies properties of the polylogarithm functions $Li_n(z)$, defined by $Li_0(z) = -\log z$, $Li_1(z) = -\log(1-z)$ and $Li_n(z) = \int_0^z Li_{n-1}(z)/z\, dz, n > 1$. LC

Analysis, P. *Constructive Theory of Multivariate Functions with an Application to Tomography.* Manfred Reimer. Bibliographisches Institut, 1990, 280 pp, (P). [ISBN: 3-411-14601-X] Contains constructive theory using polynomial restrictions. Divided into three parts: properties and relations of multivariate polynomials, multivariate approximations by the use of linear operators, and an application in the reconstruction problem of tomography. More than fifty problems and solutions. DH

Algebraic Geometry, P. *Algebraic-Geometric Codes.* M.A. Tsfasman, S.G. Vlăduţ. Math. & Its Applic., V. 58. Kluwer Academic, 1991, xxiv + 667 pp, $229. [ISBN: 0-7923-0727-5] Algebraic geometry meets coding theory. Introductory chapters on each, followed by methods of constructing codes from algebraic curves, modular curves, and others. Last chapter integrates results about sphere packings, number theory, and algebraic-geometric codes. Note price! AD

Differential Geometry, S*(16-18), P, L*. *The Theory of Singularities and Its Applications.* V.I. Arnold. Univ of Cambridge (US Distr: 40 W. 20th St., New York 10011), 1991, 72 pp, $19.75 (P). [ISBN: 0-521-422809] An exposition of singularity theory, which "describes the birth of discrete objects from smooth, continuous sources." Aims to illustrate by examples and key theorems (stated but not proved) the "remarkable discovery" that simple general laws govern qualitative change, whether in manifolds, differential equations, or abelian integrals. LAS

Geometry, S*(12-16), L.** *Fractals: Endlessly Repeated Geometrical Figures.* Hans Lauwerier. Transl: Sophia Gill-Hoffstädt. Princeton Univ Pr, 1991, xiv + 209 pp, $14.95 (P); $49.50. [ISBN: 0-691-02445-6; 0-691-08551-X] An engaging elementary introduction to the geometry of fractals in the plane—spirals, trees, stars—together with simple algebraic analysis of their properties. Concludes with discussion of Julia sets and the Mandel-

brot set. Appendix gives numerous Basic programs to implement ideas from the text. Requires no more than high school algebra; ideal resource for math club projects. LAS

Operations Research, T(16-18: 1, 2), L. *Game Theory.* Drew Fudenberg, Jean Tirole. MIT Pr, 1991, xxiii + 579 pp, $42. [ISBN: 0-262-06141-4] An introduction to game theory focusing on those aspects that have been most useful in the study of economic problems. Although no prior study of game theory is presumed, informal acquaintance with basic ideas such as Nash equilibrium, subgame-perfect equilibrium, and incomplete information is helpful. AO

Operations Research, P. *Multi-Objective Programming in the USSR.* Elliot R. Lieberman. Stat. Model. & Decision Sci. Academic Pr, 1991, xxviii + 368 pp, $59.95. [ISBN: 0-12-449660-1] Summarizes and analyzes the work of Soviet researchers over the last thirty years in the area of multi-objective programming. AO

Operations Research, T(16-17: 1). *Linear Programming.* Howard Karloff. Progress in Theoret. Comput. Sci. Birkhäuser, 1991, viii + 142 pp, $24.95. [ISBN: 0-8176-3561-0] A self-contained, concise introduction to the basic theory of linear programming covering the simplex algorithm, duality, the ellipsoid algorithm, and Karmarkar's algorithm. The viewpoint is that of a theoretical computer scientist. Proofs that the ellipsoid algorithm and Karmarkar's algorithm run in polynomial time are included. AO

Statistical Methods, T(17: 1, 2), P. *Statistical Analysis of Reliability and Life-Testing Models: Theory and Methods, Second Edition.* Lee J. Bain, Max Engelhardt. Stat.: Textbooks & Mono., V. 115. Marcel Dekker, 1991, vii + 496 pp, $115. [ISBN: 0-8247-8506-1] Presents properties and techniques for distributions that are useful in reliability and life-testing, particularly the exponential, Weibull, gamma and logistic distributions, including censored sampling results whenever possible. Has a new chapter on reliability for repairable systems. Assumes a background of probability theory and mathematical statistics, which are briefly reviewed in the first two chapters. RSK

Statistical Methods, P. *Regression Diagnostics.* John Fox. Quantitat. Applic. in Soc. Sci., V. 79. Sage Pub, 1991, 92 pp, $8.50 (P). [ISBN: 0-8039-3971-X] Monograph summarizing the most common procedures for dealing with problems in regression such as multicollinearity, outliers and influential data, non-normality, heteroscedasticity, and nonlinearity. Technical details are relegated to an appendix. RSK

Statistical Methods, P. *Proceedings of the Thirty-Sixth Conference on the Design of Experiments in Army Research, Development, and Testing.* Report 91-2. US Army Research Office (POB 12211, Research Triangle Park, NC), 1991, xi + 384 pp, (P). Contains most of the papers presented at the Conference held at the University of Delaware in October 1990. Also contains notes from a two-day tutorial by Russell R. Barton on "Graphical Design of Experiments" held before the Conference started. RSK

Computational Statistics, S*(13-15), C. *A MINITAB Companion with Macros.* Peter W. Zehna. Addison-Wesley, 1992, xvi + 382 pp, $24.95 (P) with diskette. [ISBN: 0-201-55580-8] Introduces and illustrates standard Minitab commands. Also presents a variety of macros, all of which are provided on an enclosed diskette, to handle procedures not included in Minitab. Includes problems with answers. RSK

Statistics, P. *The Chronological Annotated Bibliography of Order Statistics, Volume III: 1960-1961.* H. Leon Harter. Amer. Ser. in Math. & Management Sci. American Sciences Pr, 1991, vi + 214 pp, $95 (P). [ISBN: 0-935950-21-4] *Volume I* (1978) covered 1800 years; *Volume II* (1983) covered ten years (1950-59). This volume covers only two years of an exploding literature, with additional two-year bibliographies to follow. Arranged by year, not by sub-topic. Each title contains a full summary, usually quoted from abstract of original article. Includes additional pre-1960 references to supplement *Volumes I* and *II.* LAS

Programming, T(13-14), C. *Microsoft Q-Basic.* David I. Schneider. Dellen, 1991. *An Introduction to Structured Programming, Second Edition,* xiii + 536 pp, $30 (P) and diskette, [ISBN: 0-02-407591-4]; *An Introduction to Structured Programming for Engineering, Mathematics, and the Sciences,* xi + 578 pp, (P) and diskette. [ISBN: 0-02-407605-8] Complete introduction to QBasic, the language included, without written documentation, with DOS 5.0 for IBM-compatible microcomputers. Numerous exercises, practice problems, and programming projects. Emphasis on structured programming. QBasic disks included. Texts differ primarily in content focus of exercises and projects. MW

Algorithms, T(14-15: 1, 2). *Data Structures & Their Algorithms.* Harry R. Lewis, Larry Denenberg. Harper Collins, 1991, xv + 509 pp. [ISBN: 0-673-39736-X] Appropriate for a CS 7-type course. Emphasizes practically useful techniques, including some that are relatively new (e.g., skip lists and splay trees). An informal analysis of almost every algorithm is presented in a style that avoids that use of sophisticated mathematics. AO

Computer Systems, T(15-16), P. *VAX/ VMS: Operating System Concepts.* David Donald Miller. Digital Pr, 1992, xx + 550 pp, $44.95. [ISBN: 1-55558-065-3] This is an introductory text on operating system concepts including such topics as input/output systems, process and memory management, security pro-

tection, and privacy. It uses for its examples the VMS operating system for the VAX family of computers manufactured by the Digital Equipment Corporation. GMS

Computer Systems, P. *Managing NFS and NIS.* Hal Stern. O'Reilly & Assoc, 1991, xxiv + 410 pp, $27.95 (P). [ISBN: 0-937175-75-7] The NFS (Network File System) and NIS (Network Information System) protocols provide transparent access to files and other information in a distributed computing environment. This volume describes how NFS and NIS work and provides practical hints for computer system administrators and network managers. AO

Computer Graphics, L. *Graphics Gems II.* Ed: James Arvo. Graphics Gems Ser. Academic Pr, 1991, xxxii + 643 pp, $49.95. [ISBN: 0-12-064480-0] A collection of practical techniques and methods for computer graphics programmers. Some "gems" are new ways of solving well-known problems while others present useful mathematical machinery. C code for many of the algorithms discussed are included in an appendix. AO

Theory of Computation, P. *Very Large Scale Computation in the 21st Century.* Ed: Jill P. Mesirov. SIAM, 1991, xvi + 327 pp, $48.50. [ISBN: 0-89871-279-3] Twenty papers addressed to challenging computational problems in physics, chemistry, fluid dynamics, astrophysics, biology, engineering, algorithm and system design. Features approaches via parallel processing, especially on the Connection Machine. Authors stress need to re-think fundamental algorithms to fit new architectures. LAS

Computer Science, L*. *Research Directions in Computer Science: An MIT Perspective.* Eds: Albert R. Meyer, *et al.* MIT Pr, 1991, 490 pp, $40. [ISBN: 0-262-13257-5] Twenty forward-looking, broad-brush papers on computer systems, policy, theory, and artificial intelligence from a 25th anniversary celebration of MIT's Project MAC. Introduced by a banquet address by John Updike reflecting on the "flourishing opulence" and "ramifying vitality" of the MIT lab, "where money and energy gather." LAS

Applications (Fluid Dynamics), P. *New Trends in Nonlinear Dynamics and Pattern-Forming Phenomena: The Geometry of Nonequilibrium.* Eds: Pierre Coullet, Patrick Huerre. NATO ASI Ser. B. 237. Plenum Pr, 1990, x + 357 pp, $85. [ISBN: 0-306-43692-2] Proceedings of a NATO workshop held August 1988 in Cargèse, France to investigate systems driven far from equilibrium. Most contributors are physicists modelling fluid dynamics either discretely, with cellular automata and coupled map lattices, or continuously, with partial differential equations. Handful of papers on chemical waves, crystal growth, and materials instabilities. SK

Applications (Fluid Dynamics), P. *Multigrid and Defect Correction for the Steady Navier-Stokes Equations Application to Aerodynamics.* B. Koren. CWI Tract, V. 74. Centrum voor Wiskunde en Informatica, 1991, 127 pp, Dfl. 39 (P). [ISBN: 90-6196-391-5] A monograph summarizing recent results on numerical methods for solving the equations describing high-speed gas flow. RWN

Applications (Physics), S(18), P. *Nematics: Mathematical and Physical Aspects.* Eds: Jean-Michel Coron, Jean-Michel Ghidaglia, Frédéric Hélein. NATO ASI Ser. C, V. 332. Kluwer Academic, 1991, xiii + 428 pp, $140. [ISBN: 0-7923-1113-2] Proceedings of a workshop held at l'Universite de Paris Sud (Orsay) in May 1990 on the science of nematic liquid crystals. Note price. MU

Applications (Quality Control), P. *Statistical Process Control in Automated Manufacturing.* Eds: J. Bert Keats, Norma Faris Hubele. Quality & Reliability, V. 15. Marcel Dekker, 1989, xv + 294 pp, $79.95. [ISBN: 0-8247-7889-8] Most of the thirteen papers in this text are based on presentations made at a symposium sponsored by Arizona State University in November 1986. Papers focus primarily on fundamental issues in statistical process control (SPC), and the application of time series and expert systems techniques to SPC. RWJ

Applications, P. *Lecture Notes in Mathematics-1463: Singularity Theory and its Applications, Warwick 1989, Part II.* Eds: M. Roberts, I. Stewart. Springer-Verlag, 1991, viii + 322 pp, $33 (P). [ISBN: 0-387-53736-8] Contains sixteen papers from a year-long symposium held at the University of Warwick in 1988-1989. OJ

Applications, P. *Production Research: Approaching the 21st Century.* Eds: Mark Pridham, Christopher O'Brien. Taylor & Francis, 1991, xiii + 841 pp, $218. [ISBN: 0-85066-753-4] One-hundred papers from a 1989 international conference at the University of Nottingham dealing with optimization and control of production processes. Papers divided into seven sections covering production processes, management, productivity, human factors, automation, expert systems, and computer-aided manufacture. An impressive array of mathematical methods in the service of productive activity. Note price! LAS

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LC: Laura Chihara, St. Olaf; DH: Deanna Haunsperger, St. Olaf; OJ: Ockle Johnson, St. Olaf; RWJ: Roger W. Johnson, Carleton; SK: Steve Kennedy, St. Olaf; RSK: Richard S. Kleber, St. Olaf; LCL: Loren C. Larson, St. Olaf; RWN: Richard W. Nau, Carleton; AO: Arnold Ostebee, St. Olaf; MLR: Margaret L. Reese, St. Olaf; GMS: G. Michael Schneider, Macalester; LAS: Lynn Arthur Steen, St. Olaf; MU: Milton Ulmer, Carleton; MW: Martha Wallace, St. Olaf.

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Carsten Thomassen received his Master's degree in 1972 at University of Aarhus, Denmark, and his Ph.D. in 1976 at University of Waterloo, Canada. Since 1981 he has been a professor of mathematics at the Technical University of Denmark. He is coeditor in chief of the *Journal of Graph Theory*.

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Clifford E. Weil got his Ph.D. from Purdue University under the direction of Casper Goffman in 1963. He was a Fine instructor at princeton University, 1963–64 and a Dickson instructor at the University of Chicago, 1964–66 before moving to Michigan State University in 1966. Since 1982 he has been one of the managing editors of the *Real Analysis Exchange*. His research specialty is in the theory of generalized derivatives.

A. M. Bruckner is Professor of Mathematics at U.C.S.B. where he has been since 1959. He wrote his doctoral dissertation on superadditive functions under the direction of John Green at U.C.L.A. Most of his work has been in the area of real functions, and most of that work has dealt with questions related to differentiation theory.

Stories About Maxima and Minima

V.M. Tikhomirov,
Translated by Abe Shenitzer

We are pleased to announce the first volume in a new series of expository books translated from the Russian KVANT Library, published jointly with the American Mathematical Society.

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David Jones in *New Scientist*

Mathematical Magic Show begins with a chapter on nothing, and finishes with a chapter on everything. In between we visit most of the prime sites of recreational mathematics—game theory, factorial...puzzles, playing cards, finger arithmetic, Möbias bands, polyominoes, perfect numbers, the knight's tour, trees, and dice. Gardner always has new facts and ideas to add interest to even the most well-trodden areas. For example, he extends his discussion of the knight's tour to bring in the cook's tour (a cook travels three squares forward and then one square right or left) and then goes on to include camels, asps, and giraffes. The chapter on dice has some very useful hints on cheating at craps, and how not to get caught.

Harvey Mellar in *Times Literary Supplement*

Over the 30 or so years that (Martin Gardner's) column ran in Scientific American, he built up an enthusiastic readership which included practically every professional mathematician that I have met, and no doubt the countless many I have not. There must be a great many professionals whose own interest in the subject was awakened by his rare gift of being able to put across to the outsider the deepest developments in an extremely difficult and abstract subject, and to convey the enjoyment and excitement that lies within mathematics proper.

Keith Devlin in *The Guardian*

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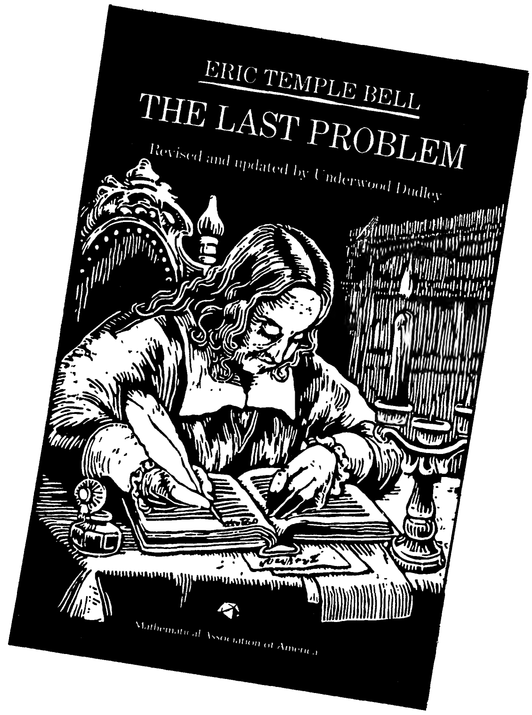

THE LAST PROBLEM

E. T. Bell

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What T.A.A. Broadbent said about Bell's work applies to THE LAST PROBLEM.

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Patricia Clark Kenschaft, Editor

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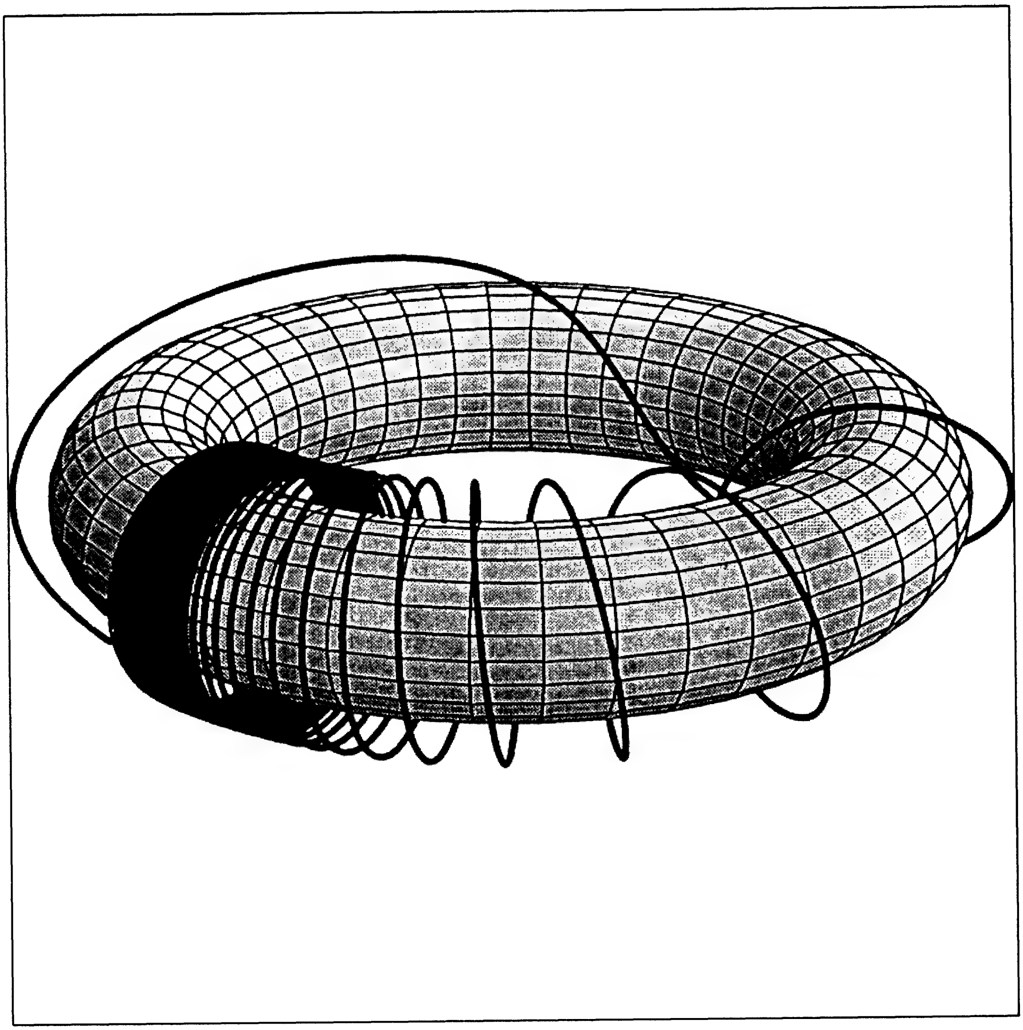
The American Mathematical Monthly

Volume 99, Number 3 / MARCH 1992



The Gauss Map (p. 205)

NOTICE TO AUTHORS

The *Monthly* publishes articles, notes, and other features about mathematics and the profession The readership of the *Monthly* is intended to include everybody who is mathematically inclined, including of course professional mathematicians and students of mathematics at all collegiate levels While no single article or feature is likely to appeal to everyone, material should interest and be accessible to a large number of readers This is the most important criterion for acceptance

Articles may be expositions of old results or presentations of new ones They may concern all of mathematics or one small area, a broad development or a single application, historical reminiscences or one important event While some articles may contain the author's new research, the novelty of material and generality of the results is far less important than the clarity of exposition and general interest Discussing one illuminating case of a well known result is far better than providing all the details of an obscure but new proposition Articles in the *Monthly* are supposed to inform and to entertain, they are meant to be read rather than archived

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Since Plato, mathematicians have tended to be smug about mathematics. ("Let no one unskilled in Geometry enter here." How come? Is it really necessary to understand geometry to read Plato?) Philosophers from Bacon to Kant have touted the virtues of mathematics for developing the mind. (I've always wondered why these same philosophers knew only a little mathematics.) By the turn of the century, mathematicians were so taken with themselves that a famous German mathematician (Möbius) proclaimed: "Mathematicians bear the same relation to the rest of mankind that those who are academically trained bear to those who are not." Those with great mathematical talent must have supreme intellect, the argument goes, and that gives them insight far beyond normal people.

Baloney. Great mathematicians are sometimes ignoble human beings. History is full of examples, and by now mathematicians should not find this simple truth surprising. Mathematicians are human beings, with the complete range of human imperfections. Only our mathematical arrogance suggests otherwise.

One of the greatest mathematicians of modern times, I. R. Shafarevich, has written a long (and dull) treatise about science and the Soviet Union. This rather prosaic work includes a number of silly comments about society and history, and in particular it includes many common anti-semitic slogans. A group of mathematicians (led by Irwin Kra, SUNY at Stony Brook) has written and circulated an open letter, which was signed by hundreds of others.

We are saddened by the numerous anti-semitic sentiments appearing in your work "Russophobia" and in your public comments on the current political situation.

We have applauded your defense of individuals during the dark chapters of recent Russian history. We respect your profound and lasting contributions to mathematics. A mind capable of seeing the beauty of our discipline, a mind that can further our science, should also be able to see the emptiness, futility and absence of reason in the conspiratorial theory to which you subscribe.

Your espousal of long discredited allegations about the role of Jews in world history, and in particular about their role in Russian history, can only have a chilling effect on your interactions with Jewish and non-Jewish mathematicians and on the recently improved relations between East and West. Your writing can be used to give an intellectual foundation to a theory of hate that has in the past and can again in the future lead to mass murder.

We ask that you reassess your position and we urge a public disclaimer of your anti-Semitic polemic.

No response has been received.

Should we be sad that a great mathematician can write and say foolish things? Absolutely. Should we be outraged? Of course. Should we be surprised? Hardly. Mathematicians—even great mathematicians—are fallible human beings whose wisdom should be as questionable as that of a politician or an actor. Mathematicians need to learn more humility. Here is one more lesson.

—John Ewing

Continued Fractions and Chaos

R. M. Corless

1. INTRODUCTION. This paper is meant for the reader who knows something about continued fractions, and wishes to know more about the theory of chaotic dynamical systems;[1] it is also useful for the person who knows something about chaotic dynamical systems but wishes to see clearly what the effects of numerical simulation of such a system are. This paper is not purely introductory, however: there are new dynamical systems results presented here and also in the companion paper (Corless, Frank & Monroe [1989]), which presents some discussion of dynamical reconstruction techniques and dimension estimates.

The theory of continued fractions goes back at least to *c.* A.D. 500 to the work of Āryabhata, and possibly as far back as *c.* 300 B.C. to Euclid. The theory of chaotic dynamical systems is relatively recent, going back only to the work of Poincaré [1899] and Birkhoff [1932]. The foundations of the theory of continued fractions, as we know it now, are well established due to the work of Euler, Lagrange, Gauss, and others, while the foundations of chaotic dynamical systems are still evolving. This paper will use the well-established theory of simple continued fractions to explore some current results of the theory of chaotic dynamical systems.

Olds [1963] gives a good introduction to the classical theory of simple continued fractions, by which we mean continued fractions of the form

$$n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cdots}}}$$

where the n_i are all positive integers, except n_0 which may be zero or negative. We will denote this as $n_0 + [n_1, n_2, n_3, \dots]$, and in what follows n_0 will usually be zero. Simple continued fractions have found applications in Fabry-Perot interferometry (Ikeda & Mizuno [1984]), and the concept of "noble" numbers used in orbital stability and quasi-amorphous states of matter (Schroeder, [1984]). For other uses of simple continued fractions in chaos, see Devaney [1985]. Other types of continued fraction exist, for example, Gautschi [1970], Henrici [1977], Jones and Thron [1980], and others, use functional or analytic continued fractions in approximation theory, since analytic continued fractions can be very effective for computation. We will not be concerned with such continued fractions. We will summarize in the next section all the classical results that we need, without proof. Proofs can be found in Olds [1963], Hardy and Wright [1979], Niven [1956], Khinchin [1963], Billingsley [1963], and Mañé [1987].

[1]One referee has remarked that "This describes the referee, who admits to having found the paper interesting. Though, I suspect, now, more people know about chaos than continued fractions." The author is inclined to agree, and hopes that this paper will interest some of these people in continued fractions.

2. SUMMARY OF CLASSICAL RESULTS

The Gauss Map. We begin with the classical method for finding the continued fraction representation of a number γ. We put n_0 equal to the integer part of γ, by which we mean the greatest integer less than or equal to γ. If the fractional part of γ is not zero, we put γ_0 equal to the fractional part of γ. We then invert γ_0, and put n_1 equal to the integer part of $1/\gamma_0$. Similarly we put γ_1 equal to the fractional part, and repeat. Note that n_0 may be positive, negative, or zero, but that all the subsequent n_i will be positive, and that each γ_i is in the interval $[0, 1)$. This process gives us unique continued fraction for each starting point γ, and the process terminates if and only if γ is rational. (For any rational γ there is one other simple continued fraction which is only trivially different from the one generated by this algorithm.) This algorithm is related to the Euclidean algorithm for finding the greatest common divisor (gcd) of two integers k and m (Olds [1963]), in that if we use this method to find the continued fraction of k/m, then the integer parts that arise are precisely the quotients that arise in the Euclidean algorithm, and in fact the last nonzero remainder from the Euclidean algorithm appears as the numerator of the last nonzero fractional part. This remainder is of course the gcd of k and m. Further, this algorithm can easily be seen to terminate in $O(\log(\min(k, m)))$ operations. Classically, most attention has been paid to the integers generated by this algorithm, which make up the continued fraction itself. However, Gauss was apparently the first to study the other part of this algorithm, which we present as the following map, called the Gauss map (Mañé [1987]) (see FIGURE 1):

$$G(x) = \begin{cases} 0 & \text{if } x = 0 \\ \dfrac{1}{x} \bmod 1 & \text{otherwise.} \end{cases}$$

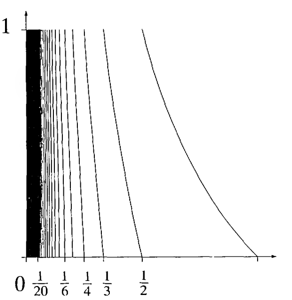

Figure 1. The graph of the Gauss Map $G(x)$. Note that there are an infinite number of jump discontinuities at values of $x = 1/n$, for integers n. In addition, there is a pole at the origin. The darkening of the curve towards the origin is suggestive of the fractional nature of the capacity dimension.

We use the notation "mod 1" to mean taking the fractional part. In terms of the Gauss map G, our algorithm then becomes

$$\begin{aligned} \gamma_{k+1} &= \text{fractional part of } 1/\gamma_k = G(\gamma_k) \\ n_{k+1} &= \text{integer part of } 1/\gamma_k, \quad \text{for } k = 0, 1, 2, 3, \ldots \end{aligned}$$

and we see that the continued fraction is generated as a byproduct of the iteration of the Gauss map. Thus we expect that any classical results on continued fractions will have implications for the dynamics of the Gauss map.

Note that the jump discontinuities occuring at $x = 1/n$ (for each integer n) may all be removed by mapping onto the circle with the transformation $e^{2\pi i x}$. After this is done, we see that the Gauss map ($e^{2\pi i x} \to e^{2\pi i/x}$) is a map of the circle onto the circle, and may be pictured on a torus, as in FIGURE 2. The singularity at the origin is not removed by this transformation. For convenience, the singularity is dealt with by artificially making zero a fixed point of the map (this makes our difficulties no worse). Most theorems on the dynamics of discrete maps assume continuity, which is thus violated here.

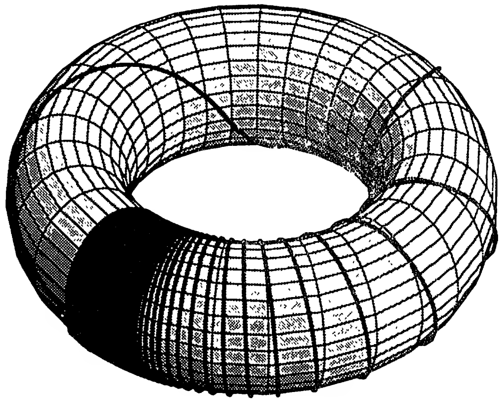

Figure 2. The graph of the Gauss Map $G(x)$ on the torus. Note that all the jump discontinuities have been removed, but that the pole at the origin remains. The darkening of the curve towards the singularity again gives an idea of the fractional nature of the capacity dimension.

We make the following observation: if we represent a point in the interval $[0, 1)$ by its continued fraction, $\gamma_0 = [n_1, n_2, n_3, \dots]$, then a simple induction shows that $G(\gamma_0) = \gamma_1 = [n_2, n_3, n_4, \dots]$, $G(\gamma_1) = \gamma_2 = [n_3, n_4, n_5, \dots]$, $G(\gamma_2) = \gamma_3 = [n_4, n_5, n_6, \dots]$, and so on. This makes a connection between the Gauss map and the "shift map" of symbolic dynamics (Devaney, 1985). We will not explore this connection further here, but we note that the shift maps normally studied are slightly different than the Gauss map, in that here the size of the numbers in the list being "shifted" is not bounded.

An analogy is illuminating: if we think of our space as a circular hoop with the origin at one point O on the hoop, our initial point as a dimensionless bead on the hoop, and the Gauss map is taking the bead from its current position clockwise past O at least once to its next position on the hoop, then the integers n_i are the number of times the bead passes O on the ith iteration (in general the maximum such number is called the "winding number" of the map, and here this is obviously infinite), and the γ_i are the coordinates of the bead on the hoop once it comes to rest. If the bead comes to rest close to the origin on one side, with a small γ_i, then on the next iteration it will be pushed many times around the hoop. If it comes to rest close to the origin on the other side, with a γ_i close to 1, then it will only go past the origin once on its next iteration. We may think of the bead as being pushed around the circle, with the strength of the push being inversely proportional to the distance measured counterclockwise from the point O.

3. DYNAMICAL SYSTEMS TERMINOLOGY. In what follows we give a compact introduction to the terminology used in the study of discrete dynamical systems.

For more details, see Devaney [1985]. To begin with, a **discrete dynamical system** is a recurrence relation $x_{k+1} = G(x_k)$, with the index k playing the role of a discrete "time". Note that the points x_k may be multi-dimensional. The sequence $\{x_k\}_{k=0}^{\infty}$ is called the **orbit** of the initial point x_0 under the **map** $x \to G(x)$, and is denoted as $\mathrm{orb}(x_0)$. Any points x that satisfy $x = G(x)$ are called **fixed points** of the map, and more generally if $x = G^n(x)$ where $G^n(x) = G(G^{n-1}(x))$ then x is called a **periodic point** of the map. If N is the least such number n, then as usual we say x has period N. The **α-limit set** of $\mathrm{orb}(x_0)$ is the set of all initial points whose orbits approach $\mathrm{orb}(x_0)$ as "time" increases; to be precise, an initial point y_0 is in the α-limit set of $\mathrm{orb}(x_0)$ if there exist m and n such that for all $\varepsilon > 0$ there exists K such that $k \geqslant K$ implies $|x_{m+k} - y_{n+k}| < \varepsilon$. The **$\omega$-limit set** of $\mathrm{orb}(x_0)$ is the set of accumulation points of $\mathrm{orb}(x_0)$. An **attractor** of a map is a set of points which "attracts" orbits, from some set of initial points of nonzero probability of being selected. To be precise, an attractor of a map is an **indecomposable** closed **invariant** set Λ with the property that, given $\varepsilon > 0$, there is a set U of positive Lebesgue measure in the ε-neighbourhood of Λ such that if x is in U then the ω-limit set of $\mathrm{orb}(x)$ is contained in Λ and the orbit of x is contained in U (Guckenheimer & Holmes, [1983]). An invariant set is a set such that $G(\Lambda) = \Lambda$, and an indecomposable set is one which cannot be broken into two or more pieces which are distinct under G. A map G is said to be **sensitive to initial conditions** (SIC) if initially close initial points have orbits that separate at an exponential rate. A map that is SIC is also said to be **chaotic**. The possible average exponents of these rates of separation are called the **Lyapunov exponents** of the map. Osledec's theorem (Osledec, [1968]) states that for a wide class of maps, and for almost all initial points, there are only finitely many possible Lyapunov exponents (in fact, only n for an n-dimensional map).

4. CLASSICAL RESULTS INTERPRETED IN DYNAMICAL SYSTEMS TERMINOLOGY PERIODIC AND FIXED POINTS OF THE GAUSS MAP. The following classical theorem, interpreted in a modern dynamical sense, identifies the fixed and periodic points of the Gauss map.

Theorem (Galois). *The number γ has a purely periodic continued fraction, including the first integer n_0, if and only if γ is a "reduced quadratic irrational", which means that γ is a root of a quadratic equation with integer coefficients and, further, that its algebraic conjugate (i.e. the other root of the quadratic) lies in the interval* $(-1, 0)$.

Corollary. *The periodic points of the Gauss map are the reciprocals of the reduced quadratic irrationals.*

For a proof of the theorem, see Olds [1963], or Hardy and Wright [1979]. To prove the corollary, we note that $\gamma = [n_1, n_2, n_3, \dots]$ is periodic under the Gauss map if and only if its continued fraction is periodic, starting at n_1, by the shift property mentioned in the previous section.

An example of particular interest is τ, the golden ratio, which satisfies $\tau^2 - \tau - 1 = 0$. The other root of this quadratic is $-1/\tau$ which is in the desired interval. The continued fraction of τ is $\tau = 1 + [1, 1, 1, 1, \dots]$, so $1/\tau$ has the continued fraction $[1, 1, 1, 1, \dots]$, which shows that $1/\tau$ is a point of period 1 of the Gauss map. We will return to this example later.

There are general results in the theory of chaotic dynamical systems, with which we could hope to establish the character of the set of periodic points of the Gauss

map (Šaarkovskii [1964], Štefan [1977], Li and Yorke [1975]). However, these results deal with the characterisation of the behaviour of *continuous* maps of the interval, extended by Block to maps of the circle (Block [1980]), and the Gauss map has a singularity at the origin. Thus the hypotheses of these theorems are not weak enough to apply. However, the results of these theorems hold, as will be seen by direct methods.

We note here that there are infinitely many points of each period. For example, $[n_1, n_2, \dots, n_k, n_1, n_2, \dots, n_k, \dots]$ has period k, for any choice of integers $n_1, n_2, \dots, n_k$. Having points of arbitrary period is one characteristic of a chaotic map (Li and Yorke [1975]). However, we would like to see if the map is sensitive to initial conditions (SIC) in that nearby initial points have orbits that separated at an exponential rate. This again can be established in an elementary fashion by using a classical result.

Theorem (Lagrange). *γ has an ultimately periodic continued fraction, which means that $\gamma = [a_1, a_2, a_3, \dots, a_i, n_1, n_2, \dots, n_k, n_1, n_2, \dots, n_k, \dots]$ with transients $a_1, a_2, a_3, \dots, a_i$ at the start of a periodic continued fraction, if and only if γ is a quadratic irrational (γ is a root of a quadratic with integer coefficients).*

Corollary. *The Gauss map is S.I.C.*

For a proof of Lagrange's theorem, see Hardy and Wright [1979]. To prove the corollary, we note that every rational initial point is "attracted" to the artificial fixed point at 0, while every quadratic irrational is ultimately "attracted" to a periodic orbit. Both sets are dense in the interval $[0, 1)$. The rate of separation may be checked by considering all points in a small interval I, of width ε. By the pigeonhole principle, this interval must contain a rational number of the form p/n, where n is the smallest integer larger than $1/\varepsilon$. The number of iterations of the Gauss map required to reach zero for this initial point is, by the speed of the Euclidean algorithm, $O(\log(n))$, and thus $O(\log(\varepsilon))$. To construct a specific initial point in this interval that does something different under G, first expand p/n into its finite continued fraction: $p/n = [a_1, a_2, a_3, \dots, a_i]$. Then for large enough N, the following infinite continued fraction is the continued fraction expansion of a point in I: $[a_1, a_2, a_3, \dots, a_i, N, 1, 1, 1, 1, \dots]$. Clearly, the orbit of G starting at this initial point winds up on the fixed point at $1/\tau$. Q.E.D.

Aperiodic Points. Of course, non-quadratic irrationals have continued fraction expansions, too. By Lagrange's theorem, these continued fractions are **aperiodic**, and hence the orbit of these initial points under the Gauss map is aperiodic. Note that most numbers in $[0, 1)$ are thus aperiodic. We examine some beautiful examples, beginning with one due to Euler:

1. e (the base of the natural logarithms) has an aperiodic continued fraction expansion $e = 2 + [1, 2, 1, 1, 4, 1, 1, 6, \dots]$. The elements of the orbit of this initial point are always of the form $[1, 2N, 1, 1, \dots]$, $[2N, 1, 1, \dots]$, or $[1, 1, 2N, \dots]$, which tend to 1, 0, and 1/2, respectively. Thus the ω-limit set of this orbit is the set $\{1, 0, 1/2\}$, which, unlike the ω-limit sets of continuous maps, is *not* invariant under the Gauss map since $G(1) = G(1/2) = 0$, so G applied to this set simply gives 0. In other words, we have an asymptotically periodic orbit which is not asymptotic to a real orbit of the map. This cannot happen for a discrete dynamical system with a continuous map.

2. (Stark [1971]). If x is the positive root of $x^3 - 3600x^2 + 120x - 2 = 0$, then

$$x = 3599 + [1, 28, 1, 7198, 1, 29, 388787400, 23, 1, 8998, 1, 13, 1, 10284, 1, 2, 35400776804, 1, 1, \dots]$$

which has very large entries placed irregularly throughout. This intermittency is a typical feature of a chaotic system (Guckenheimer and Holmes [1983]).

3. (Lambert, 1770—*cf* Olds [1963]). The continued fraction for π is not known, in the sense that no pattern has been identified. It begins $\pi = 3 + [7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, \dots]$ and some 17,000,000 elements of this continued fraction have been computed by Gosper (Borwein and Borwein, [1987]). There are many open questions about this continued fraction—for example, it is not known if the elements of the continued fraction are bounded.

Lyapunov exponents. We showed earlier that the separation of orbits initially close to each other occurred at an exponential rate. We would like to examine the Lyapunov exponents of the Gauss map, to see if we can explicitly measure the *rate* of separation. The Lyapunov exponents of orbits of the Gauss map are defined as (Devaney [1985])

$$\lambda(\gamma) = \lim_{n \to \infty} \frac{1}{n} \ln\left(\prod_{i=0}^{n} |G'(\gamma_i)| \right)$$

whenever this limit exists. Nearby orbits will separate from the orbit of γ at an average rate of $e^{\lambda k}$, after k iterations of G. Khinchin [1963] derived a remarkable theorem with which we could show the Lyapunov exponent of almost all (in the sense of Lebesgue measure) orbits can be shown to be $\pi^2/6 \ln 2$. Easier ways have since been found to establish this result, using ergodic theory. We summarize the ergodic results in the next section. In this section we simply note that for any rational initial point, the above limit does *not* exist. Further, for any periodic orbit the calculation can be made explicitly, to give Lyapunov exponents that *differ* from the almost-everywhere value. For example, the fixed points $\alpha_N = [N, N, N, N, \dots]$ have Lyapunov exponents

$$\lambda(\alpha_N) = 2\ln(1/\alpha_N) \sim \ln(N) + N^{-2} - \tfrac{3}{2}N^{-4} + O(N^{-6})$$

so that there are orbits with *arbitrarily large* Lyapunov exponents, i.e., orbits that are arbitrarily sensitive to perturbations in the initial point. Note also that for the orbit of e, the limit defining the Lyapunov exponent is *infinite*. The special case $N = 1$ gives τ, the golden ratio. Thus $\lambda(1/\tau) = 2\ln\tau = 0.96\dots$, which is smaller than the almost-everywhere Lyapunov exponent. In fact, we have the following:

Theorem. *No orbit of the Gauss map has a Lyapunov exponent smaller than* $\lambda(1/\tau) = 2\ln\tau$.

Proof: Let $\gamma = [n_1, n_2, n_3, \dots]$ be any initial point in $(0,1)$ such that $\lambda(\gamma)$ exists. We will show that the product $\prod_{i=0}^{N}(1/\gamma_i^2)$ which appears in the definition of $\lambda(\gamma)$ must be at least τ^{2N} (for N sufficiently large) which will prove the theorem. We consider two subsequent elements γ_k and γ_{k+1} of the orbit of γ. If $k = N$, enlarge the product by one term. Note γ_k and γ_{k+1} are related by $\gamma_k = 1/(n_{k+1} + \gamma_{k+1})$. If $\gamma_k \leqslant 1/\tau$ then the contribution of γ_k^{-2} to the product is at least τ^2. If instead $\gamma_k > 1/\tau$ then $\gamma_k \cdot \gamma_{k+1} = \gamma_{k+1}/(n_{k+1} + \gamma_{k+1}) = 1 - n_{k+1}\gamma_k \leqslant 1 - \gamma_k < 1 - 1/\tau = 1/\tau^2$ so the contribution of $1/\gamma_k^2\gamma_{k+1}^2$ to the product is at least τ^4. This proves the theorem.

Remark. There are infinitely many initial points γ in $(0, 1)$ with this Lyapunov exponent. For example, all the numbers $\gamma = [n_1, n_2, n_3, \dots, n_k, 1, 1, 1, 1, \dots]$, that is, all the numbers whose continued fractions ultimately end in 1's, have Lyapunov exponent $2 \ln \tau$. These are the so-called noble numbers (Schroeder [1984]), noticed for their resistance to chaos, and we see here that they all share the (still positive) minimum possible Lyapunov exponent under the Gauss map.

Ergodic results. The Gauss map is well-known in ergodic theory (see Billingsley [1963] or Mañé [1987]). The results are summarized here, for contrast with the results of the sections previous and following. This section is meant more as incentive for the reader to investigate ergodic theory than as exposition. The Gauss map preserves the Gauss measure

$$\mu(A) = \frac{1}{\ln 2} \int_A \frac{1}{1+x} \, d\lambda,$$

where λ is the Lebesgue measure. Thus the Gauss map is ergodic, and almost all (in the sense of either the Lebesgue or Gauss measure) initial points have orbits which have the interval $[0, 1]$ as ω-limit set. Thus the *only* attractor whose basin of attraction has nonzero measure is the interval $[0, 1]$. By the ergodicity of the map, we may explicitly calculate the Lyapunov exponent as follows:

$$\lambda(\gamma) = -2 \lim_{n \to \infty} \frac{1}{n} \left(\sum_{i=0}^{n} \ln(\gamma_i) \right) = \frac{-2}{\ln 2} \int_0^1 \frac{\ln(x)}{1+x} \, d\lambda = \frac{\pi^2}{6 \ln 2} = 2.3731 \dots,$$

which holds for almost all initial points γ. This is of interest, since there are few nontrivial maps for which the Lyapunov exponent can be calculated explicitly.

5. THE FLOATING-POINT GAUSS MAP. All of the results of the previous sections are valid for the familiar domain of the real numbers. However, when we work in any fixed-precision system, we have two difficulties:

1. Not all real numbers are even representable in the system, and
2. Arithmetic doesn't have the properties we are used to.

For example, defining $\mathbf{u}$ as the smallest machine representable number which when added to 1 gives a number different from 1 when stored, we see that $G(\delta)$ is computed as 0, whenever δ is any number between 0 and $\mathbf{u}$. This effectively limits the power of the singularity of the Gauss map.

To return to the analogy of the introduction, we consider the domain of machine representable numbers not as a smooth circle but as a slotted ring, with the number of slots on the ring corresponding to the number of machine-representable numbers in the interval $[0, 1)$, where all numbers in $[0, \mathbf{u})$ are "lumped together". In this analogy, $\mathbf{u}$ corresponds to the approximate width of the slots. Now our bead can only occupy one of the slots on the ring, and not just any arbitrary position, and the floating-point Gauss map takes the bead from one slot, winding around the ring as many times as are indicated by the integer part, and finally putting the bead into another slot. We see now that the maximum winding number of the floating-point Gauss map is finite, and the slot next to the origin is the one with this winding number.

A more evident difficulty is that all of the representable points are *rational*, and we know that the exact Gauss map takes these initial points to zero eventually. So

if we define a floating-point Gauss map as

$$\hat{G}(x) = \begin{cases} 0 & \text{if } x = 0 \\ \dfrac{1}{x} \bmod 1 & \text{otherwise.} \end{cases}$$

where now the operations of division and "mod 1" take place over the floating-point domain, with attendant roundoff error, we have to answer some new dynamical questions:

1. Are there any orbits which *don't* go to 0?
2. Is the behaviour of the floating-point Gauss map similar to the exact Gauss map? In particular, is $\hat{G}$ chaotic?
3. Can we define an appropriate Lyapunov exponent for this map?
4. Is numerical work with $\hat{G}$ useful at all for study of G?

Not surprisingly, some orbits under $\hat{G}$ do terminate at 0, though often not when we expect them to. However, on some machines, some orbits never hit 0, being periodic. For example on the HP28S the initial point $\gamma_0 = 0.3$ gives $\gamma_1 = 0.3333333333$, $\gamma_2 = 0.0000000003$, and $\gamma_3 = 0.3 = \gamma_0$, with period 3. Note that under the exact Gauss map the second iterate (γ_2) of this initial point is zero. Since the set of machine-representable numbers is finite, *all* orbits are ultimately periodic (perhaps with period 1, as at $x = 0$). Note that the behaviour of $\hat{G}$ depends strongly on the floating-point implementation. For example, with the Apple SANE numerics implementation, the starting point $\gamma_0 = 0.3$ gives an orbit with either a long transient regime or a long period, with no regularity detected in the first 65,000 elements of the orbit.

Since all orbits are ultimately periodic, and there are only a finite number of such orbits, the floating-point Gauss map (and indeed *any* machine simulation of *any* dynamical system) is **not** chaotic in the usual sense. Arbitrarily small perturbations in the initial conditions are not even allowed, so the sensitivity of the map to such perturbations is moot. The definition of the Lyapunov exponent for the exact Gauss map seems not to be relevant here: the presence of the derivative $G'(x)$ in the definition of Lyapunov exponent measures the effect of such arbitrarily small perturbations. However, if we define an approximate Lyapunov exponent for the first N iterations of the floating point Gauss map as

$$\lambda_N(\gamma) = \frac{1}{N}\ln\left(\prod_{i=0}^{N}|\hat{G}'(\gamma_i)|\right),$$

whenever the elements of the orbit are nonzero, then this in some sense measures the average sensitivity of the first N elements of the corresponding orbit under the *exact* Gauss map to arbitrarily small perturbations. This "Lyapunov exponent" is what is calculated in practice for a great many numerical simulations of dynamical systems, and if it is positive this is taken as evidence for chaos in the underlying system (Guckenheimer and Holmes [1985]).

But what if the calculated orbit has no counterpart in the exact system? If roundoff errors introduced into the calculation produce an orbit that is unlike any in the exact system, this approximate Lyapunov exponent would be spurious. We will give a proof in the following section, which uses the techniques of backward error analysis, that shows orbits under the floating-point Gauss map are "machine close" to corresponding orbits under the exact Gauss map. A general theorem of this nature has been proved for hyperbolic invariant sets, by Bowen (Guckenheimer

& Holmes [1985]). Here a direct proof is more appropriate and informative. This means that the approximate Lyapunov exponent defined above will accurately reflect the Lyapunov exponent of *some* orbit of the exact Gauss map, provided N is large enough that transient effects have been minimized, and not so large that accumulated roundoff error in the sum degrades the result.

We contrast this behaviour with what happens when continuous maps are made discrete by (e.g.) finite difference schemes. Yamaguti & Ushiki [1981] and Ushiki [1982] have shown that finite difference formulae applied to non-chaotic continuous systems may produce chaotic numerical solutions if the stepsize h is not too small, assuming the calculations are carried out *exactly*. In contrast we have shown here that a chaotic discrete map becomes nonchaotic when implemented in fixed-precision arithmetic.

A further difficulty is that all of the orbits of $\hat{G}$ are ultimately periodic, and periodic orbits of G have Lyapunov exponents that are different from the almost-everywhere value (which is usually the exponent of physical interest). It is not immediately clear that these Lyapunov exponents calculated from $\hat{G}$ will tell us anything useful about the exact map G.

On closer examination, however, we see that if the period of an orbit is long, then the orbit behaves for a long time as if it were aperiodic, reflecting the effect of "nearby" initial points that are aperiodic. Hence we may expect that the computed Lyapunov exponent of a long period orbit will be close to $(\pi^2/6)\ln 2 = 2.373\ldots$. This is what happens in practice, since many initial points seem to give long period orbits. For example, if we compute the first 100,000 elements of the orbit of 0.73 under $\hat{G}$ on the HP28S, we get a computed $\lambda = 2.36992$. This is within 0.2% of the expected value of the Lyapunov exponent of the exact Gauss map. Notice, though, that the Lyapunov exponent of the orbit of the exact map G starting at 0.73 is not even *defined*—we *rely* on the roundoff error to give us our results, which is somewhat unusual. We will expand more on this in a later section.

Orbits under $\hat{G}$ are close to orbits under G. The following theorem justifies the remarks of the previous section. The basic idea of its proof is that given some initial point $\hat{y}$ the floating-point Gauss map also generates an initial point y whose continued fraction is exactly equal to $[a_1, a_2, a_3, \ldots]$, where the a_k are all (machine representable) integers. This initial point y has a G-orbit that is everywhere within a small multiple of $\mathbf{u}$, the machine epsilon, of the $\hat{G}$-orbit of $\hat{y}$. The technique of the proof is of interest for more than just the Gauss map, because similar techniques can be used to prove that numerical simulations of orbits of some continuous systems are machine close to exact orbits of some nearby initial point (for a descriptive review of work by Yorke, Grebogi, and Hammel establishing similar results for continuous maps see Cipra [1988]).

Theorem. *If $x_0, x_1, x_2, x_3, \ldots$ is the sequence of iterates of $\hat{G}$, and $a_1, a_2, a_3, \ldots$ is the sequence of (machine representable) integers that arise in the process, then $y = [a_1, a_2, a_3, \ldots]$ has an orbit under G whose elements are close to $x_0, x_1, x_2, x_3, \ldots$ in a sense to be made precise, and, in particular, y is close to x_0.*

We will show first that we may approximate an element of the orbit of y by a certain rational number. We then show, using a common model of floating-point arithmetic, that the corresponding x_k is "machine close" to this same rational number. This last will be seen to depend on the fact that if you run the Gauss map backwards, errors are damped instead of amplified.

Proof: Consider $y_k = [a_{k+1}, a_{k+2}, a_{k+3}, \ldots]$. The rational numbers $p_n/q_n = [a_{k+1}, a_{k+2}, a_{k+3}, \ldots, a_{k+n}]$ satisfy

$$\left| y_k - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}$$

and $q_n \geqslant F_n$ where F_n is the nth Fibonnacci number, from elementary properties of simple continued fractions (see Olds [1963] or Hardy and Wright [1979] for details). This means that given an $\varepsilon > 0$, we can find an n so that $|y_k - p_n/q_n| < \varepsilon$.

To prove the second part, we use the common model of floating-point division that states that if the floating point numbers a, b, and c satisfy $a \div b = c$, where the division takes place over the floating-point numbers, then there is a number δ with $|\delta| < \mathbf{u}$ so that $c(1 + \delta) = a/b$ exactly. Note that we do not model the addition, since this will be seen to be unnecessary.

If the orbit $x_0, x_1, x_2, x_3, \ldots$ has been produced by a floating-point system satisfying this model, then for each n there is a number δ_{k+n} with $|\delta_{k+n}| < \mathbf{u}$ such that

$$(1 + \delta_{k+n}) x_{k+n} = \frac{1}{a_{k+n+1} + x_{k+n+1}},$$

where we may consider the addition as exact, since a_{k+n+1} is a machine representable integer, defined by this process, and x_{k+n+1} is a machine representable floating point number. If we put $\varepsilon_{k+n+1} = x_{k+n+1}/a_{k+n+1}$ then we have

$$(1 + \varepsilon_{k+n+1})(1 + \delta_{k+n}) x_{k+n} = \frac{1}{a_{k+n+1}}.$$

Now put $z_{k+m} = [a_{k+m+1}, a_{k+m+2}, a_{k+m+3}, \ldots, a_{k+n+1}]$ for $m = 1, 2, \ldots, n$, and put $\varepsilon_{k+m} = z_{k+m} - x_{k+m}$ for $m = 0, 1, 2, \ldots, n$. Note that $\varepsilon_k = z_k - x_k$ is the error we wish to estimate, since by the first part we can estimate the error $z_k - y_k$. So now

$$\begin{aligned}(1 + \delta_{k+m}) x_{k+m} &= \frac{1}{a_{k+m+1} + x_{k+m+1}} = \frac{1}{a_{k+m+1} + z_{k+m+1} - \varepsilon_{k+m+1}} \\ &= z_{k+m} \cdot \frac{1}{1 - \varepsilon_{k+m+1} \cdot z_{k+m}}\end{aligned}$$

from whence, on cross-multiplying and expanding, we get the recurrence relation

$$\varepsilon_{k+m} = \delta_{k+m} x_{k+m} - (1 + \delta_{k+m}) z_{k+m} x_{k+m} \varepsilon_{k+m+1}$$

from which we may derive an upper bound on $\varepsilon_k = z_k - x_k$, and we note at this point that z_k is one of the rationals which approximates y_k. Note that the first term in this recurrence relation is essentially the roundoff error introduced at this particular step, while the second term is the error from one level below in the continued fraction, multiplied by a "shrinkage factor" $z_{k+m} x_{k+m}$.

As in the proof that τ has the minimum Lyapunov exponent, we are unable to say anything useful about z_{k+m} directly, but we are able to bound $z_{k+m} z_{k+m+1}$, which is easily shown to be less than $1/2$. With some simple estimates on the

above recurrence this gives

$$\varepsilon_{k+m} \leqslant \begin{cases} 4\mathbf{u} + \dfrac{1-4\mathbf{u}}{2^{(n+1-m)/2}} & n-m \text{ is odd} \\ 4\mathbf{u} + \dfrac{1-3\mathbf{u}}{2^{(n-m)/2}} & n-m \text{ is even} \end{cases}$$

and since as $n \to \infty$, $z_k \to y_k$, we have at last

$$|x_k - y_k| \leqslant 4\mathbf{u}.$$

Thus there is a nearby initial point y_0 whose orbit under G follows as near as can be expected the computed orbit $x_0, x_1, x_2, x_3, \ldots$ of the floating-point Gauss map.

Our earlier example of $x_0 = 0.3$ gave a periodic orbit on the HP28S, which has $\mathbf{u} = 10^{-11}$. The nearby initial point with this orbit under $\hat{G}$ is

$$\begin{aligned} y &= [3, 3, 3333333333, 3, 3, \ldots] \\ &= \frac{1}{2}\left(\sqrt{111111111112888888889} - 33333333333\right) \\ &= 0.3 + .2999999999976 \cdot 10^{-12} + \cdots. \end{aligned}$$

As a further curiosity, we note that the machine representation of $1/\tau$ on the HP28S is an actual fixed point of $\hat{G}$, allowing us to calculate the exact continued fraction of $1/\tau$ on a finite machine.

A New method for calculating π. The observation that we can get an approximate value for the Lyapunov exponent of the exact Gauss map by calculating the average exponent from the first N elements of a numerically generated orbit gives us a new and interesting, though completely impractical, method for calculating π. We simply choose some initial point more or less at random, say $x_0 = 0.73$, and produce the first N iterates under the floating-point Gauss map, and accumulate the average Lyapunov exponent. At the end, this is supposed to be close to the exact almost-everywhere Lyapunov exponent of the exact Gauss map, $\pi^2/6\ln 2 = 2.373\ldots$. Well, if we know $\ln 2$ and can take square roots, this gives us the value of π. Using the HP28S and 100,000 iterates of the floating-point Gauss map with the above initial point, we get $\pi \approx 3.13945$. Note that this method *relies* on roundoff error, since without it this orbit terminates!

Remarks. This method is likely worse than nearly any other in existence, since it does **not** converge to the correct value in any particular fixed-precision system, since all orbits are ultimately periodic, and the Lyapunov exponent of a periodic orbit is the logarithm of an algebraic number, which can't be π^2 unless e^{π^2} is an algebraic number[2]. Yet this qualifies as a genuine method, since in principle you could implement higher and higher precision floating-point systems and achieve the desired accuracy by longer and longer runs with this high-precision arithmetic. Of course this is impractical, perhaps even ridiculous. There is also the problem of choosing "good" initial points—if we are lucky, the first initial point we choose for whatever floating-point system we have will do the trick—but there is no guarantee, and indeed the computed Lyapunov exponent may converge to something totally different (or worse, something only slightly different).

[2]This is a well known unsolved problem.

This method is clearly related to the Monte Carlo methods, with the roundoff error associated with the floating-point arithmetic playing the part of the random number generator required. The author knows of no other case in mathematics where roundoff error plays a useful role in an actual calculation.

6. CONCLUSIONS. The Gauss map has been shown to be a good example of a chaotic discrete dynamical system, in that it exhibits in an accessible fashion all the common features of such systems. The map is simple enough that the relationship of numerical simulation of the map to the exact map can be explored effectively. We find that the numerical simulation of the map behaves significantly differently, in that the numerical simulation is not chaotic, but is still useful in that the Lyapunov exponent of the exact map can be accurately calculated from the simulation. We have in fact shown that this behaviour of numerical simulation is general. We have also exhibited a new (though impractical) method for the calculation of π.

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Civilization advances by extending the number of important operations which we can perform without thinking.

—*Alfred North Whitehead*

A Strengthening of the Schwarz-Pick Inequality

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1. THE RESULT. The unit disc Δ in the complex plane carries the hyperbolic metric ρ derived from the line element $2|dz|/(1-|z|^2)$: for example,

$$\rho(0,z) = \log\left(\frac{1+|z|}{1-|z|}\right),$$

or, equivalently,

$$\tanh\left(\tfrac{1}{2}\rho(0,z)\right) = |z|. \tag{1}$$

The (orientation preserving) isometries for this metric are precisely the conformal self maps of the disc, these are the Möbius transformations

$$z \mapsto \frac{az+\bar{c}}{cz+\bar{a}}; \qquad |a|^2 - |c|^2 = 1.$$

The well-known Schwarz-Pick lemma asserts that if f is an analytic map of the unit disc into itself, then f is either an isometry, or a strict contraction, relative to the metric ρ. We shall show how a simple argument gives the following stronger version of this classical result.

Theorem. *Let $f\colon \Delta \to \Delta$ be analytic. Then, for all z and w in Δ,*

$$\rho(f(z), f(w)) \le \log(\cosh \rho(z,w) + \|f'(w)\| \sinh \rho(z,w)). \tag{2}$$

Here, $\|f'(w)\|$ is the hyperbolic change of scale of f at w; that is,

$$\|f'(w)\| = \frac{|f'(w)|(1-|w|^2)}{1-|f(w)|^2}.$$

As $\|f'(w)\| \le 1$, the right hand side of (2) is at most $\rho(z,w)$ and this recaptures the classical inequality. However, if f is not an isometry, then $\|f'(w)\| < 1$ and the right hand side of (2) shows how a particular value of $\|f'(w)\|$ exerts a stronger, global, influence on the contracting effect of f: this idea is illustrated in the next section.

2. AN APPLICATION. Consider any analytic map $f\colon \Delta \to \Delta$ with $f(0) = 0$ and f not an isometry. Then the classical Schwarz lemma shows that $|f'(0)| < 1$ so 0 is an attractive fixed point. The stronger version of the Schwarz-Pick inequality given above enables us to draw the stronger conclusion that the iterates f^n of f converge *uniformly* to 0 on compact subsets of Δ. (This result is usually shown by using a normal families argument.)

For, if we define a function $\phi\colon [0,\infty) \to [0,\infty)$ by

$$\phi(R) = \log(\cosh R + \|f'(0)\| \sinh R),$$

then ϕ is strictly increasing with $\phi(R) < R$ for $R > 0$ and 0 as the only fixed point.

Therefore the iterates $\phi^n(R)$ decrease and tend to 0 as n increases for each R. The inequality (2) shows that, if $\rho(0, z) \le R$, then

$$\rho(0, f(z)) \le \phi(R)$$

and, by induction,

$$\rho(0, f^n(z)) \le \phi^n(R).$$

Thus f^n converges uniformly to 0 on the disc $\{z: \rho(0, z) \le R\}$.

Although f^n need not converge uniformly on all of Δ, the inequality (2) does allow us to make a uniform statement about the global convergence on Δ. For, if $|z|$ is nearly 1, then

$$\rho(0, f(z)) \le \phi(\rho(0, z)) \simeq \rho(z, 0) - k$$

where $k = \log(2/(1 + |f'(0)|))$. This shows that, while z is near to the boundary of Δ, each application of f moves z towards the origin by at least a *fixed hyperbolic distance*.

3. THE PROOF. First let g be any analytic map of Δ into itself. Then

$$\begin{aligned}\rho(0, g(z)) &\le \rho(0, g(0)) + \rho(g(0), g(z)) \\ &\le \rho(0, g(0)) + \rho(0, z),\end{aligned}$$

and, applying the function $x \mapsto \tanh \frac{1}{2}x$ to both sides of this inequality, we obtain

$$|g(z)| \le \left(\frac{r + a}{1 + ar}\right), \tag{3}$$

where $a = |g(0)|$ and $r = |z|$.

We shall now establish (2). Without loss of generality we may assume that $w = 0 = f(w)$. Consider any analytic map f of Δ into itself such that $0 = f(0)$. The map g defined by $g(z) = f(z)/z$ maps Δ into itself and, applying (3), we obtain

$$|f(z)| \le r\left(\frac{r + a}{1 + ar}\right),$$

where now we have $a = |f'(0)|$. Using this, the inequality (2) follows directly from (1).

In conclusion, we remark that equality holds in (2) if, and only if, f is either an isometry, or a Blaschke product of degree two with w lying on the geodesic segment between z and the critical point of f. To see this, observe first that equality holds in (2) if, and only if, there is equality at all stages in the argument above. The case where f is an isometry (and g is constant) is trivial. Otherwise we must have $\rho(g(0), g(z)) = \rho(0, z)$ and $g(0)$ lying on the geodesic segment $[0, g(z)]$. This holds if, and only if, g is an isometry and 0 lies on the geodesic segment $[g^{-1}(0), z]$. If g is an isometry then

$$f(z) = z\left(\frac{az + \bar{c}}{cz + \bar{a}}\right), \qquad |a|^2 - |c|^2 = 1.$$

The assertion concerning the critical point now follows by computation.

Note that, if R is any hyperbolic Riemann surface, then any universal covering map π: $\Delta \to R$ enables us to transfer the hyperbolic metric to R. The theorem above then clearly applies to analytic maps between any two hyperbolic Riemann surfaces when they are each given the hyperbolic metric.

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A Simple Proof for Sturm's Separation Theorem

Géza Makay

Consider the second-order linear differential equation

$$y'' + f(x)y' + g(x)y = 0, \tag{1}$$

where $f, g\colon \mathbf{R} \to \mathbf{R}$ are continuous. Sturm's theorem says that the zeros of two linearly independent solutions of (1) separate one another. The standard proof of this theorem is based upon some properties of the *Wronsky determinant* (see e.g. [**1**, p. 124]). In this note we present a proof using only the elementary calculus.

Theorem A. *Let y_1, y_2 be linearly independent solutions of equation* (1). *If ξ, η ($\xi < \eta$) are successive zeros of y_1, then y_2 has one and only one zero in the interval (ξ, η).*

Proof: Suppose the contrary. Then we can assume, without loss of generality, that $y_1(x) > 0$ and $y_2(x) > 0$ for all $x \in (\xi, \eta)$ (if y is a solution, then $-y$ is also a solution). Since y_1, y_2 are linearly independent solutions, $y_2(x) > 0$ for all $x \in [\xi, \eta]$, too. Define the set

$$\mathscr{C} = \{c \in \mathbf{R}\colon y_1(x) \le cy_2(x) \text{ for every } x \in [\xi, \eta]\}.$$

Since y_1 is bounded above and y_2 is bounded away from zero, $\mathscr{C}$ is not empty. Let $c_0 = \inf \mathscr{C}$. It is easy to see that $y_1(x) \le c_0 y_2(x)$ on the interval $[\xi, \eta]$ and there is an $x_0 \in (\xi, \eta)$ with $y_1(x_0) = c_0 y_2(x_0)$. Obviously, $y_1'(x_0) = c_0 y_2'(x_0)$; therefore, solutions $y_1, c_0 y_2$ satisfy the same initial conditions at x_0. By uniqueness we obtain $y_1 = c_0 y_2$, a contradiction to the linear independence of y_1, y_2.

Since the role of y_1 and y_2 can be changed, the zero of y_2 on (ξ, η) is unique.

This proof works also for some nonlinear equations.

Theorem B. *Suppose that the second-order equation*

$$F(y'', y', y, x) = 0 \quad (F\colon \mathbf{R}^4 \to \mathbf{R} \textit{ continuous}) \tag{2}$$

satisfies the following conditions:

(a) *If y is a solution of* (2), *then cy is also a solution for all $c \in \mathbf{R}$*;

(b) *the solutions of initial value problems for* (2) *are unique*.

Then Theorem A is true for the equation (2).

Example. Consider the second order nonlinear equation

$$y''(y')^2 + y^3 = 0. \tag{3}$$

Let the function S_0 be defined by

$$x = \int_0^{S_0(x)} \frac{ds}{\sqrt[4]{1 - s^4}}$$

for $x \in [0, \hat{\pi}/2]$, where

$$\hat{\pi} := \frac{\pi/2}{\sin(\pi/4)}.$$

It can be seen [**3**] that the $2\hat{\pi}$-periodic function

$$S(x) := \begin{cases} S_0(x), & \text{if } 0 \leq x < \hat{\pi}/2 \\ S_0(\hat{\pi} - x), & \text{if } \hat{\pi}/2 \leq x < \hat{\pi} \\ -S_0(x - \hat{\pi}), & \text{if } \hat{\pi} \leq x < 3\hat{\pi}/2 \\ -S_0(2\hat{\pi} - x), & \text{if } 3\hat{\pi}/2 \leq x < 2\hat{\pi} \end{cases}$$

is the solution of equation (3) satisfying the initial condition $S(0) = 0$, $S'(0) = 1$. The function S can be considered as a generalization of function sin. Since the equation is autonomous and it satisfies conditions (a) and (b) in Theorem B (see [**2**]), every solution is of the form

$$y(x) = cS(x + \alpha) \quad (c, \alpha \in \mathbf{R}).$$

According to Theorem B, the zeros of two linearly independent solutions of equation (3) separate one another.

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Major Theorems on Compactness: a Unified Exposition

Jerzy Dydak and Nathan Feldman[1]

The purpose of this article is to present a unified approach to the following, seemingly unrelated, four major results on compactness: the Stone-Weierstrass Theorem, the Tychonoff Theorem, the Stone-Čech compactification (more generally, classification of all compactifications of Tychonoff spaces) and the Tietze-Urysohn Extension Theorem. It originated from the authors' realization that typical proofs of these results encountered in textbooks (see [**1**] or [**2**]) are quite tricky even though they may be short, as in the case of the Stone-Weierstrass Theorem. At the same time, there are interconnections between those basic theorems: it is well known that the Tychonoff Theorem is a consequence of the properties of the Stone-Čech compactification (see [**2**]), but less well known that Stone-Čech compactification plus the Stone-Weierstrass Theorem imply the Tietze Extension Theorem (see [**3**] and our proof in this paper). This makes one wonder if we can prove (in a natural way) a single result which would imply the four theorems. This was precisely our goal when we started to collaborate during the Research Experience for Undergraduates Program at the University of Tennessee (Summer 1989). We think that our solution to the problem offers these essential benefits to students:

a. they see a miniature theory at work,
b. they can appreciate the power of functorial approach to problems in topology,
c. the proofs and constructions are a series of logical steps rather than effective but unexpected tricks,
d. demonstrate that the main theorem, by implying the four major theorems, is a good lesson in the pyramidal structure of mathematics,
e. *one can easily* convert this text to a collection of problems in classes where the Moore Method is used.

This article is organized as follows: the first part is devoted to stating and proving the main theorem. Then the proofs of four major theorems follow.

The only nontrivial fact used (and a standard proof of it can be found in any textbook on topology) is the following:

Urysohn Lemma. *Given two disjoint closed subsets A and B of the normal space X there is a continuous function $f\colon X \to [0,1]$ with $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$.*

[1]This work was done while N. Feldman participated in the Research Experience for Undergraduates Program at the University of Tennessee (summer 1989).

Suppose X is a topological space. Our goal is to understand all the maps from X to compact Hausdorff spaces. Given such a map $f\colon X \to Y$, the closure Y' of $f(X)$ is also compact Hausdorff, so our original goal can be reduced to maps $f\colon X \to Y$ such that $cl(f(X)) = Y$. Then f induces a function f^* from the set $C(Y)$ of all real-valued continuous maps on Y to the set $C^*(X)$ of all bounded real-valued continuous maps on X via the formula $f^*(g) = g \circ f$. The image of f^* is denoted by P_f.

1. Proposition. *P_f is a closed subalgebra of $C^*(X)$ and $f^*\colon C^*(Y) \to P_f$ is an isometry of algebras.*

Proof: Obviously, $f^*\colon C^*(Y) \to P_f$ is onto. The reason it is an isometry is because $f(X)$ is dense in Y, and $|g \circ f(x) - g' \circ f(x)| \leqslant a$ for all $x \in X$ implies $|g(y) - g'(y)| \leqslant a$ for all $y \in cl(f(X)) = Y$. Now P_f is complete (as isometric to a complete space $C^*(Y)$), so it is closed in $C^*(X)$. □

Thus each map $f\colon X \to Y$ with $cl(f(X)) = Y$ being compact Hausdorff, selects a closed subalgebra P_f of $C^*(X)$ which is isomorphic to $C^*(Y)$ and contains all the constant functions. The following theorem essentially means that P_f is all we need to know to identify the map f:

Main Theorem. *Suppose P is a closed subalgebra of the algebra $C^*(X)$ of bounded real-valued continuous functions on a topological space X such that $1 \in P$. Then there is a compact Hausdorff space $\mathscr{M}(P)$ and a map $i_P\colon X \to \mathscr{M}(P)$ such that the function $i_P^*\colon C^*(\mathscr{M}(P)) \to C^*(X)$ given by $i_P^*(g) = g \circ i_P$ is one-to-one and its image is P. The space $\mathscr{M}(P)$ is unique in the following sense: for each map $f\colon X \to Y$ with $cl(f(X)) = Y$ being compact Hausdorff and $P_f = P$, there is a homeomorphism $h\colon Y \to \mathscr{M}(P)$ with $h \circ f = i_P$.*

Moreover, if $f\colon X \to Y$ is a map and Q is a closed subalgebra of $C^*(Y)$ such that for any $g \in Q$ the composition $g \circ f$ belongs to P and $1 \in Q$, then there is a unique map $f_*\colon \mathscr{M}(P) \to \mathscr{M}(Q)$ making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle i_P} & & \downarrow{\scriptstyle i_Q} \\ \mathscr{M}(P) & \xrightarrow{f_*} & \mathscr{M}(Q) \end{array}$$

commutative.

Notice that $i_P(X)$ must be dense in $\mathscr{M}(P)$ (otherwise, there would be a nonconstant map $\alpha\colon \mathscr{M}(P) \to R$ vanishing on $i_P(X)$, contradicting i_P^* being one-to-one). To simplify the notation, put $Y = \mathscr{M}(P)$ and $i = i_P$. How does one create $Y = \mathscr{M}(P)$ out of P? Notice that each $y \in Y$ can be identified with the subalgebra $\tau_y := \{g \in C^*(Y) | g(y) = 0\}$ of $C^*(Y)$. Thus, Y can be replaced as a set by $Y' := \{\tau_y | y \in Y\}$. In order to assign a topology to Y' (so as to make Y' homeomorphic to Y) notice that the family $\alpha^{-1}(R - \{0\})$, $\alpha \in C^*(Y)$, is a basis of the topology of Y. Thus, one needs $\{\tau_y \in Y' | y \in \alpha^{-1}(R - \{0\})\}$, $\alpha \in C^*(Y)$ to be a basis of the topology on Y'. Notice that $\{\tau_y \in Y' | y \in \alpha^{-1}(R - \{0\})\} = \{\tau_y \in Y' | \alpha(y) \neq 0\} = \{\tau \in Y' | \alpha \notin \tau\}$. Thus, our task of identifying Y' (and therefore Y) will be completed once we can isolate sets $\{\alpha \cdot i | \alpha \in \tau_y\}$, $y \in Y$, among all of

subalgebras of P. Indeed, the topology on Y' will be given by stipulating that $N(\alpha) := \{\text{subalgebras not containing } \alpha\}$, $\alpha \in P$, form a basis.

The obvious feature of all the functions in τ_y is that they have the same root (namely, y). The trouble is $\{\alpha \circ i | \alpha \in \tau_y\}$ may not have this property if $y \in Y - i(X)$. Notice, however, that each function in $\{\alpha \circ i | \alpha \in \tau_y\}$ has values arbitrarily close to 0. This leads to the following:

2. *Definition*. Given a subalgebra P of $C^*(X)$ let $\mathscr{M}(P)$ be the set of all subalgebras τ of P which are maximal with respect to the following property:

($*$) given $\alpha \in \tau$ the set $\alpha^{-1}(-\varepsilon, \varepsilon)$ is not empty for all $\varepsilon > 0$.

Remark. In [2] the maximal ideals of P are considered. We think that one arises at Condition ($*$) more naturally and the proofs are easier in this case.

3. *Example*. $\tau_a := \{f \in P | f(a) = 0\} \in \mathscr{M}(P)$.

Proof: Obviously τ_x satisfies Condition ($*$). If $\alpha(a) \neq 0$, then $\beta(x) := \alpha(x) - \alpha(a) \in \tau_a$ and $(\alpha^2 + \beta^2)^{-1}(-\varepsilon, \varepsilon) = \varnothing$ for $\varepsilon < \alpha(a)^2/4$. Thus, τ_a is maximal. □

4. Proposition. *If X is compact, then $\mathscr{M}(P) = \{\tau_x | x \in X\}$.*

Proof: For any finite number $f_1, \ldots, f_k$ of elements of $\tau \in \mathscr{M}(P)$, the sum $\sum_{i=1}^k f_i^2$ belongs to τ and attains its absolute minimum m at $x \in X$. $m > 0$ would contradict condition ($*$), so $m = 0$. Thus $f_1, \ldots, f_k$ possess a mutual root and, by compactness of X, all the functions in τ have a mutual root. □

5. *Definition*. Given $\alpha \in P$ let $N(\alpha) = \{\tau \in \mathscr{M}(P) \colon \alpha \notin \tau\}$.

6. Proposition. $N(f) \cap N(g) = N(f \cdot g)$.

Proof: This is equivalent to $\mathscr{M}(P) - N(f \cdot g) = (\mathscr{M}(P) - N(f)) \cup (\mathscr{M}(P) - N(g))$, which is the same as the equivalence of $f \cdot g \in \tau$ with ($f \in \tau$ or $g \in \tau$). This may be recognized as saying that τ is a prime ideal of P. If $f \in \tau$ and $g \in P$, we choose $M > 1$ such that $|g(x)| < M$ for all $x \in X$. Given $\varepsilon > 0$, $a \in R$ and $h \in \tau$ there is x_0 such that $f^2(x_0) + h^2(x_0) < \min(\varepsilon^2/4M^2a^2, \varepsilon^2/4)$ (since $f, h \in \tau$). Then $|(af \cdot g + h)(x_0)| < \varepsilon$ which means that the subalgebra $\{af \cdot g + h \mid g \in P, h \in \tau, a \in R\}$ satisfies condition ($*$). Thus, $\tau \supset \{af \cdot g + h \mid g \in P, h \in \tau, a \in R\}$ and $f \cdot g \in \tau$. Assume $f \cdot g \in \tau$ but $f \notin \tau$ and $g \notin \tau$. Then, the subalgebra $\{af + h \mid h \in \tau, a \in R\}$ does not satisfy condition ($*$) and $\inf_{x \in X}\{|af(x) + h(x)|\} > 0$ for some $h \in \tau$ and $a \in R$. Similarly, $\inf_{x \in X}\{|bg(x) + h'(x)|\} > 0$ for some $h' \in \tau$ and $b \in R$. Therefore, $\inf_{x \in X}\{|(af(x) + h(x)) \cdot (bg(x) + h'(x))|\} > 0$ contradicting $(af + h) \cdot (bg + h') = abfg + afh' + bgh + hh' \in \tau$ (recall that τ is an ideal). ■

Proposition 6 implies that we can form a topology out of the family $\{N(f)\} f \in P$.

7. Proposition. a. *$\mathscr{M}(P)$ with the topology $\{N(\alpha) | \alpha \in P\}$ is compact.*

b. *$i_P \colon X \to \mathscr{M}(P)$ defined by $i_P(x) = \tau_x$ is continuous and $i_P(X)$ is dense in $\mathscr{M}(P)$.*

c. *If $1 \in P$ and P separates the points of X (which means that for any two points $x \neq y$ in X there is $\alpha \in P$ with $\alpha(x) \neq \alpha(y)$), then i_P is one-to-one. Moreover, if*

$(\alpha^{-1}(R-\{0\})_{\alpha\in P}$ *is a basis of* X (*e.g.*; X *is Tychonoff and* $P = C^*(X)$), *then* i_P: $X \to i_P(X)$ *is a homeomorphism.*

Proof: a. Suppose $\mathscr{M}(P) = \bigcup_{s\in S} N(\alpha_s)$ and $\mathscr{M}(P) - \bigcup_{s\in A} N(\alpha_s) \neq \varnothing$ for each finite subset A of S. This implies that $\{\alpha_s | s \in A\}$ is contained in some $\tau_A \in \mathscr{M}(P)$ for all finite subsets A of S. Let τ be the subalgebra of P generated by $\{\alpha_s | s \in S\}$. Since each element of τ is contained in some τ_A, τ satisfies condition $(*)$. Thus, $\tau \subset \tau' \in \mathscr{M}(P)$ and $\tau' \in \mathscr{M}(P) - \bigcup_{s\in S} N(\alpha_s)$, a contradiction.

b. Notice that $i_P^{-1}(N(\alpha)) = \alpha^{-1}(R-\{0\})$, so i_P is continuous. Suppose $\alpha \in P$ and $N(\alpha) \cap i_P(X) = \varnothing$. Then $\alpha \in \tau_x$ for all $x \in X$, which means $\alpha \equiv 0$. In such a case $\alpha \in \tau$ for all $\tau \in \mathscr{M}(P)$ and $N(\alpha) = \varnothing$. Thus, $i_P(X)$ is dense in $\mathscr{M}(P)$.

c. If $1 \in P$, then all the constant functions belong to P (P is a vector subspace of $C^*(X)$). Now, P separating the points of X means $\tau_x = \tau_y$ is equivalent to $x = y$. Thus, i_P is one-to-one. Notice that $i_P^{-1}(N(\alpha)) = \alpha^{-1}(R-\{0\})$ means $N(\alpha) \cap i_P(X) = i_P(\alpha^{-1}(R-\{0\})$ if i_P is one-to-one. Thus, if $\{\alpha^{-1}(R-\{0\})_{\alpha\in P}$ is a basis of X, then i_P: $X \to i_P(X)$ is open. □

8. *Examples.* a. i_P: $X \to \mathscr{M}(P)$ is a homeomorphism for any compact Hausdorff space X and $P = C^*(X)$.

b. i_P: $[a,b] \to \mathscr{M}(P)$ is a homeomorphism for any closed subalgebra P of $C^*[a,b]$ containing all the polynomials.

Proof: a. By Proposition 4, i_P is onto and by Proposition 7, i_P: $X \to i_P(X) = \mathscr{M}(P)$ is a homeomorphism.

b. Since $[a,b]$ and $[-1,1]$ are homeomorphic via a linear function we may assume $a = -1$ and $b = 1$ (unless $a = b$ in which case there is nothing to prove). In view of Proposition 7 it suffices to show that for any interval (c,d) there is $\alpha \in P$ with $\alpha^{-1}(R-\{0\} = (c,d) \cap [-1,1]$. The special case $c = 0$, $d = 2$ is taken care by:

Claim. The function p: $[-1,1] \to R$, $p(x) = x + |x|$, is the uniform limit of some polynomials p_n.

Proof of claim. Notice that $\lim_{n\to\infty} x^{2n}$ is 0 if $x^2 \neq 1$ and is 1 if $x^2 = 1$. Our method of constructing the sequence of polynomials is to improve the sequence x^{2n} so as to make it convergent to p. We start with $p_1 = x^2$ and we want $p_{n+1} = p_n \cdot (1 + q_n)$. We need the sequence $\{p_n(x)\}_{n\geqslant 1}$ to be increasing for $x > 0$ and decreasing for $x < 0$. Thus, we need $q_n \geqslant 0$ on $[0,1]$ and $q_n \leqslant 0$ on $[-1,0]$. Also, $|q_n|$ should be small on $[0,1]$ (to insure convergence of $p_n(x)$ to x) and not so small on $[-1,0]$. Our approach hands out such a function: $x - p_n(x)$. For technical reasons we define q_n as $(x - p_n(x))/2$. Then

$$x - p_{n+1}(x) = (x - p_n(x)) \cdot (1 - p_n(x)/2)$$

and $x \geqslant p_{n+1} \geqslant 0$ on $[0,1]$. Also, $p_n \leqslant 1$ implies $1 \geqslant q_n \geqslant -1$ and $p_{n+1} \geqslant 0$ on $[-1,0]$. Thus the sequence $\{p_n(x)\}_{n\geqslant 1}$ is increasing for $x > 0$ and decreasing for $x < 0$. Put $q(x) = \lim_{n\to\infty} p_n(x)$. Then $q(x) = q(x) \cdot (1 + (x - q(x))/2)$, so $q(x) = 0$ for $x \leqslant 0$ and $q(x) = x$ for $x \geqslant 0$. It remains to show that p_n approaches p uniformly. Given $1 > \varepsilon > 0$ notice that $|p(x) - p_n(x)| < \varepsilon$ for $|x| < \varepsilon$. If $x > \varepsilon$, then $x - p_{n+1}(x) = (x - p_n(x))(1 - p_n(x)/2) \leqslant (x - p_n(x))(1 - \varepsilon^2/2)$ (because $p_1(x) = x^2 \leqslant p_n(x)$). Also, for $x < -\varepsilon$ we have $p_{n+1}(x) = p_n(x)(1 + q_n(x)) \leqslant p_n(x)(1 - \varepsilon/2)$ as $x - p_n(x) \leqslant x < -\varepsilon$. Thus, for n sufficiently large, $|p(x) - p_n(x)| < \varepsilon$ for all x.

Given c and d the map $\alpha(x) := p(x-c)\cdot p(d-x)$ satisfies $\alpha^{-1}(R - \{0\} = (c,d) \cap [-1,1]$. $\square$

We now turn to functorial properties of our construction. Given a map f: $X \to Y$, a subalgebra P of $C^*(X)$ and a subalgebra Q of $C^*(Y)$ such that for any $g \in Q$ the composition $g \cdot f$ belongs to P, we would like to construct a map f_*: $\mathscr{M}(P) \to \mathscr{M}(Q)$ such that $f_* \cdot i_P = i_Q \cdot f$. The most natural choice for $f_*(\tau)$ is $\tau' := \{g \in Q | g \circ f \in \tau\}$. The difficulty is with showing that τ' is maximal and we are able to do it only in case where Y is compact:

9. Proposition. *Suppose f: $X \to Y$ is a map, P is a subalgebra of $C^*(X)$ and Q is a subalgebra of $C^*(Y)$ such that for any $g \in Q$ the composition $g \cdot f$ belongs to P.*

a. *If for every $\tau \in \mathscr{M}(P)$ the set $\tau' := \{g \in Q | g \circ f \in \tau\}$ belongs to $\mathscr{M}(Q)$, then the map f_*: $\mathscr{M}(P) \to \mathscr{M}(Q)$ defined by $f_*(\tau) = \tau'$ is continuous and the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_P"] & Y \arrow[d, "i_Q"] \\ \mathscr{M}(P) \arrow[r, "f_*"] & \mathscr{M}(Q) \end{tikzcd}$$

is commutative.

b. *If Y is compact Hausdorff and $Q = C^*(Y)$ or $Y = [a,b]$ and Q is the closure of all polynomials on $[a,b]$, then for every $\tau \in \mathscr{M}(P)$ the set $\tau' := \{g \in Q | g \circ f \in \tau\}$ equals $\tau_y \in \mathscr{M}(Q)$ for some $y \in Y$.*

Proof: a. In this case $f_*^{-1}(N(\alpha)) = N(\alpha \circ f)$ for any $\alpha \in Q$, so f_* is continuous. Also, if $x \in X$ then $f_* \circ i_P(x) = f_*(\tau_x) = \tau_{f(x)} = i_Q \circ f(x)$.

b. Given τ we will show that there is a unique $y \in Y$ so that $\tau' \subset \tau_y$. Suppose that for each $y \in Y$ there is $\alpha_y \in Q$ such that $\alpha_y(y) \neq 0$ and $\alpha_y \in \tau'$. Choose a neighborhood U_y of y in $\alpha_y^{-1}(R - \{0\})$. By compactness of Y we can find (by choosing a finite subcovering of $\{U_y\}_{y \in Y}$) finitely many functions $\alpha_1, \dots, \alpha_m \in \tau'$ with $\sum_{i=1}^m \alpha_i^2 > \varepsilon > 0$, which contradicts the fact that τ satisfies condition $(*)$. Suppose $y \neq z$ and $\tau' \subset \tau_y, \tau' \subset \tau_z$. Choose $g, g' \in Q$ such that $g \cdot g' = 0$ and $g(y) \neq 0$, $g'(z) \neq 0$. Thus, $g \notin \tau_y$ and $g' \notin \tau_z$. Then $g \circ f \in \tau$ or $g' \circ f \in \tau$ as τ is a prime ideal (see Proposition 6) and $g \in \tau'$ or $g' \in \tau'$, a contradiction.

It remains to show that $\tau_y \subset \tau'$. Suppose U is an open neighborhood of y and $g|U \equiv 0$, $g \in \mathscr{M}(Q)$. Choose a neighborhood V of y in U with $cl(V) \subset U$. Let $h \in Q$ with $h(y) = 1$ and $h|Y - V \equiv 0$. Since $h \cdot g \equiv 0$, $h \circ f \in \tau$ or $g \circ f \in \tau$ (see Proposition 6). In view of $h(y) = 1$ and $\tau' \subset \tau_y$, we have $g \circ f \in \tau$ and $g \in \tau'$. Finally, notice that any $g \in Q$, $g(y) = 0$ is a limit of $g_n \in Q$, $n \geqslant 1$, with each g_n vanishing on some neighborhood of y (if Q is the closure of polynomials use the fact that $|x| \in Q$ – see the Claim in Example 8). Thus, $g_n \in \tau'$. If $g \circ f \notin \tau$, there is $h \in \tau$ with $\inf_{x \in X}\{|g \circ f(x) + h(x)|\} > \varepsilon > 0$. Choose g_n so that $|g - g_n| < \varepsilon/2$. Then, $\inf_{x \in X}\{|g_n \circ f(x) + h(x)|\} > \varepsilon/2$, a contradiction. ■

10. Corollary. *Suppose P is a closed subalgebra of $C^*(X)$ containing* 1.

a. *For any $\alpha \in P$, α: $X \to [a,b]$ there is α': $\mathscr{M}(P) \to [a,b]$ with $\alpha' \circ i_P = \alpha$ and $N(\alpha) = (\alpha')^{-1}(R - \{0\})$. In particular, $\mathscr{M}(P)$ is Hausdorff.*

b. *If* $\alpha\colon X \to Y$ *is a map from* X *to a compact Hausdorff space such that* P_α *is contained in* P, *then there is a unique map* $\alpha'\colon \mathscr{M}(P) \to Y$ *with* $\alpha = \alpha' \circ i_P$.

Proof: Put $Q = C^*(Y)$ in the case (b) and let Q be the closure of all polynomials on $[a, b]$ in the case (a). By Example 8, i_Q is a homeomorphism.

a. Since P is an algebra containing constant functions, $g \circ \alpha \in P$ for any polynomial g. Thus $g \circ \alpha \in P$ for any $g \in Q$ (P is closed). By Proposition 9 there is a map $\alpha_*\colon \mathscr{M}(P) \to \mathscr{M}(Q)$ with $i_Q \circ \alpha = \alpha_* \circ i_P$.

b. Use Proposition 9 to produce a map $\alpha_*\colon \mathscr{M}(P) \to \mathscr{M}(Q)$ with $i_Q \circ \alpha = \alpha_* \circ i_P$.

Put $\alpha' = (i_Q)^{-1} \circ \alpha_*$. To show that $\mathscr{M}(P)$ is Hausdorff assume $\tau_1 \neq \tau_2 \in \mathscr{M}(P)$ and choose $\alpha \in \tau_1 - \tau_2$. Then $\tau_1' = \{g \in Q | g \circ \alpha \in \tau_1\}$ contains the identity function $id(x) = x$ and $\tau_2' = \{g \in Q | g \circ \alpha \in \tau_2\}$ does not contain id. Thus, $\alpha'(\tau_1) \neq \alpha'(\tau_2)$ and $\mathscr{M}(P)$ is Hausdorff. □

Corollary 10 establishes the fact that P_f contains P for $f = i_P\colon X \to \mathscr{M}(P)$ provided P is closed and contains all the constant functions. The next result claims that P_f is contained in P:

11. Proposition. *Suppose* P *is a closed subalgebra of* $C^*(X)$ *containing* 1. *Then, for any* $g\colon \mathscr{M}(P) \to R$ *the map* $g \circ i_P$ *belongs to* P.

Proof: Given any map $h \in P$, let $h^*\colon \mathscr{M}(P) \to R$ be the unique map satisfying $h = h' \circ i_P$ (see Corollary 10). Essentially we need to prove that $\{h^* | h \in P\} = C^*(\mathscr{M}(P))$. Suppose $\varepsilon > 0$ and $g\colon \mathscr{M}(P) \to R$ is continuous. Choose for each $y \in \mathscr{M}(P)$ a neighborhood $U_y = N(\alpha_y)$ of y such that $|g(z) - g(z')| < \varepsilon$ for $z, z' \in U_y$. Choose finitely many points $y_1, \ldots, y_k$ with $\bigcup_{i=1}^k U_i = Y$, where $U_i := U_{y_i}$ and $\alpha_i = \alpha_{y_i}$ for $i \leqslant k$. Notice that $|g(y_i) - g \circ i_P(x)| < \varepsilon$ if $i_P(x) \in U_i$ and $\alpha_i(x) = 0$ otherwise. Thus $|\Sigma(g(y_i) - g \circ i_P(x)) \cdot \alpha_i(x)| < \varepsilon \cdot \Sigma|(\alpha_i(x)|$ for all $x \in X$. Our task will be completed if we can choose the functions $\{\alpha_i\}$ in such a way that $\Sigma\alpha_i(x) = 1$ and $\alpha_i \geqslant 0$. Indeed, $g' := \Sigma g(y_i) \cdot \alpha_i$ belongs to P and $|g'(x) - g \circ i_P(x)| = |\Sigma(g(y_i) - g \circ i_P(x)) \cdot \alpha_i(x)| < \varepsilon \cdot \Sigma|(\alpha_i(x)| = \varepsilon$.

First of all we may replace each α_i by α_i^2 in view of $N(\alpha_i^2) = N(\alpha_i) \cap N(\alpha_i) = N(\alpha_i)$ (see Proposition 6). Now, since $\mathscr{M}(P)$ is compact, $\alpha = (\Sigma\alpha_i)^*$ is bounded. Notice that $\alpha(y) \neq 0$ for any $y \in \mathscr{M}(P)$. Indeed, $\{h | h^*(y) = 0\} \subset N(\alpha_i)$ for some i which means $\alpha_i^*(y) \neq 0$. Thus $\alpha\colon \mathscr{M}(P) \to [a, b]$, where $a > 0$. Notice that the map $r(x) = x^{-1}$, $x \in [a, b]$, is the limit of polynomials

$$\left(\frac{1}{x} = \frac{1}{b} \cdot \frac{1}{1 - \dfrac{b - x}{b}} = \frac{1}{b} \cdot \sum_{n=0}^{\infty} \left(\frac{b - x}{b}\right)^n\right).$$

Consequently, $\beta = 1/(\Sigma\alpha_i) \in P$ and $N(\beta) = \mathscr{M}(P)$ ($\mathscr{M}(P) = N(1) = N(\beta) \cap N(\Sigma(\alpha_i))$, so $N(\beta) = \mathscr{M}(P)$). Now, we can replace each α_i by $\beta \cdot \alpha_i$ in view of $N(\beta \cdot \alpha_i) = N(\alpha_i) \cap N(\beta) = N(\alpha_i)$. □

Proof of main theorem. Corollary 10 and Proposition 11 establish that $P_{i_P} = P$. If $f\colon X \to Y$ is a map with $cl(f(X)) = Y$ being compact Hausdorff and $P_f = P$, then the map $f'\colon \mathscr{M}(P) \to Y$ with $f = f' \circ i_P$ (see Corollary 10b) must be a homeomorphism. Indeed, $(f')^*\colon C^*(Y) \to C^*(\mathscr{M}(P))$ is an isomorphism of algebras, so f' must be onto (otherwise $g \circ f' = 0$ for a nontrivial $g\colon Y \to [0, 1]$) and it must be

one-to-one (otherwise a pair of points in $\mathcal{M}(P)$ could not be separated by a real-valued function).

Suppose f: $X \to Y$ is a map and Q is a closed subalgebra of $C^*(Y)$ such that for any $g \in Q$ the composition $g \circ f$ belongs to P and $1 \in Q$. Consider $\alpha = i_Q \circ f$: $X \to \mathcal{M}(Q)$. If g: $\mathcal{M}(Q) \to R$, then $g \circ i_Q \in Q$ by Proposition 11, so $g \circ \alpha \in P$. By Corollary 10b there is a unique map f_*: $\mathcal{M}(P) \to \mathcal{M}(Q)$ such that $f_* \circ i_P = \alpha$. Then the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_P"] & Y \arrow[d, "i_Q"] \\ \mathcal{M}(P) \arrow[r, "f_*"] & \mathcal{M}(Q) \end{tikzcd}$$

is commutative. $\square$

Stone-Weierstrass Theorem. *Suppose X is a compact Hausdorff space. If P is a closed subalgebra of $C^*(X)$ which contains* 1 *and separates the points of X, then $P = C^*(X)$.*

Proof: The map i_P: $X \to \mathcal{M}(P)$ is onto (recall that its image is dense in $\mathcal{M}(P)$) and is one-to-one (otherwise $f(x) = f(y)$ for all $f \in P$ and some $x \neq y$). Thus i_P is a homeomorphism and $P = C^*(X)$. $\square$

Stone-Čech Compactification. *Suppose X is a Tychonoff space. Then there is a compact Hausdorff space βX containing X as a dense set such that any map f: $X \to Y$ from X to a compact Hausdorff space Y extends over βX.*

Proof: Put $\beta X = \mathcal{M}(C^*(X))$. By Proposition 7 we may consider X to be a subset of βX. Then $cl(f(X)) = \mathcal{M}(P)$ (up to a homeomorphism) for some $P \subset C^*(X)$, so there is a map from βX to $\mathcal{M}(P)$ extending f. $\square$

Tietze-Urysohn Extension Theorem. *If A is a closed subset of a normal space X, then any continuous function f: $A \to R$ extends over X.*

Proof: In the special case of X being compact, this means precisely that $P = \{f \circ i | f$: $X \to R\}$ equals $C^*(A)$, where i: $A \to X$ is the inclusion. Using the Stone-Weierstrass Theorem one gets $cl(P) = C^*(A)$, so it suffices to show that P is closed. Suppose $f_n \circ i$ converges uniformly to f. We may assume $|f_{n+1}(a) - f_n(a)| < 2^{-n}$ for all $a \in A$. Let r_n: $R \to [-2^{-n}, 2^{-n}]$ be the map defined by $r_n(x) = x$ for $-2^{-n} \leqslant x \leqslant 2^{-n}$, $r_n(x) = -2^{-n}$ for $x < -2^{-n}$ and $r_n(x) = 2^{-n}$ for $x > 2^{-n}$. Then $g_n = f_1 + \sum_{k=1}^{n} r_k \circ (f_{k+1} - f_k)$ converge uniformly to g: $X \to R$ with $g(a) = f(a)$ for $a \in A$.

Notice that i_*: $\beta A \to \beta X$ is one-to-one: Given $x \neq y$ in βA, choose two closed and disjoint sets C, D in βA with $x \in \operatorname{int} C$ and $y \in \operatorname{int} D$. Let g: $X \to [0, 1]$ be a map with $g(C \cap A) = \{0\}$ and $g(D \cap A) = \{1\}$. There is an extension g': $\beta X \to [0, 1]$ of g and an extension g'': $\beta A \to [0, 1]$ of $g|A$. Since A is dense in βA, we have $g'' = g' \circ i_*$ and $i_*(x) \neq i_*(y)$.

If X is not compact and f: $A \to R$ is bounded we extend f over $\beta A = \mathcal{M}(C^*(A))$. By the first part we can extend over βX and the restriction of this extension to X is the desired extension of f.

If $f\colon A \to R$ is not bounded, we identify R with $(-1,1)$ and choose an extension $g\colon X \to [-1,1]$ of f. Then consider $\alpha\colon X \to [0,1]$ with $\alpha(g^{-1}(\{-1,1\}) \subset \{0\}$ and $\alpha(A) \subset \{1\}$. Put $f'(x) = \alpha(x) \cdot g(x)$. $\square$

Tychonoff Theorem. *If $\{X_s\}_{s \in S}$ is a family of compact Hausdorff spaces, then their cartesian product $\prod_{s \in S} X_s$ is compact Hausdorff.*

Proof: Put $X = \prod_{s \in S} X_s$ and $P = C^*(X)$. For each $s \in S$ there is a map g_s: $\mathscr{M}(P) \to X_s$ such that $g_s \circ i_P = \pi_s$ is the projection $\prod_{s \in S} X_s \to X_s$. Then $g = \prod_{s \in S} g_s$: $\mathscr{M}(P) \to \prod_{s \in S} X_s = X$ is continuous and $g \circ i_P = id_X$. Since $i_P \circ g$ is identity on a dense subset $i_P(X)$ of $\mathscr{M}(P)$, $i_P \circ g = id$ and g is a homeomorphism.

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Butterfly Embedding Proof of a Theorem of König

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The following is a well-known theorem of König.

Theorem 1. *If A is a nonnegative integral matrix each of whose row and column sums is equal to a constant $k > 0$, then A can be expressed as a sum of k permutation matrices:*

$$A = P_1 + P_2 + \cdots + P_k.$$

For instance if $n = 4$ and $k = 2$, then

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{1}$$

The usual proof of Theorem 1 is to use Hall's theorem on systems of distinct representatives to find the permutation matrix P_1 and then proceed by induction on k (see e.g. Ryser [3, p. 57]). The above theorem is a special case of the following theorem also due to König (see [2]).

Theorem 2. *If A is an m by n nonnegative integral matrix each of whose row and column sums does not exceed the positive integer k, then A is a sum of k subpermutation matrices.*

Here a *subpermutation matrix* is a matrix of 0's and 1's with at most one 1 in each row and column.

Theorem 2 is usually proved as follows: Without loss of generality assume that $m \geq n$. Extend A to an m by m matrix B by including $m - n$ additional zero columns. One then shows that the entries of B can be increased by integer values to yield a matrix B' of order m with all row and column sums equal to k. By Theorem 1, $B' = P_1' + P_2' + \cdots + P_k'$, where the P_i' are permutation matrices. The proof of the theorem is completed by deleting the last $m - n$ columns of each P_i' and changing some of its 1's to 0's to get the P_i which sum to A.

The proof presented below has a similar framework but it uses an embedding technique first employed by Csima [1] for the construction of timetables. This proof is conceptually simpler and it enjoys another advantage which will be explained afterwards.

The 'butterfly'[1] *proof of Theorem 2.* Let the row sums of A be $r_1, r_2, \dots, r_m$ and let the column sums be $s_1, s_2, \dots, s_n$. We embed A in a 'butterfly' matrix

$$A' = \left[\begin{array}{cccc|cccc} k-r_1 & & & & & & & \\ & k-r_2 & & O & & & & \\ & & \ddots & & & A & & \\ O & & & k-r_m & & & & \\ \hline & & & & k-s_1 & & & \\ & & & & & k-s_2 & & O \\ & A^T & & & & & \ddots & \\ & & & & O & & & k-s_n \end{array}\right].$$

Each row and column sum of A' equals k and hence there are permutation matrices $P'_1, P'_2, \dots, P'_k$ such that $A' = P'_1 + P'_2 + \cdots + P'_k$. Let P_i be the submatrix of P'_i determined by its first m rows and last n columns. Then the P_i are subpermutation matrices and $A = P_1 + P_2 + \cdots + P_k$. $\square$

In the butterfly proof the P_i are obtained directly from the P'_i without further reference to the matrix A as is necessary in the first proof (in order to decide which 1's are to be changed to 0's). But the real advantage of the butterfly embedding proof, in contrast to the first embedding proof, is that no solutions of the equation $A = P_1 + P_2 + \cdots + P_k$ are lost as a result of the embedding. For example, let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then in the first embedding, one possible A' (in general there are many) is

$$A' = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 2 \end{bmatrix}.$$

The decomposition

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

cannot result from a decomposition of A' into a sum of four permutation matrices.

[1] The word 'butterfly' was suggested by Eric Sawyer.

In the butterfly embedding

$$A' = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

It is easy to see that all decompositions of A as a sum of 4 subpermutation matrices occur by restricting the decompositions of A'.

In general we can argue as follows. First observe that if P is a subpermutation matrix, then the corresponding butterfly matrix P' (with $k = 1$) is a permutation matrix. Further, if $A = P_1 + P_2 + \cdots + P_k$ represents A as a sum of subpermutation matrices, then $A' = P_1' + P_2' + \cdots + P_k'$ represents A' as a sum of permutation matrices. Considering the submatrices determined by the first m rows and last n columns, as in the proof of Theorem 2, we obtain $A = P_1 + P_2 + \cdots + P_k$.

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...it is now a well-established phenomenon that what is highly abstract for a generation of mathematicians is just commonplace for the next one.

—*J. Dieudonne*

A Generalization of a Congruential Property of Lucas

Richard J. McIntosh

1. INTRODUCTION. A beautiful theorem of Lucas [**8**] states that for every prime p,

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_r}{k_r} \pmod p$$

(with the convention that $\binom{a}{b} = 0$ if $a < b$), where the base p expansions of n and k are

$$n = n_0 + n_1 p + n_2 p^2 + \cdots + n_r p^r \qquad (0 \le n_i \le p-1)$$

and

$$k = k_0 + k_1 p + k_2 p^2 + \cdots + k_r p^r \qquad (0 \le k_i \le p-1).$$

Recently, there have been several articles on Lucas's theorem and its generalizations [**3**], [**5**], [**6**] and [**10**]. In this article we investigate a class of functions satisfying similar congruences.

2. THE CLASS OF LP AND DLP FUNCTIONS. We say that a function $F\colon \mathbf{N} \to \mathbf{Z}$ has the *Lucas property* (**LP**) if for every prime p,

$$F(n) \equiv F(n_0)F(n_1)F(n_2)\cdots F(n_r) \pmod p,$$

where

$$n = n_0 + n_1 p + n_2 p^2 + \cdots + n_r p^r \qquad (0 \le n_i \le p-1).$$

A function $L\colon \mathbf{N} \times \mathbf{N} \to \mathbf{Z}$ has the *double Lucas property* (**DLP**) if
(i) $L(n,k) = 0$ for $n < k$, and
(ii) For every prime p,

$$L(n,k) \equiv L(n_0,k_0)L(n_1,k_1)L(n_2,k_2)\cdots L(n_r,k_r) \pmod p,$$

where

$$n = n_0 + n_1 p + n_2 p^2 + \cdots + n_r p^r \qquad (0 \le n_i \le p-1)$$

and

$$k = k_0 + k_1 p + k_2 p^2 + \cdots + k_r p^r \qquad (0 \le k_i \le p-1).$$

Remark. It is easy to see that the definition of **LP** is equivalent to the following. For every prime p,

$$F(n) \equiv F(n_0)F(n') \qquad (\text{mod } p),$$

where

$$n = n_0 + n'p \qquad (0 \le n_0 \le p - 1).$$

Similarly, condition (ii) of the definition of **DLP** is equivalent to the following. For every prime p,

$$L(n,k) \equiv L(n_0, k_0)L(n', k') \qquad (\text{mod } p),$$

where

$$n = n_0 + n'p \qquad (0 \le n_0 \le p - 1)$$

and

$$k = k_0 + k'p \qquad (0 \le k_0 \le p - 1).$$

3. SOME EXAMPLES OF LP AND DLP FUNCTIONS. We present a cross-section of the numerous examples of **LP** and **DLP** functions that appear in the literature. In section 4 we develop the necessary machinery in order to tackle the proofs of these examples.

(1) For every $a \in \mathbf{Z}$, $F(n) = a^n$ is an **LP** function.

(2) $L(n,k) = \binom{n}{k}$ is a **DLP** function.

(3) Gessel [4] proved that the Apéry numbers $A(n) = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2$ have the Lucas property.

(4) Let

$$p_n(x) = \frac{1}{n!}\left\{\frac{d}{dx}\right\}^n x^n(1-x)^n$$

be the "shifted" Legendre polynomials ([**1**] p. 366) and note that

$$p_n(x) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}x^k.$$

Then for every $a \in \mathbf{Z}$, $F(n) = p_n(a)$ is an **LP** function.

(5) Carlitz [**2**] proved that the function $\omega(n)$ defined by

$$\frac{1}{J_0(2z^{1/2})} = \sum_{n=0}^{\infty}\omega(n)\frac{z^n}{(n!)^2}$$

is an **LP** function. He observed that $\omega(0) = 1$ and that $\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2\omega(k) = 0$ for $n \ge 1$.

Example (1) is a consequence of Fermat's little theorem and example (2) is Lucas's theorem, for which we offer a short proof.

Let $n = n_0 + n'p$ and $k = k_0 + k'p$ $(0 \le n_0,\ k_0 \le p - 1)$. Then

$$\sum_{i=0}^{n} \binom{n}{i} x^i = (1+x)^n \equiv (1+x)^{n_0}(1+x^p)^{n'} = \left[\sum_{i_0=0}^{n_0} \binom{n_0}{i_0} x^{i_0}\right]\left[\sum_{i'=0}^{n'} \binom{n'}{i'} x^{pi'}\right].$$

By equating coefficients of x^k on both sides we have

$$\binom{n}{k} \equiv \sum_{i'} \binom{n_0}{k - pi'}\binom{n'}{i'} \qquad (\mathrm{mod}\ p).$$

Since $0 \le k - pi' \le n_0 \le p - 1$ the sum on the right has at most one term $(i' = k')$ if $k_0 \le n_0$; if not, the sum is zero. Therefore

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n'}{k'} \qquad (\mathrm{mod}\ p). \qquad \square$$

The fact that the functions in examples (3), (4), and (5) satisfy the Lucas property is a consequence of the theory developed in the next section.

4. THE THEORY OF LP AND DLP FUNCTIONS. Note that in the above examples of **LP** functions $F(0) = 1$. Our first theorem shows that this is necessarily the case. The proof is straightforward and is left for the reader.

Theorem 1. *If an* **LP** *function $F(n)$ is not identically zero, then $F(0) = 1$. If a* **DLP** *function $L(n, k)$ is not identically zero, then $L(0, 0) = 1$.*

The next theorem is a direct consequence of the multiplicative nature of **LP** and **DLP** functions.

Theorem 2. (The Multiplication Principle).

(i) *A finite product of* **LP** *functions is* **LP**.

(ii) *A finite product of* **DLP** *functions is* **DLP**.

(iii) *If $G(n)$ and $H(k)$ are* **LP** *functions and $L(n, k)$ is a* **DLP** *function, then*

$$M(n, k) = L(n, k)G(n)H(k)$$

is a **DLP** *function.*

Note that $2^n = \sum_{k=0}^{n} \binom{n}{k}$, where 2^n is an **LP** function and $\binom{n}{k}$ is a **DLP** function. If $\binom{n}{k}$ is replaced by an arbitrary **DLP** function, is the sum still an **LP** function? Theorem 3 answers this question.

Theorem 3. (The Summation Principle). *If $L(n, k)$ is a* **DLP** *function, then*

$$F(n) = \sum_{k=0}^{n} L(n, k)$$

is an **LP** *function.*

Proof: It is in this result that we need condition (i) of the double Lucas property.

Let $n = n_0 + n'p$ $(0 \le n_0 \le p-1)$. Then

$$\begin{aligned}
F(n) &= \sum_{k=0}^{n} L(n,k) \\
&= \sum_{k=0}^{n_0+n'p} L(n,k) \\
&= \sum_{k=0}^{p-1+n'p} L(n,k) \\
&= \sum_{k_0=0}^{p-1} \sum_{k'=0}^{n'} L(n_0+n'p, k_0+k'p) \\
&\equiv \sum_{k_0=0}^{p-1} \sum_{k'=0}^{n'} L(n_0,k_0)L(n',k') \\
&= \left[\sum_{k_0=0}^{p-1} L(n_0,k_0)\right]\left[\sum_{k'=0}^{n'} L(n',k')\right] \\
&= \left[\sum_{k_0=0}^{n_0} L(n_0,k_0)\right]\left[\sum_{k'=0}^{n'} L(n',k')\right] \\
&= F(n_0)F(n') \qquad (\text{mod } p).
\end{aligned}$$

□

Theorem 4 shows that the class of **DLP** functions is closed under reflections in the second variable.

Theorem 4. (The Reflection Principle). *If $L(n,k)$ is a* **DLP** *function, then*

$$M(n,k) = L(n, n-k)$$

is also a **DLP** *function.*

The multiplication, reflection, and summation principles together enable us to prove that functions such as

$$F(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 2^k 3^{n-k}$$

have the Lucas property. More generally we have

Theorem 5. *If $L(n,k)$ is a* **DLP** *function and $G(n)$, $H(n)$ are* **LP** *functions, then*

$$F(n) = \sum_{k=0}^{n} L(n,k)G(k)H(n-k)$$

is an **LP** *function.*

Remark. There exists functions $L(n, k)$ that are not **DLP**, but have the property that for every pair of **LP** functions $G(n)$ and $H(n)$,

$$F(n) = \sum_{k=0}^{n} L(n,k)G(k)H(n-k)$$

defines an **LP** function.

By the Chinese remainder theorem we can construct recursively a function $L(n,k)$ that satisfies the Lucas property for every prime $p > 2$ and satisfies the condition

$$L(n,k) \equiv \begin{cases} 1 \pmod 2 & \text{if } k = 0 \text{ or } k = n, \\ 0 \pmod 2 & \text{if } 0 < k < n. \end{cases}$$

Since $L(3,1) \equiv 0 \not\equiv 1 \equiv L(1,1)L(1,0) \pmod 2$, $L(n,k)$ is not a **DLP** function. Let $G(n)$ and $H(n)$ be any two **LP** functions. By Theorem 5

$$F(n) = \sum_{k=0}^{n} L(n,k)G(k)H(n-k)$$

satisfies the Lucas property for all primes $p > 2$. Now

$$\begin{aligned} F(n) &= \sum_{k=0}^{n} L(n,k)G(k)H(n-k) \\ &\equiv \begin{cases} G(0)H(0) \pmod 2 & \text{if } n = 0, \\ G(0)H(n) + G(n)H(0) \pmod 2 & \text{if } n \geq 1. \end{cases} \end{aligned}$$

$\{0,0,0,\ldots\}$, $\{1,0,0,\ldots\}$, and $\{1,1,1,\ldots\}$ are the only sequences modulo 2 that satisfy the Lucas property for $p = 2$. A simple calculation shows that $F(n)$ satisfies the Lucas property with $p = 2$ for every pair of **LP** functions $G(n)$ and $H(n)$. This proves that $F(n)$ is an **LP** function.

In order to prove that the functions in examples (3) and (4) of section 3 satisfy the Lucas property we need a larger supply of **DLP** functions. The next theorem provides us with many nontrivial examples of such functions.

Theorem 6. *If $r_0, r_1, r_2, \ldots, r_m$ are positive integers $(m \geq 0)$, then*

$$L(n,k) = \binom{n}{k}^{r_0}\binom{n+k}{k}^{r_1}\binom{n+2k}{k}^{r_2}\cdots\binom{n+mk}{k}^{r_m}$$

is a **DLP** *function.*

Proof: A theorem of Kummer [7] states that

$$p^t \bigg\| \binom{a+b}{a},$$

where t equals the number of carries in the addition of a and b in base p arithmetic.

It is obvious that $L(n,k)$ satisfies condition (i) of the double Lucas property.

Suppose that $L(n,k) \not\equiv 0 \pmod p$. Since each r_i is positive,

$$\binom{n+jk}{k} \not\equiv 0 \pmod p \qquad (j = 0, 1, 2, \ldots, m),$$

and so by Kummer's theorem there are no carries in the addition of the $m + 1$

numbers $n, k, k, \ldots, k$ in base p arithmetic. Therefore if we write

$$n = n_0 + n_1 p + n_2 p^2 + \cdots + n_r p^r \qquad (0 \le n_i \le p-1)$$

and

$$k = k_0 + k_1 p + k_2 p^2 + \cdots + k_r p^r \qquad (0 \le k_i \le p-1),$$

then

$$0 \le n_i + mk_i \le p - 1 \qquad (i = 0, 1, 2, \ldots, r).$$

By Lucas's theorem we have

$$\binom{n+jk}{k} \equiv \binom{n_0 + jk_0}{k_0}\binom{n_1 + jk_1}{k_1}\binom{n_2 + jk_2}{k_2} \cdots \binom{n_r + jk_r}{k_r} \qquad (\text{mod } p),$$

for $j = 0, 1, 2, \ldots, m$. This implies that

$$L(n,k) \equiv L(n_0, k_0) L(n_1, k_1) L(n_2, k_2) \cdots L(n_r, k_r) \qquad (\text{mod } p).$$

Now suppose that $L(n, k) \equiv 0$ (mod p). Then for some $j \in \{0, 1, 2, \ldots, m\}$ we have

$$\binom{n+jk}{k} \equiv 0 \qquad (\text{mod } p).$$

We can assume that j is minimal, that is,

$$(*) \qquad \binom{n+lk}{k} \not\equiv 0 \quad (\text{mod } p) \qquad (l = 0, 1, 2, \ldots, j-1).$$

If $j = 0$, then by Lucas's theorem

$$\binom{n_s}{k_s} \equiv 0 \qquad (\text{mod } p)$$

for some $s \in \{0, 1, 2, \ldots, r\}$, which implies that $L(n_s, k_s) \equiv 0$ (mod p). So let us assume that $j \ge 1$. If we write

$$n = n_0 + n_1 p + n_2 p^2 + \cdots + n_r p^r \qquad (0 \le n_i \le p-1)$$

and

$$k = k_0 + k_1 p + k_2 p^2 + \cdots + k_r p^r \qquad (0 \le k_i \le p-1),$$

then

$$0 \le n_i + (j-1)k_i \le p - 1 \qquad (i = 0, 1, 2, \ldots, r)$$

because Kummer's theorem and $(*)$ imply that there are no carries in the addition of the j numbers $n, k, k, \ldots, k$ in base p arithmetic. Since

$$\binom{n+jk}{k} \equiv 0 \qquad (\text{mod } p),$$

there is at least one carry when adding k and $n + (j-1)k$ in the base p, say $n_s + jk_s \ge p$ for some $s \in \{0, 1, 2, \ldots, r\}$. This implies that

$$\binom{n_s + jk_s}{k_s} \equiv 0 \qquad (\text{mod } p),$$

and therefore $L(n_s, k_s) \equiv 0$ (mod p), which completes the proof. □

Remark. Careful inspection of the proof of Theorem 6 shows that if $r_1, r_2, r_3, \ldots, r_m$ are positive integers, then

$$B(n,k) = \binom{n+k}{k}^{r_1}\binom{n+2k}{k}^{r_2}\binom{n+3k}{k}^{r_3} \cdots \binom{n+mk}{k}^{r_m}$$

satisfies condition (ii) of the double Lucas property, and so in Theorem 6 the term $\binom{n}{k}^{r_0}$ can be replaced by any **DLP** function $M(n,k)$.

The example of Carlitz given in section 3 is a consequence of the next theorem.

Theorem 7. (The Inversion Principle). *Suppose that $F(n)$ is an* **LP** *function, $L(n,k)$ is a* **DLP** *function, and that $L(n,n) = 1$ for $n \geq 0$. If $A(n)$ is defined recursively by*

$$\sum_{k=0}^{n} L(n,k)A(k) = F(n),$$

then $A(n)$ is an **LP** *function.*

Proof: We proceed by induction on n. Suppose that $A(k)$ has the Lucas property for $0 \leq k \leq n-1$. If $n = n_0 + n'p$ $(0 \leq n_0 \leq p-1)$, then

$$\begin{aligned}
A(n) &+ \sum_{k=0}^{n-1} L(n,k)A(k) \\
&= F(n) \\
&\equiv F(n_0)F(n') \\
&= \left[\sum_{k_0=0}^{n_0} L(n_0,k_0)A(k_0)\right]\left[\sum_{k'=0}^{n'} L(n',k')A(k')\right] \\
&= \left[\sum_{k_0=0}^{p-1} L(n_0,k_0)A(k_0)\right]\left[\sum_{k'=0}^{n'} L(n',k')A(k')\right] \\
&\equiv \sum_{k_0=0}^{p-1}\sum_{k'=0}^{n'} L(n_0+n'p, k_0+k'p)A(k_0)A(k') \\
&\equiv A(n_0)A(n') + \sum_{k=0}^{n-1} L(n,k)A(k) \qquad (\mathrm{mod}\ p),
\end{aligned}$$

by induction. □

Theorem 8 shows that the intersection of the class of **LP** functions and the class of functions that are periodic modulo p for every prime p is equal to the class of exponential functions.

Theorem 8. *If an* **LP** *function $F(n)$ is periodic modulo p for each prime p, then*

$$F(n) = F(1)^n.$$

Proof: Let k be the period of $F(n)$ modulo p. By the Dirichlet box principle we can choose positive integers $i < j$ such that k divides $p^j - p^i$. Then

$$\begin{aligned}
F(n) &\equiv F(np^j) \\
&\equiv F\big(np^j - (p^j - p^i)\big) \\
&= F\big(p^i + (n-1)p^j\big) \\
&\equiv F(1)F(n-1) \qquad (\mathrm{mod}\ p),
\end{aligned}$$

by the Lucas property. By induction $F(n) \equiv F(1)^n$ (mod p). Since this holds for infinitely many primes p we have $F(n) = F(1)^n$. □

Remark. By the Dirichlet box principle any integer-valued function $F(n)$ satisfying a linear recurrence with constant coefficients is periodic modulo p for every prime p. So the only **LP** functions that satisfy such a recurrence are $F(n) = a^n$, where $a \in \mathbf{Z}$. This proves that the Apéry numbers $A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ cannot satisfy a linear recurrence with constant coefficients. In his celebrated proof of the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$ Apéry [**9**] shows that $A(n)$ satisfies the linear recurrence with polynomial coefficients

$$n^3 A(n) - (34n^3 - 51n^2 + 27n - 5)A(n-1) + (n-1)^3 A(n-2) = 0$$

for $n \geq 2$.

5. CONCLUDING REMARKS. Our theory of **LP** functions can be extended to sequences of polynomials. A classical example is the congruence of Schur.

$$P_n \equiv P_{n_0} P_{n_1}^p P_{n_2}^{p^2} \cdots P_{n_r}^{p^r} \qquad (\mathrm{mod}\ p),$$

where p is an odd prime,

$$n = n_0 + n_1 p + n_2 p^2 + \cdots + n_r p^r \qquad (0 \leq n_i \leq p-1)$$

is the base p expansion of n, and $P_n = P_n(x)$ is the Legendre polynomial of degree n. For a proof of Schur's congruence see ([**11**] Theorem 6.1). Carlitz [**2**] extended the special case of our Theorems 5 and 7 with $L(n,k) = (-1)^{n-k}\binom{n}{k}^2$ to sequences of polynomials.

The author conjectures that $\{0,0,0,\ldots\}$, $\{1,0,0,\ldots\}$, $\{1,1,1,\ldots\}$, and $\{1,2,4,8,\ldots\}$ are the only nonnegative **LP** sequences $\{u_n\}_{n=0}^{\infty}$ such that $u_n = O(b^n)$, where $b < e$.

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Mixtures and Order Statistics

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1. INTRODUCTION. Let $X(1), \dots, X(n)$ be random variables corresponding to the measurements in some sample. (In general, we are *not* insisting that the measurements be independent or identically distributed.) Let $Y(r)$ be the rth smallest of the $X(i)$; that is, $Y(r)$ is the rth order statistic of the sample. In [2] the author asserts that the distribution of the smallest sample value $Y(1)$ is a mixture of the distributions of the $X(i)$; that is,

$$F_{Y(1)} \equiv p_1 F_{X(1)} + \cdots + p_n F_{X(n)}, \tag{1}$$

where $F_X(x) = P[X \le x]$, the p_i are nonnegative, and $p_1 + \cdots + p_n = 1$. A commonly used special case of the above is when $X(1), \dots, X(n)$ are independent and identically distributed (abbreviated i.i.d.); that is, when we have a *random sample of size n*. In this case $F_{X(1)} \equiv \cdots \equiv F_{X(n)}$ and (1) reduces to the simpler claim that $Y(1)$ has the same (common) distribution as the $X(i)$'s.

The assertion would connect two of the most important tools of mathematical statistics and provide an alternative to standard formulas for the distribution of order statistics. Order statistics are used in nonparametric estimation and hypothesis testing when it is not known that the population distribution belongs to some standard family. Mixtures are encountered when one is combining information from several sources by conditioning, observing a stochastic process at random times, using prior information in Bayesian statistics, dealing with a contaminated sample, etc.

We will see that (1) is true only in trivial situations (and a similar result holds for the largest order statistic $Y(n)$). However, the result is much more interesting when we modify assertion (1) by replacing $Y(1)$, the first order statistic, with $Y(r)$, an intermediate order statistic, where $1 < r < n$. If the measurements are from a random sample (that is, i.i.d. measurements), then even the modified version of (1) can be true only when a few restricted distributions are involved. If we allow the measurements to be dependent, then those restrictions vanish and the new assertion can be true in many situations. Thus the results for smallest (largest) values are quite different from those for intermediate values and extremes are extreme in their behavior.

2. RESULTS. The assertion is easily disposed of with

Theorem 1. *Suppose* $Y(1) = \min\{X(1), \dots, X(n)\}$ *and*

$$F_{Y(1)} \equiv p_1 F_{X(1)} + \cdots + p_n F_{X(n)},$$

where $p_1 + \cdots + p_n = 1$ *and the mixture is such that each* $p_i > 0$. *Then* $P[Y(1) = X(1) = \cdots = X(n)] = 1$.

Proof: $Y(1) \le X(i)$ guarantees $F_{Y(1)}(x) \ge F_{X(i)}(x)$ for all x. If (1) holds, each $p_i > 0$, and $p_i F_{Y(1)}(x) > p_i F_{X(i)}(x)$ for some x, then summation would yield a

contradiction. Thus each $F_{X(i)} \equiv F_{Y(1)}$. But now $P[Y(1) < X(i)] > 0$ would imply (We may truncate random variables to guarantee expectations if necessary.) $E(Y(1)) < E(X(i))$ and violate the observation that $Y(1)$, $X(i)$ have the same distribution. Of course, a similar result holds for the maximum of random variables. ■

Remark. The result of Theorem 1 is essentially unchanged if we allow some $p_j = 0$. That is, the $X(i)$'s corresponding to positive weights in the mixture will be equal to one another and to $Y(1)$ with probability one. The random variables given 0 weight will be *uniformly greater* than those with positive weight; i.e., if $p_i > 0$ and $p_j = 0$, then $P[X(j) \geq X(i)] = P[X(j) \geq Y(1)] = 1$.

If the events $[Y(1) = X(1)], \ldots, [Y(1) = X(n)]$ were disjoint, then the Law of Total Probability would guarantee that

$$F_{Y(1)}(x) = \sum_{i=1}^{n} P[X(i) = x \mid Y(1) = X(i)] \cdot P[Y(1) = X(i)]$$
$$= \sum_{i=1}^{n} p_i F_{X(i) \mid Y(1) = X(i)}(x);$$

that is, the distribution of $Y(1)$ would be a mixture of conditional distributions. But the assertion cannot be completely salvaged in this fashion. Indeed, if $X(1)$, $X(2)$ are i.i.d. with $P[X(1) = 0] = \frac{1}{2} = P[X(1) = 1]$ then it is easily calculated that

$$P[Y(1) = 0] = \tfrac{3}{4} \neq \tfrac{2}{3} = P[X(1) = 0 \mid Y(1) = X(1)]p$$
$$+ P[X(2) = 0 \mid Y(1) = X(2)](1 - p).$$

If we now ask when the rth order statistics of a random sample of size n can have the same distribution as the measurements (have a distribution that is a mixture of the identical distributions of the individual measurements), we see (see Section 9.1 of [**1**]) that we are asking what distributions take only values satisfying an equation of the form

$$F(x) = \sum_{k=r}^{n} \binom{n}{k} (F(x))^k (1 - F(x))^{n-k}, \quad \text{for all } x. \tag{2}$$

Since such a polynomial equation in $F(x)$ has at most n distinct solutions, the distribution would have to be discrete. Of course, $F(x) = 0, 1$ are always solutions of (2). In fact, if $r = 1, n$ then $F(x) = 0, 1$ are the only solutions and the distribution must place all of its mass at a single point. But nontrivial solutions do exist. Indeed, if $n = 2m + 1$ and $r = m + 1$, then the Binomial Theorem and symmetry of the binomial coefficients guarantee that $F(x) = \frac{1}{2}$ is a solution. Thus the middle order statistic (sample median) of a random sample of odd size will have the same distribution as the measurements if that distribution places equal probability at two different points so that $F(x)$ takes only the values $0, \frac{1}{2}, 1$. We shall see that a few two-point distributions are the *only* nontrivial distributions that allow some intermediate order statistic from a random sample to have the same distribution as a single measurement.

Lemma. *If* $1 < r < n$, *then the equation*

$$y = \sum_{k=r}^{n} \binom{n}{k} y^k (1 - y)^{n-k} \tag{3}$$

has exactly one solution in $(0, 1)$.

Proof: Set $y = 1/(1+q)$. Then

$$\begin{aligned} y &- \sum_{k=r}^{n} \binom{n}{k} y^k (1-y)^{n-k} \\ &= \frac{1}{1+q} - \sum_{k=r}^{n} \binom{n}{k} \left(\frac{1}{1+q}\right)^k \left(1 - \frac{1}{1+q}\right)^{n-k} \\ &= \frac{(1+q)^{n-1} - \sum_{k=r}^{n} \binom{n}{k} q^{n-k}}{(1+q)^n} \\ &= \frac{\sum_{k=0}^{n-1} \binom{n-1}{k} q^{n-1-k} - \sum_{k=r}^{n} \binom{n}{k} q^{n-k}}{(1+q)^n} \\ &= \frac{-\sum_{j=0}^{n-r} \left[\binom{n}{n-j} - \binom{n-1}{n-j-1}\right] q^j + \sum_{j=n-r+1}^{n-1} \binom{n-1}{n-1-j} q^j}{(1+q)^n} \\ &= 0 \text{ if and only if} \end{aligned}$$

$$-\sum_{j=0}^{n-r} \left[\binom{n}{n-j} - \binom{n-1}{n-j-1}\right] q^j + \sum_{j=n-r+1}^{n-1} \binom{n-1}{n-j-1} q^j = 0. \quad (4)$$

Since $\binom{n}{n-j} - \binom{n-1}{n-j-1} \geq 0$, we see that the polynomial in (4) has exactly one variation (change of sign in the sequence of coefficients). Thus by Descartes' Rule of Signs (see p. 121 or p. 123 of **[3]**.) equation (4) has exactly one positive root q_0 and equation (3) has exactly one solution $p_0 = 1/(1+q_0)$ in $(0,1)$. ■

Interpreting the Lemma in terms of equation (2), we see that the rth, $1 < r < n$, order statistic of a random sample of size n will have the same distribution as any single measurement if and only if that distribution takes only the values $0, p_0, 1$ where p_0 is the unique solution of (3) in $(0,1)$. We restate this in terms of the random variables in

Theorem 2. *For $1 < r < n$, the rth order statistic of a random sample of size n will have the same distribution as any single measurement if and only if the measurement distribution is that of*

$$X = x_1 + (x_2 - x_1)W,$$

where $x_1 \leq x_2$, $P[W=0] = p_0$, $P[W=1] = 1 - p_0$, and $p_0 = p_0(n,r)$ is the unique solution in $(0,1)$ of the equation

$$y = \sum_{k=r}^{n} \binom{n}{k} y^k (1-y)^{n-k}.$$

3. AN EXAMPLE. We have noted that the first (last) order statistic of a random sample of size $n > 1$ will have the same distribution as any single measurement if and only if that distribution is completely degenerate and places all of its mass at a single point. Theorem 1 guarantees that the situation must remain trivial (with all essential measurements being equal with probability one) for extreme values even

if we drop the i.i.d. requirements of the random sample and consider any mixture of measurement distributions. For intermediate order statistics from a random sample only the very restrictive two-point distributions described in Theorem 2 can satisfy the assertion. However, if we just drop the independence requirements of the random sample we can find less restrictive examples where an intermediate order statistic and the measurements have the same distribution. Consider the unit-interval probability space with random variables

$$X^*(1)(x) = xI_{[0,1/3]}(x) + (x + 1/3)I_{(1/3,2/3)}(x) + (x - 1/3)I_{[2/3,1]}(x)$$
$$X^*(2)(x) = (x + 1/3)I_{[0,1/3]}(x) + (x - 1/3)I_{(1/3,2/3)}(x) + xI_{[2/3,1]}(x)$$
$$X^*(3)(x) = x.$$

Clearly, these random variables are not equal with probability one but $Y^*(2)$, $X^*(1)$, $X^*(2)$, $X^*(3)$ are all uniformly distributed on $[0,1]$. In fact, if F is *any* continuous distribution function that is strictly increasing on its support, then (see Section 3.2 of [**1**]) $Y(2) = F^{-1}(Y^*(2))$, $X(1) = F^{-1}(X^*(1))$, $X(2) = F^{-1}(X^*(2))$, $X(3) = F^{-1}(X^*(3))$ all have common distribution F.

Of course, this example does not completely answer the question of what might happen if we relax the conditions of Theorem 2. Can we find a similar example where the measurements are independent but not identically distributed? Is it possible to obtain other characterizations of situations where the distribution of some intermediate order statistic is a mixture of the distributions of the measurements in a sample?

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Triangles with Vertices on Lattice Points

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A triangle is called *embeddable* in $\mathbf{Z}^n$ if it is similar to a triangle whose vertices have integer coordinates in R^n. It was already known that a triangle is embeddable in $\mathbf{Z}^2$ if and only if all its angles have rational tangents. We show that a triangle is embeddable in some $\mathbf{Z}^n$ if and only if it is embeddable in $\mathbf{Z}^5$, and if and only if all its angles have tangents with rational squares. We reduce the problem of embeddability to a certain Diophantine equation. We give a complete characterization of the triangles embeddable in $\mathbf{Z}^n$ for every n. In particular, there are triangles embeddable in $\mathbf{Z}^5$ but not $\mathbf{Z}^4$, and in $\mathbf{Z}^3$ but not $\mathbf{Z}^2$, but surprisingly, the same triangles are embeddable in $\mathbf{Z}^3$ as are embeddable in $\mathbf{Z}^4$. A triangle is embeddable in $\mathbf{Z}^3$ if and only if the tangents of its angles are all rational multiples of $\sqrt{k}$ for some integer k which is a sum of three squares. The proofs use only elementary number theory.

The simplest question concerning embeddability is this: is the equilateral triangle embeddable in $\mathbf{Z}^2$? That is, are there lattice points in the plane forming the vertices of an equilateral triangle? As it turns out, there are not. Of course, the equilateral triangle is embeddable in $\mathbf{Z}^3$, with vertices at the points one unit along each of the three axes. This illustrates that more triangles may be embeddable if more dimensions are allowed. The general problem addressed in this paper is to characterize the triangles embeddable in $\mathbf{Z}^n$ for each n. We give a complete solution of this problem, as described in the preceding abstract.

The problem solved in this paper has a surprisingly long history, and is connected to the work of several other authors. These points are discussed in a separate section near the end of the paper.

Dimension Two. The following proposition is included as an introduction to the subject. (In the proposition, infinity counts as a rational tangent.)

Proposition 1. (J. McCarthy[1]). *A triangle is embeddable in $\mathbf{Z}^2$ if and only if all its angles have rational tangents.*

Proof: Let triangle ABC have its vertices on lattice points in $\mathbf{Z}^2$. Assume for the moment that neither leg of angle A is parallel to the y-axis. Let AP be a line through vertex A parallel to the x-axis. Then angle A is the difference of the angles BAP and CAP. These two angles evidently have rational tangents. But now we may use the formula

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

to conclude that angle A also has a rational tangent. In case one leg of angle A is

[1]It was John McCarthy who pointed out the result on embeddability in R^2 and asked for a generalization to R^n. Thanks are due to R. Alperin, for pointing our Lemma 7.

parallel to the y-axis, we interchange the roles of the x-axis and y-axis in this argument. This will be possible unless angle A is a right angle, in which case the conclusion is immediate.

Conversely, suppose all the angles of triangle ABC have rational tangents. If one of the angles is a right angle, the embeddability is immediate, so we assume that none of the angles is a right angle. Drop an altitude AP from vertex A to side BC (possible extended), so that P is on line BC. Then the ratios AP/BP and AP/CP are rational, being the tangents of angles B and C respectively. Express these two fractions over a common denominator as $AP/BP = u/N$ and $AP/CP = v/N$. Assume for the moment that P lies between B and C. Then triangle ABC is similar to triangle $(0, uv), (-Nv, 0), (Nu, 0)$, since two corresponding angles have the same tangent. The cases where P lies to the left of A or the right of C are similar. ■

Remark. The criterion in Proposition 1 does not extend to higher dimensions. For example, the equilateral triangle is embeddable in $\mathbf{Z}^3$ but not in $\mathbf{Z}^2$.

Embeddability of Angles and Triangles Compared. For the record, we define an angle to be embeddable in $\mathbf{Z}^n$ if it is one of the angles of a triangle embeddable in $\mathbf{Z}^n$.

Proposition 2. *If an angle θ is embeddable in $\mathbf{Z}^n$ (for any n), then $\tan^2\theta$ is rational.*

Proof: Let triangle ABC lie in $\mathbf{Z}^n$ with its vertices on lattice points. Consider sides AB and AC as vectors, and take their dot product: $AB \cdot AC = |AB|\,|AC|\cos\theta$, where θ is the angle at vertex A. Hence

$$\cos^2\theta = \frac{(AB \cdot AC)^2}{|AB|^2|AC|^2}.$$

The expression on the right hand side is a rational function of the coordinates of A, B, and C. Since those coordinates are integers, it follows that $\cos^2\theta$ is rational. Hence $\sin^2\theta = 1 - \cos^2\theta$ is also rational, and hence $\tan^2\theta = \sin^2\theta/\cos^2\theta$ is rational too. ■

The following lemma connects the embeddability of a triangle with the embeddability of its angles considered separately. (We count infinity as a rational tangent, and as a rational multiple of $\sqrt{k}$.)

Lemma 3. (i) *If the square of the tangent of each angle of a triangle T is rational, then there exists a (square-free) positive integer k such that each tangent is a rational multiple of $\sqrt{k}$.*

(ii) *Moreover, k depends only on the plane of the triangle, i.e. any two triangles in the same plane have the same k.*

(iii) *Still more generally, any two lattice angles in the same plane have the same k.*

Proof: Ad (i): Let the angles of the triangle be α, β, and γ. In case one of the angles is a right angle, say γ, then $\tan\alpha$ is the reciprocal of $\tan\beta$, so the conclusion is trivial. We may assume therefore that none of the angles is a right angle. In particular, $1 - \tan\alpha\tan\beta$ is not zero. Then

$$\tan\gamma = -\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$$

Suppose $\tan \alpha = a\sqrt{j}$ and $\tan \beta = b\sqrt{k}$ where k and j are square-free positive integers, and a and b are non-zero rationals. Then

$$\tan \gamma = -\frac{a\sqrt{j} + b\sqrt{k}}{1 - ab\sqrt{jk}}.$$

Rationalizing the denominator on the right, we have

$$\tan \gamma = -\frac{(1 + a^2 j)b\sqrt{k} + (1 + b^2 k)a\sqrt{j}}{1 - a^2 b^2 jk}.$$

Hence $\tan^2 \gamma$ is a rational plus a rational multiple of $\sqrt{jk}$, which is irrational unless $j = k$. This completes the proof of part (i) of the Lemma.

Part (ii) evidently follows from part (iii), since the six different angles of two triangles are special cases of angles in the plane.

Now for part (iii). Let two non-right lattice angles α and β be given in the same plane. Unless there are parallel sides, upon extending the sides of the angles two triangles will be formed, with a common vertex P opposite angle α in one triangle and opposite angle β in the other triangle. (It may be necessary to replace one angle with its vertical angle or a supplemental angle, which won't affect the square of the tangent.) Rescaling the figure if necessary, the intersection points of the sides, one of which is P, can be made lattice points. Assuming there are no parallel sides, it will be possible to choose the vertex P so that the angle at P is not a right angle. (Otherwise the figure must be a square with α and β diagonally opposite.) We apply part (i) of the lemma successively to the two triangles containing α and β to show that α and β have the same k.

The proof is not quite finished, because we still must consider the case in which no triangles are formed, i.e. two angles α and β in the same plane with one side of α parallel to one side of β. In this case we can translate the angles until they have a common vertex and a common side. Assume for definiteness that the common side is between the two angles. Then the sum $\alpha + \beta$ is embeddable. Therefore $\tan^2(\alpha + \beta)$ is rational. Using the same argument as we used to prove part (i), i.e. the formula for the tangent of a sum, we can show that α and β must have the same k. Similarly, using the formula for $\tan(\alpha - \beta)$, we can treat the case of parallel sides in which one angle lies inside the other. ■

One of the referees pointed out that the integer k in Lemma 3 is related to the area of the lattice triangle. Putting the matter simply, $\sqrt{k}$ is a rational multiple of the area. This observation yields another proof of part (i) of Lemma 3. The computation is a little simpler, but we can't get part (iii) without the computation given above. Since the observation about the area is interesting in its own right, we give that computation too: Let a and b be lattice points defining two adjacent sides of a triangle with angle θ at origin. Then the perpendicular from a to b is given by

$$u = a - \frac{(a \cdot b)b}{|b|^2}.$$

The area S is thus given by $4S^2 = |u|^2|b|^2 = |a|^2|b|^2 - (a \cdot b)^2$. We have

$$\tan \theta = \frac{|u|\ |b|}{a \cdot b} = \frac{2S}{acdotb}.$$

The formula shows that $4S^2$ is an integer. Write $4S^2 = m^2 k$ where k is square-free. Then $\tan\theta$ is a rational multiple of $\sqrt{k}$.

The Triangle Equations

Definition. The *triangle equations* $E(k,n)$ are

$$k\left(a_1^2 + a_2^2 + \cdots + a_n^2\right) = u_1^2 + u_2^2 + \cdots u_n^2$$
$$a_1 u_1 + a_2 u_2 + \cdots a_n u_n = 0.$$

Throughout the paper, we consider only non-trivial solutions in integers a_i and u_i of these equations.

Proposition 4. *If a triangle with an angle with tangent $\lambda\sqrt{k}$ is embeddable in $\mathbf{Z}^n$, where λ is rational, then the triangle equations $E(n,k)$ have a non-zero solution, in which the variables have no common factor.*

Conversely, if $E(n,k)$ has a non-zero solution, and if triangle ABC has all its tangents of the form $\lambda\sqrt{k}$ for rational λ, then triangle ABC is embeddable in $\mathbf{Z}^n$.

Remark. Of course there are many non-embeddable triangles with *one* tangent of the specified form, as you can fix two vertices and let the third move along one side of the triangle. Hence the second condition in the theorem is needed.

Proof: First suppose the triangle ABC has its vertices on lattice points in $\mathbf{Z}^n$. As in the previous proof, we drop the altitude from vertex B to point P on side AC. As in that proof, P has rational coordinates, and enlarging the triangle if necessary, we may assume P has integer coordinates. Performing a translation, we may assume P is the origin. We now have vector A of magnitude AP, and vector B of magnitude BP, which are orthogonal. The ratio $BP/AP = \tan A = \lambda\sqrt{k}$ by hypothesis. We thus have

$$(\tan^2 A)|A|^2 = k|(\lambda A)|^2 = |B|^2.$$

Thus $(\lambda A, B)$ solves the triangle equations $E(n,k)$.

Conversely, suppose given a solution (a,u) of the triangle equations, and a triangle ABC such that $\tan A = \lambda\sqrt{k}$ with λ rational. As before drop an altitude BP from B to side AC. Consider first the case in which P lies between A and C. Then the triangle with two vertices at $-\lambda^{-1}a$, and u will have the correct tangent $\tan A$ at vertex a. By hypothesis, the tangent at vertex C has the form $\mu\sqrt{k}$ for some rational μ. Taking the third vertex to be $\mu^{-1}a$ yields the correct tangent $\tan B$ at this vertex. Therefore the triangle is similar to the given one. The cases in which P does not lie between A and C are treated similarly. ■

Dimension Five or More

Lemma 5. *If $n \geq 5$ then the triangle equations have a non-zero solution for any k.*

Proof: It suffices to consider $n = 5$, since we can always let the variables u_i and a_i for $i > 5$ be zero. Let k be given. Then k can be written as the sum of four squares (see e.g. Hardy and Wright, p. 302):

$$k = u_1^2 + u_2^2 + u_3^2 + u_4^2.$$

Let $u_5 = 0$, and let $a_1 = 0 = a_2 = a_3 = a_4$, and $a_5 = 1$. ■

Remark. Similarly, if k is a sum of $n - 1$ squares, then the triangle equations $E(n, k)$ have a nontrivial solution.

Theorem 6. *The following are all equivalent:*

- *Triangle T is embeddable in $\mathbf{Z}^n$ for some n.*
- *All the tangents of the angles of triangle T have rational squares.*
- *For some k, all the tangents of the angles of triangle T are of the form $\lambda\sqrt{k}$ for rational λ.*
- *The triangle equations* E(n, k) *have a non-zero solution and all the tangents of the angles are of the form $\lambda\sqrt{k}$ for rational λ.*
- *Triangle T is embeddable in $\mathbf{Z}^5$.*

Proof: We show that each claim in the theorem implies the next; since the last one is a special case of the first, that will suffice. Suppose triangle T is embeddable in $\mathbf{Z}^n$. By Proposition 2, the tangents of all the angles of T have rational squares.

Now suppose all the tangents of angles of T have rational squares. By Lemma 3, there is a positive square-free integer k such that all the tangents of angles of T are rational multiples of $\sqrt{k}$.

Now suppose all the tangents are rational multiples of $\sqrt{k}$. By Proposition 4, the triangle equations $E(n, k)$ are solvable.

Now suppose $E(n, k)$ is solvable and the angles of a triangle have tangents which are rational multiples of $\sqrt{k}$. By Lemma 5, the equations are solvable already when $n = 5$. By the second half of Proposition 4, the triangle is embeddable in $\mathbf{Z}^5$. ∎

Quaternions and Orthogonal Transformations of $\mathbf{R}^4$. Background information on quaternions can be found in Hardy and Wright, p. 303. We assume the reader knows the basic properties of quaternions. A four-vector (x_1, x_2, x_3, x_4) can be regarded as a quaternion $x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$. To fix some notation: If $x = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$, then the conjugate x^* is defined by $x^* = x_1 - x_2\mathbf{i} - x_3\mathbf{j} - x_4\mathbf{k}$, the norm is defined by $|x|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$. We have $xx* = |x|^2 + 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$ which we shall identify with the scalar $|x|^2$. The multiplicative inverse of x is $x^{-1} = x*/|x|^2$.

Lemma 7. *Given a fixed quaternion α, an orthogonal transformation T_α on R^4 is defined by $T_\alpha x = x\alpha$, where on the right we mean quaternion multiplication. That is, T_α preserves orthogonality and multiplies lengths by a constant factor.*

Proof: A simple calculation. Let $x = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$, $y = y_1 + y_2\mathbf{i} + y_3\mathbf{j} + y_4\mathbf{k}$, and $\alpha = \alpha_1 + \alpha_2\mathbf{i} + \alpha_3\mathbf{j} + \alpha_4\mathbf{k}$. Note that the dot product of two vectors $x \cdot y$ is the real part of the quaternion product $xy*$. Hence $(x\alpha) \cdot (y\alpha)$ is the real part of $(x\alpha)(y\alpha)* = x\alpha\alpha * y* = x|\alpha|^2 y* = |\alpha|^2 xy*$, whose real part is $|\alpha|^2 x \cdot y$. Hence orthogonality is preserved. Taking $x = y$ we see that lengths are multiplied by $|\alpha|^2$. ∎

Dimension 4. We first characterized the triangles embeddable in $\mathbf{Z}^4$ by using a computer to show that certain triangles are not embeddable in $\mathbf{Z}^4$. This proof showed by direct search that the triangle equations $E(4, k)$ have no solutions mod 32 in which the variables have no common factor, when $k = 7, 15, 23, 31$. It is possible to prove Theorem 8 below from this result. The program we used, written in the C language, ran for several hours on an IBM PC/AT. Later we found the more insightful proof given here.

Theorem 8. *The triangle equations $E(4,k)$ are solvable iff k is a sum of three squares. Geometrically stated: A triangle is embeddable in $\mathbf{Z}^4$ if and only if all of its tangents are rational multiples of $\sqrt{k}$, where k is a sum of three squares.*

Proof: If a triangle is embeddable in $\mathbf{Z}^n$, for any n, then there is a k such that the tangents of its angles all lie in $Q(\sqrt{k})$, as has been proved above. Hence the main claim of the theorem follows from the equivalence of the first two propositions.

If k is a sum of three squares, then $E(4,k)$ is automatically solvable, as remarked after Lemma 5. Thus it will suffice to show that if $E(4,k)$ is solvable, then k is a sum of three squares. Suppose that a and u are four-vectors solving $E(4,k)$, that is $k|a|^2 = |u|^2$ and $a \cdot u = 0$. Consider the four-vectors as quaternions. Let $b = aa*$ (considered as a four-vector or quaternion, not as a scalar), and let $v = ua*$, where we mean quaternion multiplication on the right. Since quaternion multiplication preserves orthogonality, we have $b \cdot v = 0$. We have

$$\begin{aligned} k|b|^2 &= k|a|^4 \\ &= |a|^2 k|a|^2 \\ &= |a|^2|u|^2 \\ &= |a|^2 uu* \\ &= uaa*u* \\ &= ua(ua)* \\ &= vv* \\ &= |v|^2. \end{aligned}$$

Hence b and v are a new solution to the triangle equations $E(4,k)$. But $b = aa*$ has only its first component non-zero. Since v is orthogonal to b by Lemma 7, v lies in the three-dimensional subspace of vectors with zero first component. Hence we have $k|b|^2 = v_2^2 + v_2^2 + v_3^2$. Hence $k|b|^2$ is a sum of three squares.

Note that since $b = aa*$, we have $|b|^2 = |a|^4$, so $|b| = |a|^2$ is an integer. It is well-known (see e.g. LeVeque p. 187) that a number fails to be a sum of three squares if and only if it is a power of 4 times a number congruent to $7 \bmod 8$. If k were of this form, then $k|b|^2$ would also be of this form, since every odd square is congruent to $1 \bmod 8$. Hence it follows from the facts that $k|b|^2$ is a sum of three squares and $|b|$ is an integer that k is also a sum of three squares. ∎

Another proof of Theorem 8: J. McCarthy has pointed out that the fact that the tangents are rational multiples of $\sqrt{k}$ where k is a sum of three squares can be proved without use of the triangle equations, as follows: by Lemma 3, it suffices to consider only one angle θ, with vertex at origin and sides given by vectors x and y. We have

$$\begin{aligned} \tan^2\theta &= \sec^2\theta - 1 \\ &= \frac{|x|^2|y|^2}{(x \cdot y)^2} - 1 \\ &= \frac{|x|^2|y|^2 - (x \cdot y)^2}{(x \cdot y)^2}. \end{aligned}$$

Therefore, $\tan\theta$ is a rational multiple of

$$\begin{aligned} k &= |x|^2|y|^2 - (x\cdot y)^2 \\ &= \left(\Sigma x_i^2\right)\left(\Sigma y_i^2\right) - \left(\Sigma x_i y_i\right)^2. \end{aligned}$$

The proof will be completed by observing an identity which expresses the last expression as a sum of three squares:

$$\begin{aligned} &\left(\Sigma x_i^2\right)\left(\Sigma y_i^2\right) - \left(\Sigma x_i y_i\right)^2 \\ &\quad = \left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)\left(y_1^2 + y_2^2 + y_3^2 + y_4^2\right) \\ &\qquad - \left(x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4\right)^2 \\ &\quad = \left(x_1 y_2 - x_2 y_1 + x_3 y_4 - x_4 y_3\right)^2 \\ &\qquad + \left(x_1 y_3 - x_3 y_1 - x_2 y_4 + x_4 y_2\right)^2 \\ &\qquad + \left(x_1 y_4 - x_4 y_1 + x_2 y_3 - x_3 y_2\right)^2. \end{aligned}$$

McCarthy found this identity by generalizing the corresponding three-dimensional identity

$$|x|^2|y|^2 = (x\cdot y)^2 + |x \times y|^2.$$

It is really just the identity expressing the multiplicativity of the quaternion norm, applied to the two quaternions x and $y*$.

This alternate proof is interesting because it shows a uniformity in the derivation of the necessary condition on k for different dimensions; the two-dimensional case of this identity is

$$\begin{aligned} \frac{\tan^2\theta}{(x\cdot y)^2} &= \left(x_1^2 + x_2^2\right)\left(x_1^2 + y_2\right)^2 - \left(x_1 y_1 + x_2 y_2\right)^2 \\ &= \left(x_1 y_2 - x_2 y_1\right)^2 \end{aligned}$$

which explains why the tangent is rational in two dimensions.

Corollary 9. *There are triangles embeddable in* $\mathbf{Z}^5$ *but not* $\mathbf{Z}^4$. *For example, the isosceles triangles of base* 2 *and height* $\sqrt{7}$.

Dimension 3. The following very short proof took a long time to find; see the Postscript.

Theorem 10. *If* k *is a sum of three squares, then* $E(3,k)$ *is solvable in integers. Hence the same triangles are embeddable in* $\mathbf{Z}^4$ *as in* $\mathbf{Z}^3$, *and* $E(3,k)$ *is solvable if and only if* k *is a sum of three squares.*

Proof: By our result on embeddability in $\mathbf{Z}^4$, it suffices to prove the first claim. Suppose $k = x^2 + y^2 + z^2$. Define

$$\begin{aligned} a &= (z, z, -y-x) \\ u &= \left(y^2 + xy + z^2, x^2 + xy + z^2, xz - yz\right). \end{aligned}$$

One can easily check that a and u (regarded as quaternions with zero real part) are obtained by multiplying the known solution $(1,0,0,0)$ and $(0,x,y,z)$ of $E(4,k)$ on the right by the quaternion $0 + z\mathbf{i} + z\mathbf{j} + (-y-x)\mathbf{k}$. Hence, by Lemma 7, the

transformed vectors are still orthogonal and have the same ratio $\sqrt{k}$ of length, so a and u solve the triangle equations $E(3, k)$. ■

Remark. One can produce a and u *deus ex machina* and verify by a simple direct computation that they do solve the triangle equations, without ever mentioning quaternions. For example, type the three equations for k, a, and u into *Mathematica* and then ask `Simplify[a.u]` and `Simplify[k(a.a) - u.u]`.

Embeddability of regular polygons in plane lattices. We show that an old result is a corollary of our main theorem. The original proofs (there are two independent ones in the literature) are much easier than the proof of our main theorem, so the fact that it is a corollary of our theorem is only of interest for the connection, and not for the result itself. The original proofs are discussed in the next section.

Theorem (*Schoenberg [1937], Scherrer [1946]). Suppose a regular n-gon is embeddable in $\mathbf{Z}^{\mathbf{k}}$ for some k. Then $n = 3, 4,$ or 6.*

Proof: If we have an embedded n-gon, then there is an embedded isosceles triangle with one angle of $2\pi/n$. The other two angles are each $\pi/2 - \pi/n$. Their tangents are thus $\cot \pi/n$. The non-embeddability of an n-gon in any $\mathbf{Z}^{\mathbf{k}}$ will then follow from our theorem when $\cot^2(\pi/n)$ is irrational. Since

$$\cot^2 \theta = \frac{1 + \cos 2\theta}{1 - \cos 2\theta},$$

we have

$$\cos 2\theta = \frac{\cot^2 \theta - 1}{\cot^2 \theta + 1}.$$

so $\cos 2\theta$ is rational if and only if $\cot^2 \theta$ is rational. Hence an embedded n-gon is possible if and only if $\cos(2\pi/n)$ is rational. To complete the proof, we have to show that $\cos(2\pi/n)$ is rational exactly when $n = 3, 4,$ or 6.

Let $\zeta = e^{2\pi i/n}$. Then the minimal polynomial of ζ has degree $\phi(n)$, where ϕ is the Euler ϕ-function. (See for example Borevich and Shafarevich [1966], p. 326.) Since $2\cos(2\pi/n) = \zeta + 1/\zeta$, we have $f(\zeta) = 0$ for a quadratic polynomial f with coefficients in $\mathbf{Q}(\cos(2\pi/n))$. Hence the degree of the field extension $[\mathbf{Q}(\zeta)\colon \mathbf{Q}(\cos(2\pi/n)]$ is at most 2. On the other hand it is at least 2, since $\cos(2\pi/n)$ is real. Hence the degree of $\cos(2pi/n)$ over the rationals is $\phi(n)/2$. This can be one if and only $\phi(n) = 2$, that is, $n = 3, 4,$ or 6. ■

History and Related Work. The first proof that the equilateral triangle is not embeddable in $\mathbf{Z}^2$ was given (so far as I know) by E. Lucas [1878]. Lucas' proof is perhaps more accessible in Pólya and Szegö [1954], page 376 (problem 238). Since it is only a few lines, and not published elsewhere in English, it seems worth reprinting:

Put one corner of the hypothetical equilateral triangle at origin, the other corners at (a, b) and (x, y), and supposing that x, y, a, b have no common factor. Then we have

$$x^2 + y^2 = a^2 + b^2 = (x - a)^2 + (y - b)^2$$

and hence

$$2(xa + by) = x^2 + y^2 = a^2 + b^2$$
$$x^2 + y^2 + x^2 + b^2 = 4(xa + yb) \cong 0 \mod 4.$$

Since we have excluded the case of x, y, a, b all divisible by 2, they must all be odd. In that case, however, the equation

$$x^2 + y^2 = (x - a)^2 + (y - b)^2 \mod 4$$

is impossible, completing the proof.

So far as I can determine, John McCarthy was the first to state and prove (although he did not publish) the generalization of Lucas' theorem to planar polygons (Proposition 1 of this paper). One of the referees suggested that this theorem was part of the "folklore" of the subject, and should not be credited to McCarthy; but Lucas' proof is very special, and when Pólya and Szegö give it, as recently as 1954, there is no hint of a generalization, nor is this generalization mentioned in any of the other related papers discussed below, and these are all the papers I could find on the subject.

Rather than ask about arbitrary planar triangles, people seemed to have generalized Lucas' theorem in another direction, asking about arbitrary regular polygons.

Schoenberg [1937] proved that a regular n-gon with n different from 3, 4, and 6, is not embeddable in $\mathbf{Z}^{\mathbf{k}}$, or indeed any (possibly oblique) rational lattice in k-space for any k. Although it refers to k dimensions instead of a plane, it actually suffices to consider only planar lattices, since if a polygon were embeddable in $\mathbf{Z}^{\mathbf{k}}$, then the intersection of the plane of the polygon with Z^k would be a planar lattice. Schoenberg's proof is short: Let A, B, and C be three consecutive vertices of a regular lattice n-gon with center at origin. Let $P = A + C$. Then $|P| = 2|B|\cos(2\pi/n)$, so $\cos^2(2\pi/n)$ is rational. Then we can finish the proof as in the previous section, except that the cases $n = 8$ and $n = 12$ still need attention. (Schoenberg [1937], p. 50, jumps too quickly for me to follow to the conclusion that $\cos(2\pi/n)$ is rational.)

Scherrer [1946], apparently unaware of Schoenberg [1937], gave another proof of this theorem. His proof is a gem: Suppose we had an embedded n-gon (for $n > 6$). Consider the lattice vectors formed by the sides. Translate them, putting their tails all at origin. Then their heads form a *smaller* lattice n-gon, in fact smaller by at least a certain factor, namely $2\sin(\pi/n)$. Iterating this construction leads to arbitrarily small lattice n-gons, a contradiction. This proof works even for non-square lattices, which we have not considered in this paper. Scherrer also showed the case $n = 5$ is impossible, by a similar construction: Number the sides of a pentagon, considered as vectors, by $1, 2, 3, 4, 5$. Then taking them in the order $1, 3, 5, 2, 4$, place the tail of each at the head of the previous one. You will get a five-pointed star. Connecting the points, you get a smaller lattice pentagon than you started with. For square lattices, Scherrer could have ruled out $n = 3$ and $n = 6$ by Lucas' theorem.

The main point of Schoenberg [1937] is not polygons, but rather necessary and sufficient conditions for the embeddability of a regular n-simplex in $\mathbf{Z}^{\mathbf{n}}$ (it is always embeddable in $\mathbf{Z}^{\mathbf{n}+1}$, for example taking all the points with one coordinate 1 and the rest 0). Although the equilateral triangle is not embeddable in $\mathbf{Z}^2$, the tetrahedron is embeddable in the unit cube, for example at $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(1,1,1)$. Schoenberg showed that, for n even, the embedding is possible if and only if $n + 1$ is a perfect square; for $n \cong 3 \bmod 4$, it is always possible and for $n \cong 1 \bmod 4$, if and only if $n + 1$ is a sum of two squares.

The fact that the 4-simplex is not embeddable in $\mathbf{Z}^4$ refutes the idea that perhaps a polyhedron is embeddable if all of its triangles are embeddable.

Nobody seems to have considered the question of the embeddability of arbitrary triangles until the 1980's. Landau and Cremona [1987] consider the following question: given that a triangle is embeddable in $\mathbf{Z}^n$, what is the smallest embedding? That is, find the smallest triangle similar to the given one which has its vertices on lattice points in n-space. They answer the question in dimensions 3 and 4 using the greatest-common-divisor algorithm in the quaternions. Since now we know that triangles embeddable in $\mathbf{Z}^4$ are also embeddable in $\mathbf{Z}^3$, we might wonder if a smallest embedding can always be found in $\mathbf{Z}^3$. The answer (according to a letter from Landau) is no: although a lattice triangle in $\mathbf{Z}^4$ can always be rotated and dilated into $\mathbf{Z}^3$, sometimes a dilation is really required.

Postscript on the Disappearing Computer. All the proofs above use only elementary number theory. This is interesting, considering that a computer was involved throughout this research. First I used it to discover that the isosceles triangle of height $\sqrt{7}$ and base 2 is not embeddable in $\mathbf{Z}^3$; then that the same triangle is not embeddable in $\mathbf{Z}^4$; then to settle the question of non-solvability of $E(4, k)$ if k is not a sum of four squares. At first I expected to use it to find an example of a triangle embeddable in $\mathbf{Z}^4$ but not in $\mathbf{Z}^3$. Only after using it to find actual solutions of $E(3, k)$ for k a sum of three squares up to 128 did I give up my preconceptions and try to prove that the same triangles are embeddable in three-space as in four-space. When I learned how to use quaternions to describe orthogonal transformations of four-space, all my programs were displaced by the concise elegance of "real mathematics".

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Universally Nonmeasurable Subgroups of $\mathbb{R}$

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Simple proofs (by use of the axion of choice in one form or another) of the existence of subsets of the real line $\mathbb{R}$ that are not measurable with respect to the Lebesgue outer measure λ can be found in almost any textbook that discusses measure theory. Some of these examples are actually subgroups of the additive group $\mathbb{R}$. Indeed, no subgroup H of $\mathbb{R}$ for which $\mathbb{R}/H$ is countably infinite can be λ-measurable and such subgroups can be obtained by use of a Hamel basis (see [**3**]). A particularly illuminating discussion of λ-nonmeasurability is given in [**5**].

In this note we modify a transfinite induction argument given by F. Bernstein (see (10.54) of [**2**]) to show that $\mathbb{R}$ has subgroups H that are μ-measurable for no nonzero, continuous, regular Borel outer measures μ on $\mathbb{R}$. These are just the outer measures ι constructed in §9 of [**2**] (for $X = \mathbb{R}$) that satisfy $\iota(\{x\}) = 0$ for all $x \in X$ and $\iota(X) > 0$. In the present setting, these μ's are just the Lebesgue-Stieltjes outer measures obtained from arbitrary nonconstant, continuous, monotone nondecreasing functions $f: \mathbb{R} \to \mathbb{R}$ via the formula

$$\mu(E) = \inf\left\{ \sum_{j=1}^{\infty} \left[f(b_j) - f(a_j) \right] : a_j \leqslant b_j \text{ in } \mathbb{R},\ E \subset \bigcup_{j=1}^{\infty} \left]a_j, b_j\right[\right\}$$

for $E \subset \mathbb{R}$. If $f(x) = x$ for all x, then $\mu = \lambda$. Let J denote the set of all such μ and for $\mu \in J$ let $\mathscr{M}_\mu$ denote the family of all μ-measurable subsets of $\mathbb{R}$. That is, $A \in \mathscr{M}_\mu$ means $A \subset \mathbb{R}$ and

$$\mu(T) = \mu(T \cap A) + \mu(T \setminus A)$$

for every subset T of $\mathbb{R}$. It is well known (see [**2**]) that for $\mu(A) < \infty$ this is equivalent to the requirement that

$$\sup\{\mu(K): K \subset A,\ K \text{ is compact}\} = \inf\{\mu(V): A \subset V \subset \mathbb{R},\ V \text{ is open}\}.$$

Notice that if $\mu \in J$, then $\mu(\mathbb{R}) > 0$ and $\mu(C) = 0$ for each countable subset C of $\mathbb{R}$. Also, every Borel set $B \subset \mathbb{R}$ is in $\mathscr{M}_\mu$. We call a set $S \subset \mathbb{R}$ *universally nonmeasurable* if $S \in \mathscr{M}_\mu$ for *no* $\mu \in J$. F. Bernstein proved the existence of such sets S.

Theorem 1. *There exists a universally nonmeasurable subgroup H of the additive group $\mathbb{R}$ having index c. That is, card $(\mathbb{R}/H) = c$ where $c =$ card $\mathbb{R}$.*

Proof: Let $\mathscr{F}$ denote the family of all uncountable closed subsets of $\mathbb{R}$. Since each open subset of $\mathbb{R}$ is a union of open intervals having rational endpoints, it is easy to see that card $\mathscr{F} = c$. It follows from (6.66) and (6.65) of [**2**] that card $F = c$ for each $F \in \mathscr{F}$ (the continuum hypothesis is not needed). Let Δ be the least ordinal number having exactly c predecessors. Now index $\mathscr{F}$ by the elements of this

predecessor set: $\mathscr{F} = \{F_\alpha: \alpha < \Delta\}$. Since the field $\mathbb{Q}$ of rational numbers is countable and the family of all finite subsets of any infinite set has the same cardinal number as the set itself, it follows that if a subset of $\mathbb{R}$ has fewer than c points, then its linear span over $\mathbb{Q}$ also has fewer than c points. Thus, it is possible to obtain by transfinite induction a set $\{x_\alpha: \alpha < \Delta\} \cup \{y_\alpha: \alpha < \Delta\}$ of distinct real numbers that is linearly independent over $\mathbb{Q}$ and has $x_\alpha, y_\alpha \in F_\alpha$ for all α. One simply selects any $x_\alpha \in F_\alpha \setminus \operatorname{span}\{x_\beta, y_\beta: \beta < \alpha\}$ and then any

$$y_\alpha \in F_\alpha \setminus \operatorname{span}\big(\{x_\alpha\} \cup \{x_\beta, y_\beta: \beta < \alpha\}\big).$$

This done, let $H = \operatorname{span}\{x_\alpha: \alpha < \Delta\}$ and observe that $\{y_\alpha + H: \alpha < \Delta\}$ is a family of c distinct cosets of H. By way of contradiction, assume that $H \in \mathscr{M}_\mu$ for some $\mu \in J$. If $\mu(H) > 0$, choose a compact $K \subset H$ with $\mu(K) > 0$. Then K is uncountable so $K = F_\alpha$ for some $\alpha < \Delta$. But then $y_\alpha \in H$ contrary to the independence of $\{x_\beta, y_\beta: \beta < \Delta\}$. Thus $\mu(H) = 0$. Likewise, if $H' = \mathbb{R} \setminus H$ and $\mu(H') > 0$, then $x_\gamma \in F_\gamma \subset H'$ for some $\gamma < \Delta$ even though $x_\alpha \in H$ for all α. Thus $\mu(H') = 0$ too. Consequently $\mu(\mathbb{R}) = 0$ which contradicts $\mu \in J$. Thus H is universally nonmeasurable. □

Next we prove a surprising fact which shows the pervasive nonmeasurability carried by any universally nonmeasurable subgroup of $\mathbb{R}$.

Theorem 2. *Let H be any universally nonmeasurable subgroup of $\mathbb{R}$ and let B be any uncountable Borel set of $\mathbb{R}$. For $x \in \mathbb{R}$ put $B_x = B \cap (x + H)$. Consider any $\mu \in J$ for which $0 < \mu(B) < \infty$. Then for all $x \in \mathbb{R}$ we have $\mu(B_x) = \mu(B)$ and $B_x \notin \mathscr{M}_\mu$. Thus, for all such μ, the family $\{B_x: x \in \mathbb{R}\}$ partitions B into card $(\mathbb{R}/H)$ distinct sets none of which is μ-measurable.*

Proof: Assume that there is an $x \in \mathbb{R}$ such that $\mu(B_x) < \mu(B)$. Then there is some open $V \subset \mathbb{R}$ with $B_x \subset V$ and $\mu(V) < \mu(B)$. Since $B \setminus V \in \mathscr{M}_\mu$ and $\mu(B \setminus V) > 0$, there exists a compact $K \subset B \setminus V$ with $\mu(K) > 0$. Define ν by

$$\nu(E) = \mu(K \cap (x + E))$$

for $E \subset \mathbb{R}$. Noting that $\nu(K - x) > 0$, one checks that $\nu \in J$. But this is impossible because K and $x + H$ are disjoint so $\nu(H) = 0$ and hence $H \in \mathscr{M}_\nu$ contrary to the universal nonmeasurability of H. This proves that $\mu(B_x) = \mu(B)$ for all $x \in \mathbb{R}$. Next assume that $B_x \in \mathscr{M}_\mu$ for some x. Since $H \neq \mathbb{R}$ there is a $y \in \mathbb{R}$ such that B_y is disjoint from B_x. Then $2\mu(B) = \mu(B_x) + \mu(B_y) \leqslant \mu(B_x) + \mu(B \setminus B_x) = \mu(B) < \infty$ contrary to $\mu(B) > 0$. Thus no B_x is μ-measurable. Since no B_x is empty and the family $\mathbb{R}/H$ of cosets of h is pairwise disjoint, the last sentence of the theorem follows. □

Remarks. (a) It is known that any B as in Theorem 2 contains a subset that is homeomorphic to the Cantor space $\{0, 1\}^{\mathbb{N}}$ (see p. 268 of **[1]**). Thus, by considering infinite product measures, we see that there exist exactly c different $\mu \in J$ such that $\mu(B) = 1$. Note that card $J = c$.

(b) For any nondiscrete locally compact group G it can be asked whether G has a universally nonmeasurable subgroup H. By use of entirely different and much less elementary methods than those used here, it is established in **[4]** that if G is also abelian, σ-compact, and metrizable, then such an H exists if and only if for each prime number p the subgroup $G(p) = \{x \in G: px = 0\}$ is either open or discrete. Of course, for such G, our simple proof of Theorem 1 produces such an

H if, in addition, G is a vector space over some countable field. As a consequence of this fact from [**4**], we see that if $\mathbb{Z}_m$ is the discrete cyclic group of order $m > 1$ and $G = \mathbb{Z}_m^{\mathbb{N}}$, then such an H exists if and only if m is a prime.

(c) Some authors call a subset S of $\mathbb{R}$ a *Bernstein set* if neither S nor its complement S' has an uncountable compact subset. It is easy to see that S has this property if and only if S is universally nonmeasurable. Indeed, inner regularity of elements of J at their measurable sets shows that this property fails for S if S fails to be universally nonmeasurable. On the other hand, if K is an uncountable compact subset of $\mathbb{R}$, then [see (a)] there exists $\mu \in J$ with $\mu(K') = 0$; hence, if K lies in either S or S', then either $\mu(S') = 0$ or $\mu(S) = 0$ so $S \in \mathscr{M}_\mu$.

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Mathematics is the pursuit of necessary consequences of arbitrary axioms about meaningless things.

—*Anonymous*

A Combinatorial Generalization of a Putnam Problem

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As a part of the thirty-fourth William Lowell Putnam Mathematical Competition, the following problem appeared in the MONTHLY [2]:

Let $a_1, a_2, \ldots, a_{2n+1}$ be a sequence of integers such that, if any of them is removed, the remaining ones can be divided into two sets of n integers with equal sums. Prove $a_1 = a_2 = \cdots = a_{2n+1}$.

Here we give a combinatorial proof of a generalization of this problem. The arguments rely on a matrix theoretic formulation of the original problem and elementary properties of cyclotomic polynomials.

Theorem 1. *Let ξ be a primitive q-th root of unity where $q = p^r$, p prime. Suppose we are given a sequence S of $qn + 1$ complex numbers $z_1, z_2, \ldots, z_{qn+1}$ with the property that for every i, $1 \leqslant i \leqslant qn + 1$, $S \setminus \{z_i\}$ can be partitioned into q equal size subsets $S_{i,0}, S_{i,1}, \ldots, S_{i,q-1}$ with*

$$\sum_{k=0}^{q-1} \xi^k \sum_{z_j \in S_{i,k}} z_j = 0. \tag{1}$$

Then $z_1 = z_2 = \cdots = z_{qn+1}$.

Note that the original problem is a special case of Theorem 1 in which $p = 2$, $r = 1$ and each z_i is an integer.

Proof: For each i fix a partition $S_{i,0}, S_{i,1}, \ldots, S_{i,q-1}$ of $S \setminus \{z_i\}$ satisfying (1). Let $N = qn$ and consider the $(N+1) \times (N+1)$ zero diagonal matrix $\mathbf{A} = \|a_{ij}\|$ where for $i \neq j$, $a_{ij} = \xi^k$ if and only if $z_j \in S_{i,k}$. If we put $\bar{\mathbf{z}} = [z_1, z_2, \ldots, z_{N+1}]^T$, then $\bar{\mathbf{z}}$ is a solution of the linear system $\mathbf{Az} = \mathbf{0}$. Since $\sum_{k=0}^{q-1} \xi^k = 0$, $\mathbf{A}$ is singular with zero row sums and $[1, 1, \ldots, 1]^T$ is in the kernel of $\mathbf{A}$. Thus to prove the theorem, it suffices to show that rank($\mathbf{A}$) $= N$.

Let $f(x)|_{x^k}$ denote the coefficient of the term x^k in a polynomial $f(x)$. Then up to sign, $\det(x\mathbf{I} - \mathbf{A})|_{x^r}$ is the sum of the $(N+1-r) \times (N+1-r)$ principal minors of $\mathbf{A}$. We will show that $\det(x\mathbf{I} - \mathbf{A})|_x$ must be nonzero, and hence rank($\mathbf{A}$) $= N$. We argue as follows.

Let M_j be the $N \times N$ principal minor of $\mathbf{A}$ corresponding to the jth diagonal entry. In the expansion of M_j from first principles, we have

$$M_j = \sum_{\sigma} (-1)^{i(\sigma)} \prod_{\substack{i=1 \\ i \neq j}}^{N+1} a_{i\sigma_i}, \tag{2}$$

in which the summation is over all permutations (in fact derangements) σ of the index set $\{1, \ldots, j-1, j+1, \ldots, N+1\}$, and $(-1)^{i(\sigma)}$ is the sign of σ. Clearly the nonzero terms in the sum in (2) are of the form $\pm\xi^e$, for various $e \in \{0, 1, \ldots, q-1\}$. Since $\mathbf{A}$ has zero diagonal and nonzero off-diagonal entries, the sum $\Sigma(-1)^{i(\sigma)}$ over such terms in M_j is given by

$$\det(\mathbf{J} - \mathbf{I}) = (-1)^{N-1}(N-1)$$

where $\mathbf{J}$ is the $N \times N$ matrix of 1's and $\mathbf{I}$ is the $N \times N$ identity matrix. Since this is true for every M_j, we conclude that

$$\det(x\mathbf{I} - \mathbf{A})|_x = \sum_{j=1}^{N+1} M_j = c_{q-1}\xi^{q-1} + \cdots + c_1\xi + c_0,$$

with

$$c_{q-1} + \cdots + c_1 + c_0 = (-1)^{N-1}(N-1)(N+1). \tag{3}$$

Now by way of contradiction, assume that

$$c_{q-1}\xi^{q-1} + \cdots + c_1\xi + c_0 = 0.$$

Setting

$$f(t) = c_{q-1}t^{q-1} + \cdots + c_1 t + c_0,$$

we then have $f(\xi) = 0$. Furthermore, $f(t)$ has integral coefficients. Therefore, the q-th cyclotomic polynomial $\Phi_q(t)$ must divide $f(t)$. Note also from (3) that $f(1) \equiv (-1)^N \pmod p$. Writing $f(t) = \Phi_q(t)h(t)$, we must have that $\Phi_q(1)h(1) \equiv (-1)^N \pmod p$. In particular, $\Phi_q(1) \not\equiv 0 \pmod p$. But we can easily show that for $m = p^r$ with $r > 0$ and p prime, we must have $\Phi_m(1) = p$. To see this, recall that

$$t^m - 1 = \prod_{d|m} \Phi_d(t)$$

(see, for example, [**3**]), and thus, by Möbius inversion,

$$\Phi_m(t) = \prod_{d|m} (t^d - 1)^{\mu(\frac{m}{d})}. \tag{4}$$

In (4), μ is the Möbius function defined by

$$\mu(m) = \begin{cases} 1 & \text{if } m = 1 \\ (-1)^\nu & \text{if } m \text{ is a product of } \nu \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

It immediately follows that for $m = p^r$, $r > 0$,

$$\Phi_m(t) = \frac{t^{p^r}}{t^{p^{r-1}} - 1} = 1 + t^{p^{r-1}} + t^{2p^{r-1}} + \cdots + t^{(p-1)p^{r-1}},$$

and so $\Phi_m(1) = p$. This gives us the desired contradiction.

We note that the property of $\Phi_m(1)$ for $m = p^r$ that we have made use of is a special case of the following more general result

$$\Phi_m(1) = \begin{cases} 0 & \text{iff } m = 1 \\ p & \text{iff } m = p^r,\ p \text{ prime},\ r > 0 \\ 1 & \text{iff } m \text{ has two or more prime factors,} \end{cases}$$

which can be found in [**1**]. □

In proving Theorem 1 we used the fact that the row sums of the matrix $\mathbf{A}$ vanish only to show that rank($\mathbf{A}$) $< N + 1$. The same argument used in the proof also provides a combinatorial proof of the following linear algebra result:

Theorem 2. *Suppose* $\mathbf{A}$ *is an* $N \times N$ *zero diagonal matrix whose off-diagonal entries are q-th roots of unity for some* $q = p^r$, *p prime*, $r > 0$. *If* $N \not\equiv 1 \pmod p$, *then* $\mathbf{A}$ *is nonsingular.*

Remarks. Note that Theorem 2 and its proof apply more generally to a matrix whose diagonal entries are algebraic integers which are merely divisible by the prime p.

Furthermore, if q is not a prime power, then we can show that the conclusion of Theorem 1 is false. In this case $q = uv$ with $\gcd(u, v) = 1$. Using the Chinese remainder theorem, pick $t < q$ with $t \equiv 0 \pmod u$ and $t \equiv 1 \pmod v$. Take $z_1 = \cdots = z_t = 1$ and $z_{t+1} = \cdots = z_{qn+1} = 0$. Then the twin identities

$$1 + \xi^v + \cdots + \xi^{v(t-1)} = 0, \qquad 1 + \xi^u + \cdots + \xi^{u(t-2)} = 0$$

show that no matter which z_i is discarded, the remaining ones can be multiplied by q-th roots of 1 using n copies of each root in such a way that they sum to 0.

Finally, we can consider the variant of the problem in which the classes $S_{i,0}, S_{i,1}, \ldots, S_{i,q-1}$ are not required to have the same cardinality. In this case Theorem 2 implies that the solution, if it exists, must be unique up to scalar multiples. It is easy to see that the sequence $1, 1, 1, 3, 3$ for example, admits a solution in this general sense.

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A Sufficient Condition for All the Roots of a Polynomial To Be Real

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Let

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

be a polynomial of degree $n \geq 2$ with real coefficients. If all the roots of P_n are real, a result due to Newton (see [2], §2.22 and §4.3, for two proofs) implies that the coefficients of P_n satisfy the following concavity condition:

$$a_i^2 - \frac{n-i+1}{n-i}\,\frac{i+1}{i}\,a_{i-1}a_{i+1} \geq 0, \qquad i = 1, 2, \ldots, n-1. \tag{1}$$

If the roots of P_n are not all equal, these inequalities are strict.

A question naturally arises: is the converse of Newton's result true? That is, if the coefficients of P_n satisfy (1) (or some similar concavity condition), must all the roots of P_n be real? When $n = 2$, (1) becomes

$$a_1^2 - 4a_0a_2 \geq 0,$$

the familiar necessary and sufficient condition for the roots of a quadratic to be real. Note that if the inequality is strict, the roots are distinct. For $n = 3$, (1) becomes

$$a_1^2 - 3a_0a_2 \geq 0, \qquad a_2^2 - 3a_1a_3 \geq 0.$$

Unfortunately, these are not sufficient to guarantee real roots for the cubic below satisfies these inequalities and has a pair of non-real roots:

$$P_3(x) = 5x^3 + 39x^2 + 92x + 58 = (x+1)(5x^2 + 34x + 58).$$

Other similar examples may be constructed for $n > 3$, so the concavity condition (1) is not sufficient to imply that all the roots of P_n are real, $n \geq 3$. However, all is not lost, for it turns out that a stronger concavity condition is sufficient. Before we state and prove such a result, we need a preliminary Lemma:

Lemma 1. *Let P_n be a polynomial of degree $n \geq 2$, with real coefficients and all of whose roots have negative real parts. If P_n has a repeated real root, then for some i, $1 \leq i \leq n-1$,*

$$a_i^2 - 4a_{i-1}a_{i+1} \leq 0.$$

Proof: The Lemma is clearly true for $n = 2$. Suppose $n > 2$ and that P_n has a repeated real root $-a$, $a > 0$. We write

$$P_n(x) = (x+a)^2\left(b_{n-2}x^{n-2} + b_{n-3}x^{n-3} + \cdots + b_0\right).$$

Then all the b's are positive (since they are real and the roots lie in the left

half-plane) and we have

$$P_n(x) = \sum_{j=0}^{n} \left(b_{j-2} + 2ab_{j-1} + a^2 b_j\right) x^j,$$

where $b_i \equiv 0$ if $i < 0$ or $i > n-2$. The case $n = 3$ will give us a clue to the proof in the general case. Suppose that $a_1^2 - 4a_0a_2 > 0$ and $a_2^2 - 4a_1a_3 > 0$. But this means that

$$b_0 - 4ab_1 > 0 \quad \text{and} \quad ab_1 - 4b_0 > 0,$$

an impossibility. Now let $n > 3$. Suppose that for $1 \le k \le n-1$, $a_k^2 - 4a_{k-1}a_{k+1} > 0$. Then for these values of k we have

$$a^4b_k^2 + b_{k-2}^2 - 4a^3b_kb_{k-1} - 4ab_{k-1}b_{k-2} - 14a^2b_kb_{k-2}$$
$$-8a^3b_{k+1}b_{k-2} - 4a^2b_{k+1}b_{k-3} - 8ab_kb_{k-3} - 4b_{k-3}b_{k-1} - 4a^4b_{k-1}b_{k+1} > 0. \tag{2}$$

For $1 \le k \le n-1$, let

$$q_k = ab_k - 4b_{k-1}, \qquad r_k = b_{k-1} - 4ab_k.$$

Using this notation and ignoring the last 6 terms in (2) we obtain

$$a^3b_kq_k + b_{k-2}r_{k-1} > 0, \qquad 1 \le k \le n-1. \tag{3}$$

When $k = 1$ we have $a^3b_1q_1 > 0$ so $q_1 > 0$. But this implies $r_1 < 0$. When $k = n-1$ we have

$$b_{n-3}r_{n-2} > 0$$

so $r_{n-2} > 0$. Let ν be the smallest integer such that $r_\nu > 0$. We see that

$$1 < \nu < n-1.$$

Then (3) implies

$$a^3b_\nu q_\nu + b_{\nu-2}r_{\nu-1} > 0.$$

Since $r_{\nu-1} \le 0$, $q_\nu > 0$. But this implies $r_\nu < 0$, a contradiction.

Now for our main result.

Theorem 1. *Let P_n be a polynomial of degree $n \ge 2$ with positive coefficients. If*

$$a_i^2 - 4a_{i-1}a_{i+1} > 0 \qquad i = 1, 2, \ldots, n-1 \tag{4}$$

then all the roots of P_n are real and distinct.

Proof: The proof will be by induction on n. Clearly the theorem is true for $n = 2$. Let $n > 2$ and let $P_n(x)$ be a polynomial of degree n with positive coefficients for which (4) holds. Put

$$Q(x) = P_n(x) - a_0 = xR(x).$$

Now $R(x)$, by the induction hypothesis, has distinct real roots, all of which are negative. Thus $Q(x)$ has n distinct real roots, the largest of which is 0. Consider $Q_\lambda(x) = Q(x) + \lambda$ where $0 \le \lambda$. Let $N(\lambda)$ be the number of distinct real roots of $Q_\lambda(x)$. Note that $N(0) = n$. Let $S = \{\lambda : \lambda > 0, N(\lambda) < n\}$. Clearly $S \neq \varnothing$ and bounded below, so let λ_0 be the greatest lower bound of S. If $\lambda_0 > a_0$ we are done, so suppose that $\lambda_0 \le a_0$. Since the roots of a polynomial vary continuously as its coefficients vary (see, for example, [1]) and $Q(x) = Q_0(x)$ has n distinct real roots, there exists an $\varepsilon > 0$ such that if $0 \le \lambda < \varepsilon$, $Q_\lambda(x)$ also has n distinct real roots. Thus $\lambda_0 > 0$. If $N(\lambda_0) = n$ then (by the same reasoning) there exists an

$\varepsilon > 0$ such that $(\lambda_0, \lambda_0 + \varepsilon) \cap S = \varnothing$, which is impossible; thus $N(\lambda_0) < n$ and hence $Q_{\lambda_0}(x)$ has a repeated real root or some non-real roots. Suppose that $Q_{\lambda_0}(x)$ has some non-real roots. Using the continuity of the roots again, there exists an $\varepsilon > 0$ such that $0 < \lambda_0 - \varepsilon$ and $Q_{\lambda_0 - \varepsilon}(x)$ also has some non-real roots. This means $\lambda_0 - \varepsilon \in S$, which contradicts our assumption that λ_0 is the greatest lower bound of S. Thus $Q_{\lambda_0}(x)$ has a repeated real root. Since all the roots of $Q_{\lambda_0}(x)$ are negative we can apply Lemma 1 and conclude that for some i, $a_i^2 - 4a_{i-1}a_{i+1} \le 0$. But if $\lambda_0 \le a_0$ then the coefficients of $Q_{\lambda_0}(x)$ satisfy (4), a contradiction. Thus $a_0 < \lambda_0$ and $P_n(x)$ has distinct real roots.

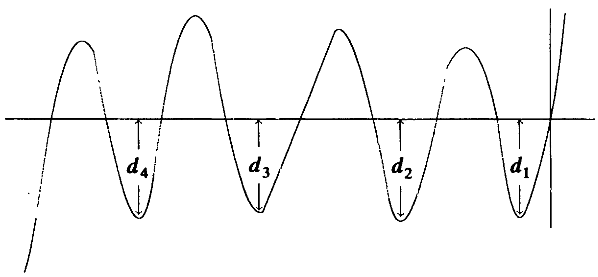
The idea in this proof gives rise to an interesting geometrical interpretation. Suppose that

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x$$

is a polynomial of degree at least two, with positive coefficients (except for the constant term, which is 0) which satisfy

$$a_i^2 - 4a_{i-1}a_{i+1} > 0 \quad \text{for } i = 2, 3 \ldots, n-1.$$

Theorem 1 shows that $P(x)$ has $n - 1$ distinct negative real roots and hence its graph looks like



Let $d_1, d_2, \ldots, d_{[(n-1)/2]}$ be the depths of the local minima of $P(x)$ and set

$$d = \min\{d_i : 1 \le i \le [(n - 1/2]\}.$$

Then $P(x) + \lambda$ will have distinct real roots as long as $\lambda < d$. If

$$\frac{a_1^2}{4a_2} > d,$$

then Theorem 1 implies that $P(x) + d$ has $n - 1$ distinct real roots, which it does not, since at least one root is repeated. Thus we have the following estimate for the depth of the relative minima:

$$d_i \ge \frac{a_1^2}{4a_2}, \qquad 1 \le i \le [(n-1)/2].$$

Can the coefficient 4 in inequalities (4) of Theorem 1 be improved? The following theorem shows that it cannot.

Theorem 2. *Given $\varepsilon > 0$ and an integer $n \ge 2$, there is a polynomial with positive coefficients of degree n which has some non-real roots and whose coefficients satisfy*

$$a_i^2 - (4 - \varepsilon)a_{i-1}a_{i+1} > 0, \qquad 1 \le i \le n - 1.$$

Proof: First, we introduce the following notation. If

$$P(x) = \sum_{j=0}^{n} b_j x^j,$$

we define

$$S(P,i) = \frac{b_i^2}{b_{i-1}b_{i+1}}.$$

The proof is by induction on n. Clearly the theorem is true for $n = 2$, so let $n > 2$ and suppose that the result is true for $n - 1$. Let $\varepsilon > 0$ and let

$$P_{n-1}(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0$$

be a polynomial with positive coefficients of degree $n - 1$ with some non-real roots and whose coefficients satisfy

$$a_i^2 - (4 - \varepsilon/2)a_{i-1}a_{i+1} > 0, \qquad 1 \le i \le n-2.$$

We put

$$P_\mu(x) = (\mu x + 1)P_{n-1}(x).$$

Then

$$S(P_\mu, 1) = \frac{\mu a_0^2 + 2a_0a_1 + \mu^{-1}a_1^2}{a_0a_1 + \mu^{-1}a_0a_2},$$

$$S(P_\mu, n-1) = \frac{a_{n-2}^2 + \mu^{-1}(2a_{n-2}a_{n-1}) + \mu^{-2}a_{n-1}^2}{a_{n-1}a_{n-3} + \mu^{-1}a_{n-1}a_{n-2}},$$

and for $i = 2, 3, \ldots, n-2$,

$$S(P_\mu, i) = \frac{a_{i-1}^2 + \mu^{-1}(2a_{i-1}a_i) + \mu^{-2}a_i^2}{a_ia_{i-2} + \mu^{-1}(a_{i-2}a_{i+1} + a_ia_{i-1}) + \mu^{-2}a_{i-1}a_{i+1}}.$$

Since

$$\lim_{\mu \to \infty} S(P_\mu, i) = S(P_{n-1}, i-1) \text{ for } i = 2, 3, \ldots, n-1$$

and

$$\lim_{\mu \to \infty} S(P_\mu, 1) = \infty,$$

we may choose μ large enough so that $S(P_\mu, i) > 4 - \varepsilon$, $i = 1, 2, \ldots, n-1$, which completes the proof.

The requirement for positive coefficients is necessary, for

$$x^3 - 5x^2 + 6x + 1$$

has two non-real roots even though the coefficients satisfy the concavity condition (4). Of course, Theorem 1 can be easily extended to the cases where all the coefficients have the same sign or the coefficients alternate in sign.

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Improving an Approximation for Pi

Daniel Shanks

Let P be an approximation to pi, good to n decimals; n is whatever you wish. we want a simple explicit function of P correct to $3n$ decimals. The answer is

$$P + \sin P.$$

For example, if

$$P = 3.142$$

then

$$P + \sin P = 3.14159265360.$$

The proof is easy: let $P = \text{pi} + x$, then evaluate $\sin P$ as a power series in x.

Many variations are readily apparent: for example,

$$P + 2\cos(P/2)$$

is somewhat better since the trigonometric argument is reduced by a factor of 2 and the error by a factor of 4. And further, iteration of

$$\cos x = 2\cos^2(x/2) - 1$$

can make the trigonometric argument as small as one wishes. If one wants $5n$ correct decimals that can be done with

$$P + (2\sin P - \tan P)/3.$$

PROBLEMS AND SOLUTIONS

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An asterisk ($*$) after the number of a problem, or part of a problem, indicates that no solution is currently available. Partial solutions will be useful in such cases. Otherwise, the published solution is likely to be based on a solution which is complete and correct. Of course, an elegant partial solution or a method leading to a more general result is always useful and welcome. In addition, references to other appearances of MONTHLY problems or to solutions of these problems in the literature are also solicited.

PROBLEMS

10202. *Proposed by Juan Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let A', B', C' be the feet of the altitudes of $\triangle ABC$ and let X, Y, Z be the centers of the circumscribing rectangles of $\triangle ABC$ with edges BC, CA, AB respectively. Prove that $\triangle XYZ$ is a dilation of $\triangle A'B'C'$.

10203. *Proposed by Ivan Vidav, University of Ljubljana, Ljubljana, Yugoslavia.*

Suppose that a, b, c and d are positive integers satisfying the two relations

$$b^2 + 1 = ac \quad \text{and} \quad c^2 + 1 = bd.$$

Prove that $a = 3b - c$ and $d = 3c - b$.

10204. *Proposed by Edgar A. Ramos and Douglas B. West, University of Illinois, Urbana, IL.*

Given a strongly connected directed graph G, let $s(G)$ be the length of the shortest closed walk visiting every vertex. Determine, for each positive integer n,

the maximum value of $s(G)$ over strongly connected directed graphs with n vertices.

10205. *Proposed by Richard Sinkhorn, University of Houston, Houston, TX.*

In elementary linear algebra, two different definitions of the word "adjoint" are used. The adjoint of a square matrix A with complex entries is either:

(I) the matrix whose (i, j)-entry is the cofactor of a_{ji} in A; or,

(II) the complex conjugate of the transpose of A.

Under what conditions on the matrix A will these two definitions yield the same matrix?

10206. *Proposed by David M. Bloom, Brooklyn College of CUNY, Brooklyn, NY.*

If m and k are positive integers, prove that

$$\sum_r \binom{r}{k-r}\binom{m}{r} = \sum_j \binom{\lfloor j/2 \rfloor}{k-j}\binom{m-k+\lfloor 3j/2 \rfloor}{j}.$$

10207. *Proposed by Eric Freden (student), Brigham Young University, Provo, UT.*

Find a closed form for $\Sigma_{n=0}^{\infty}\mathrm{Vol}(B^n)$ where B^n is the unit ball in $\mathbf{R}^n$ (and $\mathrm{Vol}(B^0)$ is taken to be 1).

10208. *Proposed by Solomon Golomb, University of Southern California, Los Angeles, CA.*

Let $1 \le a_1 < a_2 < a_3 < \dots$ be an increasing sequence of positive integers.

(a) Is there such a sequence $\{a_k\}$ having the property that, for all integers n (positive, negative, or zero), $\{a_k + n\}$ contains only finitely many primes?

(b)* Is there such a sequence $\{a_k\}$ and a constant $B > 0$ having the property that $\{a_k + n\}$ contains no more than B primes for every integer n?

10209. *Proposed by Feng Hanqiao, Shaanxi Normal University, Xian, China, and Siu-Ah Ng, University of Hull, Hull, England.*

For each non-negative integer k, define $a_k(n)$ for non-negative integers n by

$$a_k(0) = 1 \quad \text{and} \quad a_k(i+1) = a_k(i)\left(1 + \frac{1}{k}a_k(i)\right) \qquad (i \ge 0).$$

Find $\sup_n a_{mn}(n)$ for $m = 1, 2, \dots$.

10210. *Proposed by D. H. Fremlin, University of Essex, Colchester, England.*

(a) Let f be a continuous non-negative real-valued function defined on the square $[0,1]^2$. Show that

$$\int_0^1\int_0^1\int_0^1\int_0^1 f(x_1,y_1)f(x_2,y_1)f(x_1,y_2)\,dx_1\,dx_2\,dy_1\,dy_2 \ge \left(\int_0^1\int_0^1 f(x,y)\,dx\,dy\right)^3.$$

(b) Show that there is a continuous non-negative real-valued function f defined on the cube $[0,1]^3$ such that

$$\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 f(x_2,y_1,z_1)f(x_1,y_2,z_1)f(x_1,y_1,z_2)\,dx_1\,dx_2\,dy_1\,dy_2\,dz_1\,dz_2 < \left(\int_0^1\int_0^1\int_0^1 f(x,y,z)\,dx\,dy\,dz\right)^3.$$

NOTES

(10204) A directed graph is *strongly connected* if it has a (directed) path from each vertex to every other vertex. A *closed walk* is a cyclic list of (not necessarily distinct) edges such that the head of each edge is the tail of the next edge. (10206) The sums are taken over all integer values of the indicated variable. Each is seen to be a finite sum under the usual conventions on vanishing of binomial coefficients. (10207) An obvious first step is to find a formula for $\mathrm{Vol}(B^n)$. This is rumored to be "well known", so it suffices to provide a reference. (10210) The integral on the right in (a) may be thought of as the integral of $f(x_1, y_1) \cdot f(x_2, y_2) \cdot f(x_3, y_3)$ as all variables range over the interval $[0, 1]$. The integral on the left would not be changed if one were to integrate it further over $[0, 1]$ with respect to variables x_3 and y_3 not occurring in the expression. The problem deals with comparing two expressions, each of which is the integral over the same space of a product of three variants of $f(x, y)$. The general question underlying this problem is to determine when the relation of the size of the integrals depends on the variants being integrated and not on the function f.

SOLUTIONS

Some Definite Integrals and Infinite Series

E 3372 [1990, 151]. *Proposed by W. A. Bassali, Kuwait University, Safat, Kuwait.*

Suppose $\delta_n = \binom{2n}{n}4^{-n}$ for $n = 0, 1, 2, \ldots$. Prove that

$$\int_0^{\pi/2} \sin^{-1}(\sin^2\theta)\, d\theta = \int_0^{\pi/2} \cos^{-1}\sqrt{\cos\theta}\, d\theta$$

$$= \int_0^1 \frac{\sinh^{-1} x}{x}\, dx = 2\int_0^{\pi/2} \frac{\theta\, d\theta}{\sqrt{\sec^2\theta + 1}}$$

$$= \frac{1}{2}\int_0^{\pi/2} \theta\sqrt{\operatorname{cosec}\theta + 1}\, d\theta = \sum_{n=0}^{\infty} \frac{(-1)^n \delta_n}{(2n+1)^2}$$

$$= \frac{\pi^2}{8} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{\delta_{2n+1}}{(2n+1)^2} = \frac{\pi}{2}\sum_{n=0}^{\infty} \frac{\delta_n\, \delta_{2n+1}}{2n+1}.$$

Solution by Kee-Wai Lau, Hong Kong and the editors. Let us denote the eight expressions of the problem by A, B, C, D, E, F, G, H respectively. To prove them all equal we require seven steps. Since $\delta_n \sim (\pi n)^{-1/2}$, the series in F, G, and H

converge absolutely. We shall require the three expansions:

$$(1-x)^{-1/2} = \sum_{n=0}^{\infty} \delta_n x^n \qquad (|x| < 1), \tag{1}$$

$$\sin^{-1} x = \int_0^x (1-t^2)^{-1/2}\, dt = \sum_{n=0}^{\infty} \delta_n x^{2n+1}/(2n+1) \qquad (|x| \le 1), \tag{2}$$

$$\sinh^{-1} x = \int_0^x (1+t^2)^{-1/2}\, dt = \sum_{n=0}^{\infty} (-1)^n \delta_n x^{2n+1}/(2n+1) \qquad (|x| \le 1), \tag{3}$$

the last two of which converge absolutely and uniformly on the closed unit disc. We also need Wallis' definite integral

$$\int_0^{\pi/2} \sin^{2k}\theta\, d\theta = (\pi/2)\,\delta_k \qquad (k = 1, 2, 3, \ldots). \tag{4}$$

Step 1, $A = B$. Using the substitution $y = \sin^{-1}(\sin^2\theta)$, that is,

$$\theta = \sin^{-1}\sqrt{\sin y} = \pi/2 - \cos^{-1}\sqrt{\sin y},$$

and integrating by parts, we obtain

$$\begin{aligned} A &= \int_0^{\pi/2} \sin^{-1}(\sin^2\theta)\, d\theta = \int_0^{\pi/2} y\, d\left(-\cos^{-1}\sqrt{\sin y}\right) \\ &= -y\cos^{-1}\sqrt{\sin y}\,\Big|_0^{\pi/2} + \int_0^{\pi/2} \cos^{-1}\left(\sqrt{\sin y}\right) dy \\ &= \int_0^{\pi/2} \cos^{-1}\sqrt{\cos\theta}\, d\theta = B. \end{aligned}$$

Step 2, $A = D$. Using the substitution $\phi = \pi/2 - \theta$ and integrating by parts, we get

$$\begin{aligned} A &= \int_0^{\pi/2} \sin^{-1}(\cos^2\phi)\, d\phi \\ &= \phi\sin^{-1}(\cos^2\phi)|_0^{\pi/2} + 2\int_0^{\pi/2} \frac{\phi\sin\phi\cos\phi\, d\phi}{\sqrt{1-\cos^4\phi}} \\ &= 2\int_0^{\pi/2} \frac{\phi\cos\phi\, d\phi}{\sqrt{1+\cos^2\phi}} = D. \end{aligned}$$

Step 3, $A = E$. Using the substitution $\phi = \sin^{-1}(\sin^2\theta)$, that is, $\theta = \sin^{-1}\sqrt{\sin\phi}$, we get

$$A = \int_0^{\pi/2} \frac{\phi\cos\phi\, d\phi}{2\sqrt{\sin\phi}\,\sqrt{1-\sin\phi}} = \int_0^{\pi/2} \frac{\phi\sqrt{1+\sin\phi}\, d\phi}{2\sqrt{\sin\phi}} = E$$

Step 4, $A = H$. Using (2) and (4), we have

$$A = \int_0^{\pi/2} \sum_{n=0}^{\infty} \frac{\delta_n}{2n+1} \sin^{4n+2}\theta\, d\theta = \sum_{n=0}^{\infty} \frac{\delta_n}{2n+1} \int_0^{\pi/2} \sin^{4n+2}\theta\, d\theta = H.$$

Step 5, $C = F$. Using (3), we obtain

$$C = \int_0^1 \frac{\sinh^{-1} x\, dx}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n \delta_n}{2n+1} \int_0^1 x^{2n}\, dx = F.$$

Step 6, $B = F$. We require the absolutely and uniformly convergent expansion

$$\sin^{-1}\sqrt{1-\cos\theta} = \sum_{n=0}^{\infty} \frac{(-1)^n \delta_n \sin(2n+1)\theta}{2n+1} \qquad (0 \le \theta \le \pi/2), \qquad (5)$$

which can be proved as follows. Let $\alpha = \sinh^{-1}(e^{-i\theta})$, $\beta = \sinh^{-1}(e^{-i\theta})$, where by (3) and (2) we have

$$|\alpha| = |\beta| \le \sum \delta_n/(2n+1) = \pi/2.$$

Since

$$\begin{aligned} \cos\theta + i\sin\theta &= \sinh(\operatorname{Re}\alpha + i\operatorname{Im}\alpha) \\ &= \sinh(\operatorname{Re}\alpha)\cos(\operatorname{Im}\alpha) + i\cosh(\operatorname{Re}\alpha)\sin(\operatorname{Im}\alpha) \end{aligned}$$

and $\sin\theta \ge 0$, we must have $\sin(\operatorname{Im}\alpha) \ge 0$ and so $0 \le \operatorname{Im}\alpha \le \pi/2$. By (3)

$$\sum_{n=0}^{\infty} \frac{(-1)^n \delta_n \sin(2n+1)\theta}{2n+1} = \operatorname{Im}\alpha.$$

Now

$$\begin{aligned} 2\sin^2(\operatorname{Im}\alpha) &= 2\sin^2\{(\alpha-\beta)/(2i)\} \\ &= 1 - \cosh(\alpha-\beta) \\ &= 1 - \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta \\ &= 2 - \cosh\alpha\cosh\beta. \end{aligned}$$

Since $\cosh\alpha\cosh\beta = |\cosh\alpha|^2 \ge 0$, we have

$$\begin{aligned} 2\sin^2(\operatorname{Im}\alpha) &= 2 - \sqrt{(1+\sinh^2\alpha)(1+\sinh^2\beta)} \\ &= 2 - \sqrt{(1+e^{2i\theta})(1+e^{-2i\theta})} \\ &= 2 - \sqrt{(e^{i\theta}+e^{-i\theta})^2} \\ &= 2 - 2\cos\theta. \end{aligned}$$

Since $0 \le \operatorname{Im}\alpha \le \pi/2$, we have

$$\sin(\operatorname{Im}\alpha) = \sqrt{1-\cos\theta}$$

or

$$\operatorname{Im}\alpha = \sin^{-1}\sqrt{1-\cos\theta}\,.$$

Thus (5) is proved. Using (5) we have

$$\begin{aligned} B &= \int_0^{\pi/2} \cos^{-1}\sqrt{\cos\theta}\, d\theta = \int_0^{\pi/2} \sin^{-1}\sqrt{1-\cos\theta}\, d\theta \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \delta_k}{2k+1} \int_0^{\pi/2} \sin(2k+1)\theta\, d\theta = \sum_{k=0}^{\infty} \frac{(-1)^k \delta_k}{(2k+1)^2} = F. \end{aligned}$$

Step 7, $C = G$. We begin with the following formula

$$\int_0^1 (1-x^2)^{-1} \ln(1/x)\,dx = \int_0^\infty (1-e^{-2u})^{-1} u e^{-u}\,du$$

$$= \sum_{n=0}^\infty \int_0^\infty e^{-2nu} u e^{-u}\,du = \sum_{n=0}^\infty (2n+1)^{-2} = \pi^2/8.$$

Making the substitution $x = \sin\theta$, we have

$$\frac{\pi^2}{8} = \int_0^1 \frac{\ln(1/x)}{1-x^2}\,dx = \lim_{\varepsilon\to 0+} \int_0^{\pi/2-\varepsilon} \frac{\ln(\sec\theta/\tan\theta)}{\cos\theta}\,d\theta$$

$$= \lim_{\varepsilon\to 0+} \left\{ \int_0^{\pi/2-\varepsilon} \sec\theta \ln\sec\theta\,d\theta - \int_0^{\pi/2-\varepsilon} \sec\theta \ln\tan\theta\,d\theta \right\}.$$

Substituting $s = \sec\theta$ in the first integral and $s = \tan\theta$ in the second integral, we get

$$\frac{\pi^2}{8} = \lim_{\varepsilon\to 0+} \left\{ \int_1^{\sec(\pi/2-\varepsilon)} \frac{\ln s\,ds}{\sqrt{s^2-1}} - \int_0^{\tan(\pi/2-\varepsilon)} \frac{\ln s\,ds}{\sqrt{s^2+1}} \right\}.$$

Now $\sec(\pi/2-\varepsilon) - \tan(\pi/2-\varepsilon) = \tan(\varepsilon/2)$ and so we may change the upper limit of the first integral to $\tan(\pi/2-\varepsilon)$ without affecting the value of the limit. Hence

$$\frac{\pi^2}{8} = \int_1^\infty \left(\frac{\ln s}{\sqrt{s^2-1}} - \frac{\ln s}{\sqrt{s^2+1}} \right) ds - \int_0^1 \frac{\ln s}{\sqrt{s^2+1}}\,ds$$

$$= \int_1^\infty \frac{-\ln\left(s+\sqrt{s^2-1}\right) + \ln\left(s+\sqrt{s^2+1}\right)}{s}\,ds + \int_0^1 \frac{\ln\left(s+\sqrt{s^2+1}\right)}{s}\,ds,$$

where we have used integration by parts in each term.

Since $\sinh^{-1} s = \ln(s+\sqrt{s^2+1})$, the last integral is equal to C. Making the substitution $s = z^{-1/2}$ in the first integral, we get

$$\frac{\pi^2}{8} = \frac{1}{2}\int_0^1 \frac{-\ln(1+\sqrt{1-z}) + \ln(1+\sqrt{1+z})}{z}\,dz + C.$$

Using the integration formulas

$$\int \frac{dw}{w\sqrt{1\pm w}} = \ln w - 2\ln(\sqrt{1\pm w}+1),$$

we may rewrite the preceding as

$$\frac{\pi^2}{8} = \frac{1}{4}\int_0^1 \left\{ \int_0^z \left(\frac{1}{w\sqrt{1-w}} - \frac{1}{w\sqrt{1+w}} \right) dw \right\} \frac{dz}{z} + C.$$

Using (1), we find

$$\begin{aligned}
\frac{\pi^2}{8} &= \frac{1}{4}\int_0^1 \left\{ \int_0^z \left(\sum_{n=0}^{\infty} \delta_n w^{n-1} - \sum_{n=0}^{\infty} (-1)^n \delta_n w^{n-1} \right) dw \right\} \frac{dz}{z} + C \\
&= \frac{1}{2}\int_0^1 \sum_{n=0}^{\infty} \frac{\delta_{2n+1}}{2n+1} z^{2n}\, dz + C \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \frac{\delta_{2n+1}}{(2n+1)^2} + C,
\end{aligned}$$

so that $C = G$.

Editorial comment. Many other sequences of deductions are possible. For example, Jean Anglesio proved that $B = C$ as follows. By the substitution $x = -iy$ we get

$$C = \int_0^i \sinh^{-1}(-iy)\, dy/y = -i\int_0^i \sin^{-1} y\, dy/y.$$

By the Cauchy Integral Theorem we may change this to

$$C = -i\int_0^1 \sin^{-1} y\, dy/y - i\int_0^{\pi/2} \sin^{-1}(e^{it})\, d(e^{it})/e^{it},$$

since $\sin^{-1} y$ is analytic inside the unit circle and is continuous on the closed unit disc. Thus

$$C = \int_0^{\pi/2} \sin^{-1}(e^{it})\, dt - i\int_0^1 \sin^{-1} y\, dy/y,$$

so that

$$2C = C + \overline{C} = \int_0^{\pi/2} \{\sin^{-1}(e^{it}) + \sin^{-1}(e^{-it})\}\, dt.$$

By an argument similar to that by which (5) was proved in Step 6 we obtain

$$\sin^{-1}(e^{it}) + \sin^{-1}(e^{-it}) = 2\cos^{-1}\sqrt{\sin t} \qquad (0 \le t \le \pi/2).$$

Thus

$$C = \int_0^{\pi/2} \cos^{-1}\sqrt{\sin t}\, dt = \int_0^{\pi/2} \cos^{-1}\sqrt{\cos\theta}\, d\theta = B.$$

Solved also by J. Anglesio (France), R. J. Chapman (England), O. P. Lossers (the Netherlands), D. B. Tyler, and the proposer.

The Asymptotic Behavior of the Middle Binomial Coefficient

E 3373 [1990, 239]. *Proposed by Jeffrey Vaaler, University of Texas, Austin.*

Let $a_n = \binom{2n}{n} n^{1/2} 4^{-n}$. Without using Stirling's formula, prove that
(i) $\{a_n\}$ is a convergent sequence, and
(ii) if $L = \lim_{n\to\infty} a_n$, then

$$L e^{-1/(8n)} < a_n < L.$$

(Stirling's formula gives $L = \pi^{-1/2}$.)

Solution by Jean-Marie Monier, Lyon, France. A direct calculation yields

$$\frac{a_{n+1}}{a_n} = \frac{2n+1}{2\sqrt{(n+1)n}} = \left(1 + \frac{1}{4(n^2+n)}\right)^{1/2} > 1.$$

Hence, the sequence $\{a_n\}$ is strictly increasing. Since

$$\begin{aligned}\log a_{n+1} - \log a_n &= \frac{1}{2}\log\left(1 + \frac{1}{4(n^2+n)}\right) < \frac{1}{2}\cdot\frac{1}{4(n^2+n)}\\ &= \frac{1}{8}\left(\frac{1}{n} - \frac{1}{n+1}\right),\end{aligned} \tag{$*$}$$

we find that

$$\begin{aligned}\log a_n - \log a_1 &= \sum_{j=1}^{n-1}(\log a_{j+1} - \log a_j) < \frac{1}{8}\sum_{j=1}^{n-1}\left(\frac{1}{j} - \frac{1}{j+1}\right)\\ &= \frac{1}{8}\left(1 - \frac{1}{n}\right) < \frac{1}{8}.\end{aligned}$$

Thus, $a_n < a_1 e^{1/8} = (1/2)\,e^{1/8}$ for each positive integer n. Since $\{a_n\}$ is a bounded strictly increasing sequence, (i) and the upper bound on a_n in (ii) follow.

To obtain the lower bound on a_n in (ii), we observe that $(*)$ implies that

$$\log a_{n+1} + \frac{1}{8(n+1)} < \log a_n + \frac{1}{8n},$$

so that the sequence $\{\log a_n + 1/(8n)\}$ is strictly decreasing. Furthermore, $\{\log a_n + 1/(8n)\}$ converges to $\log L$ so that $\log a_n + 1/(8n) > \log L$ for all positive integers n, which implies the lower bound on a_n in (ii).

Editorial comment. Most of the solutions received were similar to the one given above. Some solvers observed that the upper bound $(1/2)\,e^{1/8}$ obtained for a_n above is a remarkably close elementary estimate. More specifically, $(1/2)\,e^{1/8} = 0.56657\ldots$, while the least upper bound is $L = 1/\sqrt{\pi} = 0.56418\ldots$.

Solved also by the proposer and 31 other readers. One partial solution was received.

The Longest Expected World Series

E 3386 [1990, 427]. *Proposed by Eugene F. Schuster, University of Texas, El Paso, TX.*

Let L be the length of a $(2N-1)$-game World Series, modeled as a sequence of independent identically distributed Bernoulli trials which terminates as soon as one team wins N games. (The length is the number of games actually played.) Prove the seemingly obvious observation that the expected length $E(L)$ of the series is maximized when the two teams are evenly matched.

Composite solution I by C. Georghiou, University of Patras, Greece, and Kumar Joag-Dev, University of Illinois at Urbana-Champaign. Let $L = N + k$, for $k \geq 0$. The probability distribution for the random variable L is given by

$$P(L = N+k) = \binom{N-1+k}{k}\left[p^N q^k + q^N p^k\right], \qquad k \geq 0,$$

where p, q are the win probabilities for the two teams in a single game. We have

$$E(L) = \sum_{k=0}^{N-1} (N+k)\binom{N-1+k}{k}[p^N q^k + q^N p^k]$$

$$= N\sum_{k=0}^{N-1}\binom{N+k}{N}[p^N q^k + q^N p^k].$$

Let $\varepsilon_N = E(L)/N$; note that $\varepsilon_1 = 1$. We claim that $\varepsilon_N - \varepsilon_{N-1} = (1/N)\binom{2N-2}{N-1}(pq)^{N-1}$, from which it follows that

$$E(L) = N\left\{1 + \sum_{k=2}^{N}(\varepsilon_k - \varepsilon_{k-1})\right\} = N\sum_{k=0}^{N-1}\frac{(pq)^k}{k+1}\binom{2k}{k}.$$

Hence $E(L)$ is maximized when pq is maximized, i.e., when $p = q = \frac{1}{2}$.

To prove the claim, we write $\varepsilon_N = \sum_{k=0}^{N-1}\binom{N+k}{N}w^k g(N-k)$, where $w = pq$ and $g(j) = p^j + q^j$. Note that $g(0) = 2$, $g(1) = 1$, and $g(j) = g(j-1) - wg(j-2)$ for $j \geq 2$. In the summation for ε_N, we separate out the last term, apply the recurrence for g to the other terms, and separate out the last term of the second resulting sum to obtain

$$\varepsilon_N = \binom{2N-1}{N}w^{N-1} + \sum_{k=0}^{N-2}\binom{N+k}{N}w^k g(N-1-k)$$
$$- \sum_{k=0}^{N-3}\binom{N+k}{N}w^{k+1}g(N-2-k) - 2\binom{2N-2}{N}w^{N-1}.$$

By collecting the terms involving w^{N-1}, shifting the index of the final summation, and applying the recurrence for the binomial coefficients, this becomes

$$\varepsilon_N = \left(\frac{2N-1}{N} - 2\frac{N-1}{N}\right)\binom{2N-2}{N-1}w^{N-1}$$
$$+ \sum_{k=0}^{N-2}\left[\binom{N+k}{N} - \binom{N-1+k}{N}\right]w^k g(N-1-k)$$
$$= \frac{1}{N}\binom{2N-2}{N-1}(pq)^{N-1} + \varepsilon_{N-1}.$$

Solution II by Fred Richman, TCI Software Research, Las Cruces, NM. We prove the stronger result that for every n, the probability that the nth game is played is maximized when $p = \frac{1}{2}$. This implies the desired result, because $E(L) = N + \sum_{n=N+1}^{2N-1} E(X_n)$, where X_n is 1 if the nth game is played and 0 otherwise. The value of $E(X_{n+1})$ is the probability that the $(n+1)$th game is played, which is the probability that the first team wins between $n - N + 1$ and $N - 1$ of the first n games. Letting $B(x, p)$ denote the cumulative probability in the binomial distribution with parameters n and p, we want to maximize $E(X_{n+1}) = B(N-1, p) - B(n-N, p)$, the middle part of the distribution.

We prove that this is maximized at $p = \frac{1}{2}$ by considering the derivative of $B(x, p)$ with respect to p. If we increase p by an infinitesimal amount, the probability that the number of successes is at most x decreases by the probability of having exactly x successes before the increase times the probability that one of the failures becomes a success when we increase p, which is $(n-x)\,dp/q$. Hence

$B(x, p + dp) = B(x, p) - \binom{n}{x} p^x q^{n-x}(n - x)\, dp/q$, or $dB(x, p)/dp = -(n - x)\binom{n}{x} p^x q^{n-x-1}$. (This differentiation formula can also be proved algebraically.) Noting that $(n - x)\binom{n}{x} = (x + 1)\binom{n}{x+1}$, we have $dE(X_{n+1})/dp = \binom{n}{N}(pq)^{n-N}(q^{2N-n-1} - p^{2N-n-1})$, which is positive if $p < \frac{1}{2}$ and negative if $p > \frac{1}{2}$

Editorial comment. It is interesting to note the appearance of the Catalan numbers $\binom{2k}{k}/(k + 1)$ in the formula for $E(L)$. K. Hinderer and M. Steiglitz refer to a discussion of this and related problems in their paper in *Didaktik der Mathematik* 15(2)(1987), 81–114 (see p. 102). The second solution above is equivalent to showing $P(L > n)$ is maximized at $p = \frac{1}{2}$ for every n, as shown directly by several solvers. John H. Lindsey II took the approach of proving the stronger result that $P(L = n + 1)/P(L = n)$ is maximized at $p = \frac{1}{2}$ for every n. Since $P(L = N + j)$ is proportional to $p^j q^N + p^N q^j$, it suffices to verify that, for every j, $(p^{j+1}q^N + p^N q^{j+1})/(p^j q^N + p^N q^j)$ has its maximum at $p = \frac{1}{2}$. This is easily proved by induction. There were a variety of other approaches.

Michael Perlman noted that any nondecreasing function of $L(N)$ has maximum expectation at $p = \frac{1}{2}$ and that similar conclusions hold for k-contestant series involving k-person games in which the series concludes when any contestant wins N of them. The fact that the expected series length is maximized when each player has probability $1/k$ of winning each game is implied by the Schur-concavity of the appropriate cumulative density function and a theorem of Y. Rinott (see *Israel J. Math.*, 15(1973) 60–77, and Marshal and Olkins' *Inequalities, Theory of Majorization and Its Applications*, Academic Press, 1979). Perlman also noted that if the series is prolonged until each contestant has won N games, then the expected length is minimized in the symmetric $1/k$ case, by Schur-convexity of the corresponding cumulative density function.

Solved also by A. Adler, R. A. Agnew, D. Callan, N. J. Fine, P. Griffin, E. Hertz, K. Hinderer & M. Steiglitz (Germany), R. D. Hurwitz, B. R. Johnson, B. G. Klein, A. Kozek (Poland), O. Krafft & M. Schaefer (Germany), K.-W. Lau (Hong Kong), J. H. Lindsey II, H. Lipman, M. D. Perlman, D. S. Romano, O. Saleh & S. Byrd, R. Stong, M. Vowe (Switzerland), D. P. Wiens, and the proposer. Three incorrect solutions were received.

Infinite Almost Everywhere

6632 [1990, 433]. *Proposed by Gilbert Muraz, Institut Fourier, Université de Grenoble I, St. Martin d'Héres, France, and Pawel Szeptycki and Fred Galvin, University of Kansas, Lawrence.*

Let E be a measurable subset of $\mathbb{R}$ modulo 1 having positive measure. For real t let N_t be the set of positive integers n such that nt modulo 1 is in E. Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive real numbers such that $\sum a_n = \infty$. Prove that

$$\sum_{n \in N_t} a_n = \infty$$

for almost all t in $[0, 1]$.

Solution by Nathan J. Fine, Deerfield Beach, Florida. By an abuse of notation we may consider E to be a subset of $[0, 1)$. Then let $E_0 = \bigcup_{j=0}^{\infty}(E + j)$, and let $\chi(t)$

be the characteristic function of E_0. Then for t in $[0,1]$ put

$$f(t) = \sum_{n \in N_t} a_n = \sum_{n \geq 1} a_n \chi(nt).$$

Let $0 \leq a < b \leq 1$, and let W be a measurable subset of (a,b) satisfying

$$m(W) - (b-a)\left(1 - \tfrac{1}{2}m(E)\right) = \Delta > 0.$$

We shall show that

$$\int_W f(t)\,dt = \infty.$$

For

$$\int_W f(t)\,dt = \sum_{n \geq 1} a_n \int_W \chi(nt)\,dt = \sum_{n \geq 1} \frac{a_n}{n} \int_{nW} \chi(t)\,dt.$$

Now

$$I_n = \int_{nW} \chi(t)\,dt = m(nW \cap E_1),$$

where $E_1 = E_0 \cap (na, nb)$. We have $I_n = m(nW \cap E_1) = m(nW) + m(E_1) - m(nW \cup E_1)$. Now $m(nW) = n \cdot m(W)$ and $m(nW \cup E_1) \leq n(b-a)$. Also

$$m(E_1) = \int_{na}^{nb} \chi(t)\,dt \geq \int_{[na]+1}^{[nb]} \chi(t)\,dt = ([nb] - [na] - 1)m(E).$$

For $n > 4/(b-a)$ we have

$$[nb] - [na] - 1 \geq nb - 1 - na - 1 > \tfrac{1}{2}n(b-a).$$

Hence, for such n,

$$\begin{aligned} I_n &\geq n \cdot m(W) + \tfrac{1}{2}n(b-a) \cdot m(E) - n(b-a) \\ &= n\Delta. \end{aligned}$$

Therefore,

$$\int_W f(t)\,dt \geq \sum_{n > \frac{4}{b-a}} \frac{a_n}{n} \cdot n\Delta = \Delta \sum_{n > \frac{4}{b-a}} a_n = \infty.$$

Finally, let $S \subset [0,1]$, $m(S) > 0$. By the metric density theorem, there is an $x_0 \in S$ and an interval (a,b) containing x_0, $a < b$, such that

$$m(S \cap (a,b)) > (b-a)\left(1 - \tfrac{1}{2}m(E)\right).$$

Choosing $W = S \cap (a,b)$, we have

$$\int_S f(t)\,dt \geq \int_W f(t)\,dt = \infty.$$

Therefore, $f(t) = \infty$ a.e.

Solved also by Adam Fieldsteel, Marcin E. Kuczma (Poland), John H. Lindsey II, Kenneth Schilling, and the proposers.

The First Orthant Must Be Penetrated

E 3395 [1990, 529]. *Proposed by Hillel Gauchman, Eastern Illinois University, Charleston, IL.*

Let A be the first orthant in n-dimensional Euclidean space E^n, i.e.,

$$A = \{x \in E^n : x = (x_1, x_2, \dots, x_n), x_1 \geq 0, \dots, x_n \geq 0\}.$$

Let S be a k-dimensional subspace through the origin in E^n, where $1 \leq k \leq n-1$, and let $S^{\perp}$ be the orthogonal complement to S through the origin. Prove that either S or $S^{\perp}$ contains a point of A other than the origin.

Solution by Dragomir. Ž. Đoković, University of Waterloo, Waterloo, Ontario, Canada. We obtain a nonnegative vector in S if $S^{\perp}$ has no nonnegative vector. Let $v_1, \dots, v_k$ be a basis of S, and let $f : E^n \to E^k$ be the linear map defined by $[f(x)]_i = x \cdot v_i$. Let $\Delta = \{x \in A : \sum x_i = 1\}$. The assumption $S^{\perp} \cap A = \{0\}$ implies $f(x) \neq 0$ for all $x \in \Delta$. Since Δ is convex in E^n and f is linear, $f(\Delta)$ is convex in E^k, so the fact that it avoids the origin guarantees a vector $y \in E^k$ such that $u \cdot y > 0$ for all $u \in f(\Delta)$.

Now let $b = \sum y_i v_i$. Since b is a linear combination of $\{v_i\}$, we have $b \in S$ and need only show $b \in A$. If $x \in \Delta$, then $x \cdot b = \sum y_i(x \cdot v_i) = y \cdot f(x) > 0$, where the final inequality follows from the choice of y. To show $b \in A$, consider the standard basis vectors $\{e_i\}$ of E^n. Since $e_i \in \Delta$, the computation above yields $e_i \cdot b > 0$, which implies that the coordinates of b are all positive, and hence $b \in A$.

Editorial comment. Many creative solutions were submitted. Methods used included induction on n, convexity theory, the Theorem of the Alternative in linear programming, the min-max theorem of matrix games, the Hahn-Banach Theorem, and a result on oriented matroids. The problem has been solved several times in the literature; readers mentioned the following: A. Ben-Israel, Notes on linear inequalities, *J. Math. Anal. Appl.* 9(1964), 303–314 (p. 308); and D. Gale, *Theory of Linear Economic Models*, McGraw-Hill, 1960, p. 48 (Corollary 1 to Theorem 2.9).

Solutions or references to solutions were submitted by I. D. Berg, F. Brulois & T. Shore, D. Callan, W. Fenton, J.-P. Grivaux (France), R. High, O. P. Lossers (The Netherlands), N. T. Peck, the late David Richman, K. Schilling, F. Schmidt, J. H. Steelman, R. Stong, Central Michigan University Problem Group, and the proposer. One incorrect solution was received.

Partitions of n into Parts Which Are Divisors of n

6640 [1990, 857]. *Proposed by Douglas Bowman (student), UCLA.*

Let $f(n)$ be the number of partitions of the positive integer n into parts taken from the set of divisors of n. Prove that

$$\{1 + o(1)\}\left\{\frac{\tau(n)}{2} - 1\right\}\log n \leq \log f(n) \leq \{1 + o(1)\}\frac{\tau(n)}{2}\log n,$$

where $\tau(n)$ is the number of divisors of n.

Solution by the Editors with the aid of Paul Erdös, Hungarian Academy of Sciences, Budapest, and Andrew M. Odlyzko, A.T.& T. Bell Labs, Murray Hill, NJ.

Clearly $f(n)$ is the coefficient of x^n in

$$\prod_{d|n}(1-x^d)^{-1},$$

and this is at most the sum of all the coefficients of the polynomial

$$\prod_{d|n}(1+x^d+x^{2d}+\cdots+x^{(n/d)d}),$$

namely

$$\prod_{d|n}(1+n/d)=\prod_{d|n}\left(1+\frac{1}{(n/d)}\right)\prod_{d|n}n/d.$$

Now

$$\prod_{d|n}\left(1+\frac{1}{(n/d)}\right)\le\exp\left(\sum_{d|n}\frac{1}{(n/d)}\right)\le\exp(C\log\log n)$$

by Theorem 323 of *An Introduction to the Theory of Numbers* by G. H. Hardy and E. M. Wright. Also

$$\left(\prod_{d|n}(n/d)\right)^2=\prod_{d|n}n/d\prod_{d|n}d=n^{\tau(n)},\text{ i.e., }\prod_{d|n}n/d=n^{\tau(n)/2}.$$

Hence

$$\log f(n)\le(\tau(n)/2)\log n+O(\log\log n),$$

an improvement on the stated upper bound.

On the other hand, $f(n)$ is greater than the number of partitions of n that use each divisor d of n strictly between 1 and n either $0,1,2,\dots,$ or $\lfloor n/(d\tau(n))\rfloor$ times and that use the part 1 enough times to produce the sum n. (Note that each divisor d strictly between 1 and n contributes at most $n/\tau(n)$ to the total.) Thus

$$\begin{aligned}f(n)&\ge\prod_{d|n,1<d<n}\left\{\left\lfloor\frac{n}{d\tau(n)}\right\rfloor+1\right\}\ge\prod_{d|n,1<d<n}\left\{\frac{n}{d\tau(n)}\right\}\\&=\frac{1}{\tau(n)^{\tau(n)-2}}\cdot\frac{1}{n}\prod_{d|n}\frac{n}{d}=\left\{\frac{n}{\tau(n)^2}\right\}^{\tau(n)/2-1}.\end{aligned}$$

It follows that

$$\begin{aligned}\log f(n)&\ge\left(\frac{\tau(n)}{2}-1\right)(\log n-2\log\tau(n))\\&=\left(\frac{\tau(n)}{2}-1\right)\left(\log n+O\left(\frac{\log n}{\log\log n}\right)\right)\end{aligned}$$

by Theorem 317 of Hardy and Wright. This is an improvement on the stated lower bound.

Solved also by O. P. Lossers (The Netherlands), L. E. Mattics, and Richard Stong.

The Period of Fibonacci Sequences Modulo m

E 3410 [1990, 916]. *Proposed by Peter Freyd, University of Pennsylvania, Philadelphia, PA.*

(a) Given a positive integer m, let $f(m)$ be the period-length of the Fibonacci sequence taken modulo m. Prove that $f(m) \leq 6m$ for all m and that equality holds for infinitely many values of m.

(b) Prove the analogous assertion for the Lucas sequence with 6 replaced by 4. (The Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ satisfies $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$; the Lucas sequence $\{L_n\}_{n=0}^{\infty}$ satisfies $L_0 = 2$, $L_1 = 1$, and $L_{n+1} = L_n + L_{n-1}$ for $n \geq 1$.)

Solution by Kevin S. Brown, Kent, WA. Let $m = \prod p_i^{a_i}$. For the period-length of a linear recurring sequence modulo m, it is immediate that $f(m) = \text{lcm}\{f(p_i^{a_i})\} \leq \text{lcm}\{p_i^{a_i - 1} f(p_i)\}$. Thus, a found for $f(m)$ is determined by the periods of the recurrence in the finite fields Z_{p_i}.

The characteristic polynomial for the Fibonacci and Lucas sequences is $q(x) = x^2 - x - 1$, which splits in the field Z_{p^2} into linear factors $x - \alpha$ and $x - \beta$. If $\alpha \neq \beta$, then the nth element in the sequence has the form $A\alpha^n + B\beta^n$ for constants A, B. If $q(x)$ splits in Z_p, then $\alpha, \beta \in Z_p$, and $f(p)$ divides $p - 1$, by Fermat's Little Theorem. On the other hand, if $q(x)$ is irreducible in Z_p, then the order of the roots of $q(x)$ can be found by noting that $\alpha^p = \beta$ and that $-1 = \alpha\beta = \alpha^{p+1}$, implying $\alpha^{2(p+1)} = 1$. Thus, $f(p)$ divides $2(p + 1)$ for "irreducible" primes. By the quadratic reciprocity law $q\{x\}$ is irreducible over Z_p if $p \equiv \pm 2 \pmod 5$ and $q(x)$ splits into distinct linear factors over Z_p if $p \equiv \pm 1 \pmod 5$.

The remaining case is when $q(x)$ has multiple conjugate roots in Z_{p^2}, which implies that $\alpha = \beta$ in Z_p. This occurs if and only if p divides the discriminant of $q(x)$, that is, if and only if $p = 5$. Then the nth term of the sequence is $(A + Bn)\alpha^n$, where A and B are again determined by the initial values. Since the periods of $A + Bn$ and $\alpha^n \pmod p$ are p and $p - 1$, the sequence in this case has period dividing $p(p - 1) = 20$. For the Fibonacci sequence $f(5) = 20$, while for the Lucas sequence $f(5) = 4$.

To maximize the value of $f(m)/m$, we should exclude any prime factors p for which $q(x)$ splits into distinct factors in Z_p, since at best they contribute a factor of $(p - 1)/p$. Therefore we need consider only products of "irreducible" primes and the special prime 5. If m is a product of only odd irreducible primes, then

$$f(m) \leq 4\, lcm\left\{\left(\frac{p_i + 1}{2}\right) p_i^{a_i - 1} : 1 \leq i \leq r\right\} \leq 4m \prod_{i=1}^{r} \left(\frac{p_i + 1}{2p_i}\right),$$

which proves that the ratio is less than 4 in this case. Noting that $f(2) = 3$, we see that for the Fibonacci sequence the maximum value of $f(m)/m$ is 6, which occurs if and only if $m = 2 \cdot 5^n$, where n is any positive integer. For the Lucas sequence, the maximum value of $f(m)/m$ is 4, which occurs if and only if $m = 6$. (We require the following easily proved facts: (i) $f(3^n) = 8 \cdot 3^{n-1}$ and $f(2^n) = 3 \cdot 2^{n-1}$ for both the Fibonacci sequence and the Lucas sequence; (ii) $f(5^n) = 4 \cdot 5^n$ for the Fibonacci sequence and $f(5^n) = 4 \cdot 5^{n-1}$ for the Lucas sequence, n being any positive integer.)

Editoral comment. F. Siwiec and L. Somer pointed out that the solution follows from theorems in [**5**]. The same theorems, derived in different ways, appear in [**4**]

(noted by D. Callan) and [**3**], and the solution also follows from results in [**1**]. A lower bound on the period appears in [**2**]. A. A. Jagers noted that the result generalizes, because the period of a series with any starting values divides the period of the series that starts with 0, 1. The assertion that $f(p)$ divides $2(p+1)$ for "irreducible" primes is proved in [**5**] without using the field of p^2 elements.

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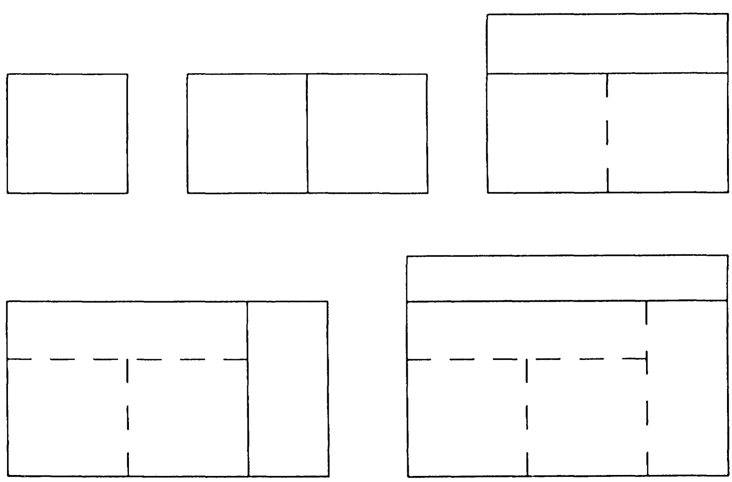
Solved also by R. Betts (student), D. M. Bloom, D. Callan, R. J. Chapman (England), A. A. Jagers (The Netherlands), M. Shirley, F. Siwiec, L. Somer, R. Stong, O. Wyler, the National Security Agency Problems Group, and the proposer.

The Limiting Shape of a Sequence of Rectangles

E 3414 [1990, 917]. *Proposed by Gerald Myerson, Macquarie University, New South Wales, Australia.*

Suppose we construct a sequence of rectangles as follows. We begin with a square of area one. We then alternate adjoining a rectangle of area one alongside or on top of the previous rectangle. The figure show the first five rectangles in the sequence.

Find the limiting ratio of length to height.



Solution by Raphael M. Robinson, University of California, Berkeley, CA. The limiting ratio is $\pi/2$. The ith rectangle in the sequence has area i. The area is increased from $2n-1$ to $2n$ by multiplying the length by $2n/(2n-1)$, and from $2n$ to $2n+1$ by multiplying the height by $(2n+1)/2n$. These steps multiply the ratio of length to height by $2n/(2n-1)$ and $2n/(2n+1)$. Hence the limiting

ratio is given by the infinite product

$$\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdots,$$

which is equal to $\pi/2$ by Wallis's formula.

Editorial comment. The Wallis product formula appears in very many calculus books. Several readers applied other methods, such as Stirling's formula, to calculate the limit. 27 of the solutions were slightly flawed, in that they dealt with the odd terms of the sequence or the even terms and ignored the other terms.

Some solvers observed that the same construction and proof yields a limiting ratio of $\pi r/2$ when the initial rectangle has length to height ratio of r. W. J. Bühler, S. K. Ghosh, and J. Lefort obtained generalizations involving the attachment of successive rectangles of different areas.

Generalizations to higher dimensions were given by W. J. Bühler, D. Chavey, A. Guetter, R. Mabry, and R. A. Young, For example, if one starts with the standard unit cube in $\mathbb{R}^d$ and successively adjoins unit volume bricks to facets orthogonal to the kth coordinate axis, $k = 1, 2, \ldots, d$, repeating the process endlessly, then the ratio of the ith edge to the $(i-1)$th edge converges to

$$\frac{\Gamma\left(\dfrac{i}{d}\right)^2}{\Gamma\left(\dfrac{i-1}{d}\right)\Gamma\left(\dfrac{i+1}{d}\right)}.$$

Solved by 116 readers and the proposer. 3 incorrect solutions were received.

A Needle in the Cartesian Plane

6644 [1990, 929]. *Proposed by Peter Rogerson, SUNY at Buffalo.*

A needle with length L between 1 and $\sqrt{2}$ is tossed at random upon the Cartesian plane. Find the probability that it comes to rest not crossing any line of the form $x = m$ or $y = n$, where m and n are integers. (The case $L < 1$ is treated on pp. 255–256 of J. V. Uspensky's *Introduction to Mathematical Probability* McGraw Hill, 1937. Uspensky attributes that case of the problem to Laplace.)

Solution by Daniel L. Stock, Troy, Michigan. The probability is

$$p = 1 - \frac{1}{\pi}\left\{4\arccos\left(\frac{1}{L}\right) + L^2 + 2 - 4\sqrt{L^2-1}\right\}.$$

To see this let θ be the angle between the needle and the horizontal. By symmetry we may assume that $0 \le \theta \le \pi/4$ (equality is irrelevant since it occurs with probability 0). If $\theta < \arccos(1/L)$ then the needle crosses at least one vertical line. Otherwise it crosses a vertical line with probability $L\cos(\theta)$. Independently of this, it crosses a horizontal line with probability $L\sin(\theta)$. Thus the desired probability is

$$p = \frac{4}{\pi}\int_{\arccos(1/L)}^{\pi/4}(1 - L\sin(\theta))(1 - L\cos(\theta))\,d\theta,$$

and elementary techniques of integration yield the stated result.

If $0 < L \leq 1$ the lower limit must be changed to zero; this yields the well-known formula

$$p = 1 - \frac{1}{\pi}(4L - L^2).$$

If the spacing of grid lines is a units apart in one direction and b units apart in the perpendicular direction with $b \geq a > 0$, then the above argument yields

$$p = \frac{2}{\pi}\int_u^v \left(1 - \frac{L}{b}\sin\theta\right)\left(1 - \frac{L}{a}\cos\theta\right) d\theta$$

for $0 \leq L \leq \sqrt{a^2 + b^2}$. Here u is 0 if $L \leq a$ and $\arccos(a/L)$ otherwise; similarly v is $\pi/2$ if $L \leq b$ and $\arcsin(b/L)$ otherwise.

Editorial comment. Both Stock and Robert N. Will evaluated the last integral above in detail by considering the three separate cases $0 < L \leq a \leq b$, $0 < a \leq L \leq b$, and $0 < a \leq b \leq L \leq \sqrt{a^2 + b^2}$. Wolfgang J. Bühler examined the 3-dimensional case with unit spacing between parallel planes and obtained

$$p = \frac{1}{4\pi}\int_0^{2\pi}\int_{-\pi/2}^{\pi/2}(1 - L|\sin\theta|)^+ (1 - L\cos\theta|\cos\phi|)^+$$
$$\cdot(1 - L\cos\theta|\sin\phi|)^+ \cos\theta\, d\theta\, d\phi.$$

He asserts that this integration may be carried out by elementary methods, but that the resulting formulas are messy. However, for $0 < L < 1$ he obtained

$$p = 1 - L\left(\frac{3}{2} - \frac{2L}{\pi} + \frac{L^2}{4\pi}\right).$$

Solved also by Michael H. Andreoli, Wolfgang J. Bühler (Germany), Robin Chapman (England), Bruce R. Johnson (Canada), Kiran S. Kedlaya, O. P. Lossers (The Netherlands), Joseph McHugh, Richard Stong, Robert N. Will, and the proposer.

Collaborating editors: *Paul T. Bateman, Bruce C. Berndt, Duane M. Broline, Barry W. Brunson, Frank S. Cater, Gulbank D. Chakerian, Michael A. Filaseta, Ira M. Gessel, Richard A. Gibbs, Douglas A. Hensley, John R. Isbell, Murray Klamkin, Daniel J. Kleitman, Fred Kochman, Frederick W. Luttmann, Marvin Marcus, Frank B. Miles, Richard Pfiefer, Stephen L. Portnoy, J. O. Shallit, John Henry Steelman, Daniel Ullman, and Edward T. H. Wang*

LETTERS

When constructing a finite field of order m as the splitting field of $f(x) = x^m - x$, one needs that the polynomial $f(x)$ is separable. Herstein [3] gave a simple argument using the factorization

$$f(x) = x^m - x = (x - \alpha) \cdot g(x),\ g(\alpha) \neq 0.$$

Gupta [2] extended the factorization to certain exponents m that are not prime powers. Herstein [3] wrote that he would be grateful to hear any specific printed source for this "trivial" proof. Unfortunately he did not live to see this resolved. But for the record let it be noted that Fraleigh [1; Lemma 45.1, p. 367] has presented essentially the same idea in his textbook. (The second and third editions of Fraleigh's book carry the cited passage without change.)

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Nievergelt's recent article on the pitfalls of numerical algebra with ill-conditioned matrices is timely and enlightening. The article does, however, contain several slight errors requiring correction.

In Section 2.2 of Yves Nievergelt, Numerical Linear Algebra on the HP-28 or How to Lie with Supercalculators, this MONTHLY 98 (1991) 539-544, the eigenvalues of the matrix

$$A = \begin{pmatrix} 888{,}445 & 887{,}112 \\ 887{,}112 & 885{,}781 \end{pmatrix}$$

are given as $\lambda_{\max} \approx 1{,}774{,}226.00002$ and $\lambda_{\min} \approx 5.63513515643 \times 10^{-7}$. These clearly cannot be right, since the eigenvalues must add up to 1774226, the trace of A. The correct eigenvalues are $\lambda_{\max} \approx 1{,}774{,}225.9999994$ (or, to 12 digits: 1,774,226.00000) and $\lambda_{\min} \approx 5.63626054404 \times 10^{-7}$. These are readily obtained from the characteristic equation

$$\lambda^2 - 1774226\lambda + 1 = 0,$$

the solution of which can be expressed in the form

$$\lambda = 887113\left[1 \pm \sqrt{(1 - 887113^{-2})}\right].$$

A binomial expansion of the square root now yields a rapidly converging series for the eigenvalues.

In Section 3 the matrix

$$S = \begin{pmatrix} 888445 - 1/3548450 & 887112 - 1/3548450 \\ 887112 - 1/3548450 & 885781 - 1/3548450 \end{pmatrix}$$

is put forth as a singular matrix indistinguishable from A when the entries are rounded to 12 digits. But a straightforward, exact calculation shows that the determinant of this S is equal to 1774224/1774225. Two signs are in error here. The singular matrix that is wanted is

$$S = \begin{pmatrix} 888445 - 1/3548450 & 887112 + 1/3548450 \\ 887112 + 1/3548450 & 885781 - 1/3548450 \end{pmatrix}.$$

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In a recent article [1], Albert Fassler makes the remark that "Despite Euler's efforts on this problem, not much more can be said today" about finding two or more primitive pythagorean triangles with a common area.

Apparently the author is unaware of Andrew Bremner's excellent article [2] on ppts where he develops linear automorphisms upon a quartic surface and shows how to develop parametric solutions providing pairs of generators and hence triangles. Subsequently he provides several higher degree parametric solutions, considerably enhancing the knowledge available about this interesting problem.

Additionally, Dan Hoey and I have made complete computer searches for sets of ppts, and I am currently having archived a table that lists all 9916 pairs for a, b (the generators of the sides) from 2,1 to 10000, 9999 where at least two pythagorean triangles (not necessarily all primitive) have a common area. Martin Gardner credits Charles Shedd for finding the first triplet listed below in 1945. I found the next three around 1986 and Hoey and I the last in 1990. Furthermore, by exhaustive search, if another triplet does occur, its smallest generators must exceed 106503,28538 and the area must be larger than 3.23×10^{19}. I conjecture the next triplet won't occur until its common area is around 10^{21} or so, truly a huge number of ppts to search through.

Triples of Primitive Pythagorean Triangles with a Common Area

Generators of Sides			Area
77,38	78,55	138,5	13123110
1610,869	2002,1817	2622,143	2570042985510
2035,266	3306,61	3422,55	2203385574390
2201,1166	2438,2035	3565,198	8943387723270
7238,2465	9077,1122	10434,731	826290896699730

The reader may take heart that perhaps another triplet does exist by virtue of the fact that all five of these triplets have two primitive triangles that occur in Whitlock's parametric solution [3]. The third triangle was found by an algorithm which uses the area. If the search is continued, the smallest generators in

Whitlock's formula must be greater than 10^6, however, as all smaller have been exhaustively checked.

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3. W. P. Whitlock Jr., Rational right triangles with equal area, *Scripta Mathematica*, IX (Sept. 1943) pp. 155–161; ibid (Dec. 1943) 265–268.

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One may be a mathematician of the first rank without being able to compute. It is possible to be a great computer without having the slightest idea of mathematics.

—*Novalis*

REVIEWS

What makes a good book review? What to one reader is a provocative, personal essay the theme for which arises naturally from the book at hand, will strike a second reader as a pointless, self-serving polemic that (to top it off) ignores the book completely. What one feels is a solid review explaining what is in the book, another views as little more than a reproduction of the book's table of contents and chapter introductions (with a list of typos). There is no one answer to the question.

That lack of consensus is good. If everyone agreed on the qualities of a good book review all book reviews would look the same. On the other hand no review is likely to please everyone. The prudent review editor proceeds pragmatically. A good book review is one that engages the curiousity of the diverse readership of the *Monthly*. The important question is: How many of those who start reading the first line will read the last? The reviewer may choose to champion the book, denounce the book, explain the subject matter, discuss what occurs when one attempts to teach from the book, compare the book to its predecessors and competitors, whatever. It is necessary only that the perspective chosen is one from which the reviewer has something interesting to say.

That being said, how does one obtain interesting reviews? Most reviews appearing in the *Monthly* are solicited. A publisher sends a book to the review editor who decides, often after conferring with others, whether to review it and to whom it should be sent for review. In this process potentially interesting books will surely be missed. An editor is always on the look out for worthy books. Readers can help and I encourage suggestions for books to be reviewed.

I look forward to my term as extended book review editor and hope to present many good reviews. I welcome your participation.

—*Darrell Haile*

Journey Through Genius: The Great Theorems of Mathematics. By William Dunham, John Wiley & Sons, New York, 1990. xiii + 300 pp.

Joe Albree and Marie Root

Often our most satisfying and insightful travels are those made in the company of great companions. Is William Dunham's *Journey Through Genius* such a companion to the historical development of mathematics? Does this *Journey* stake out any new territory in the literature of the history of mathematics?

In temporal terms, Dunham's work encompasses almost the entire sweep of the history of mathematics, from the early Greeks to the end of the nineteenth

century. But, in presenting some of mathematics' "creative milestones," he has made no attempt to compete with the general histories of Boyer [**4**], Eves [**9**], Kline [**12**], Struik [**19**], or others. These books aim to be complete tours. Dunham, by design, is very selective.

Dunham's pilgrimage has a biographical component to enhance the appreciation of the mathematics encountered. We already have several collections of biographical studies of notable mathematicians, such as Bell [**2**], Coolidge [**7**], and Osen [**15**]. In the most famous of these works, Bell's *Men of Mathematics*, great *mathematicians* are chosen, and by reading these minibiographies (some of which like Gauss's are classics) from start to finish, one can chain together a picture of the history of mathematics. On the other hand, Dunham chooses a great *theorem* for each of his chapters, and by including a brief sketch of the mathematician and his times (unfortunately, there are no women mathematicians mentioned), he intends that the same kind of reading will also result in the reader's appreciation of mathematics' "long and glorious" history. But, this *Journey* is not at all a popular cult of personalities even though some of mathematics' more "inspirational," "tragic," and "bizarre" heroes appear.

Mathematics is the primary focus of Dunham's voyage through several "mathematical masterpieces," and most of his exposition is truly lucid. For instance, just prior to his presentation of the first proposition of Archimedes' *Measurement of a Circle* (The area of a circle of radius r and circumference c is equal to the area of a triangle of base c and height r.), Dunham clearly explains the strategy of double *reductio ad absurdum* [p. 92]. And again, before launching into the derivation of Heron's formula, he points out the idiosyncratic appearance of this formula, and he warns us that its proof is both elementary and ingenious:

> As with a good Agatha Christie novel, readers of Heron's proof can be within a few lines of the end and still have no idea how the matter will be resolved. [p. 119]

Then Dunham asks us to come to grips with the mathematics by carefully retracing the arguments, step by step, allowing us to review our progress ("Here we see Heron's link between the triangle's area, K, and its semiperimeter, s." [p. 122]), and to catch our breath at key places ("these are the components of the formula we seek to prove." [p. 123]). He is like a perceptive museum guide conducting us through the key sights but also allowing us to digest and appreciate the ways in which the technical work makes manifest the major ideas.

Almost seamlessly, Dunham passes to reflections on the completed work and these contemplations enrich and fix it in our minds. The observation that Heron's formula is "certainly the most convoluted proof" [p. 126] encountered up to this point is especially meaningful. After carefully considering all of the details of the proof of Archimedes' proposition, one can clearly see that it actually was "short and simple" and truly marvel that it was overlooked by previous Greek geometers: "But simplicity is most easily perceived in hindsight." [p. 95]. This is not the minimalist writing of most mathematical authors. Would it not be enlightening if our textbooks and even some research mathematics were written in such a generous style?

Dunham's cavalcade is not bounded by just one subject area within mathematics, like Edwards [**8**], Ifrah [**11**], Ore [**14**], Stigler [**17**], or one historical or geographical region, like Berggren [**3**] or Kuratowski [**13**]. Even though his scope is definitely eurocentric, Dunham has consciously attempted to sample at least some

different branches of mathematics, such as traditional geometry, algebra, number theory, analysis, and the foundations of analysis.

In this sampling, Dunham has recognized the timeless nature of mathematics and "tried to retain virtually all of the spirit, and a good bit of the detail, of the original theorems" [p. viii]. For Johann Bernoulli's proof of the divergence of the harmonic series, as it was originally published in Jakob Bernoulli's *Tractatus de seriebus infinitis*, Dunham has reproduced a portion of the page showing the key steps. From the layout of *his* page and the similarity of notation, we can see that Dunham's argument is a faithful reproduction of the Bernoullis'. But it would be an error to compare *Journey Through Genius* with any of the source books in the history of mathematics like Callinger [**5**], Smith [**16**], or Struik [**20**].

Another recent effort at taking a new look at the history of mathematics is Stillwell whose purpose is "to give a unified view of undergraduate math ... through its history." [**18**, p. vii] For Stillwell, assuming the completion of an undergraduate mathematics major, the Pythagorean Theorem is an opportunity to introduce some of the connections between the continuous and the discrete. Dunham, on the other hand, makes this the occasion to take the generally educated reader through a tour of Book I of Euclid's *Elements*, to discuss some of the historical ramifications of parallelism, as well as to explain carefully the bride's chair, the proof of the Pythagorean Theorem as it is found in Euclid. Stillwell's unstated agenda may be to lure students into graduate work in mathematics; his style is rather terse, and even though he closes each chapter with biographical notes, they are not as integral to his book as the comparable material is in Dunham.

Clearly, Dunham considers mathematics one of the liberal arts. He observes,

> For disciplines as diverse as literature, music, and art, there is a tradition of examining masterpieces ... as the fittest and most illuminating objects of study. [p. v]

From Hippocrates of Chios (ca. 440 B.C.) to Georg Cantor (1845–1918), Dunham has chosen an even dozen great theorems and their proofs to be the centerpieces of an equal number of chapters. Dunham holds each great theorem up to the closest scrutiny for the reader. Then, he uses this mathematical masterpiece as the pretext for biographical sketches and as the occasion to fill in a considerable amount of the historical development of mathematics.

Dunham invites comparison of his work to the Norton anthologies, designed to expose the reader to the "greatness, continuity and variety" [**1**, p. xxvii] of literature. While Dunham also puts before the reader some of the greatness, continuity and variety of mathematical highlights, and his book is not like any other history of mathematics, the work is not an anthology. The Norton anthologies are textbooks, but Dunham has no exercises and more generally no designs in this direction. In range and completeness, the Norton anthologies are more comparable to the general histories of mathematics mentioned above than to Dunham's selections. In each chapter or section of the Norton anthologies, it is not always clear that one work should stand out above all the others, whereas in Dunham, there is no doubt that the great theorem is the hinge on which all of the rest of the chapter swings. Chapters and sections of the Norton anthologies are written by committees and then each book is assembled by another committee of editors. But, *Journey Through Genius* is clearly the product of Dunham's vision.

We greatly admire Dunham's new vision of the history of mathematics at the same time that we have a small number of reservations. Our most serious lament is

with his inclusion of *two* great theorems of Georg Cantor. Of Dunham's twelve great theorems encompassing the whole history of mathematics, two representatives from Euclid and two from Euler are certainly defensible, but to do the same with Cantor causes this book to be unbalanced. In a work aimed at the general, scientifically literate reader, we believe the author might have found a great theorem from probability or statistics or perhaps just one example from some area of applied mathematics. A more adventurous suggestion would be to attempt a short essay on David Hilbert's twenty-three problems as presented in his famous address of 1900 [**10**]. Hilbert's first probiem flows naturally from Cantor's proof of the non-denumerability of the continuum, the subject of Dunham's Chapter 11. With careful selections, especially from the first seven of Hilbert's problems, such a chapter would invert the great theorems theme and thereby give a more rounded picture of mathematics as a living science and a hint of some of its concerns in the twentieth century.

Dunham's writing is so enlightening, clear and appropriately serious (there are many profound ideas squarely faced) without being heavy or pedantic, that his very few slips stand out. For example, his remark at the completion of Heron's formula that "we may perhaps label his performance Hero-ic" [p. 127] depresses in a cheap and thoughtless instant the marvelous heights to which Dunham has just carried us. Then, "as we watch Euler reason his way" [p. 230] through the preliminary theorems to the refutation of Fermat's conjecture about the primality of numbers of the form $2^{2^n} + 1$, we would have preferred to have the examples *before* the theorems, rather than after them. In view of printing technology today, most of the portraits in the book are too dark and in a couple of instances (e.g. Euler on p. 209) almost unrecognizable.

Dunham closes his book with the oft-quoted opinion of Bertrand Russell, "Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere" No! No! This is just the opposite of what Dunham has spent 286 pages doing so well! A more appropriate epilogue for Dunham's worthy accomplishments might be found at the close of Hilbert's address:

> how rich, how manifold and how extensive the mathematical science of today is... For with all the variety of mathematical knowledge, we are still clearly conscious of the similarity of the logical devices, the *relationship* of the *ideas* in mathematics as a whole and the numerous analogies in its different departments. [**10**, p. 34.]

In guiding us on this journey through the centuries of relationships and ideas, Dunham has demonstrated that the universality of mathematics, its "reservoirs of wisdom" [**6**, p. xii], can be found in the embrace of its different departments.

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It is only in mathematics, and to some extent in poetry, that originality may be attained at an early age, but even then it is very rare (Newton and Keats are examples), and it is not notable until adolescence is completed.

—*Havelock Ellis*

TELEGRAPHIC REVIEWS

Edited by
Lynn Arthur Steen
with the assistance of
the Mathematics Departments of Carleton, Macalester, and St. Olaf Colleges

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General, S, C, P. *A Guide to MathWriter: The Scientific Word Processor for the Macintosh, Version 2.0.* J. Robert Cooke, E. Ted Sobel. Brooks/Cole, 1991, xiv + 331 pp, $395 (Professional Version) (P); $99.95 (Educational Version) (P). [ISBN: 0-534-13560-9] A "wysiwyg" (what-you-see-is-what-you-get) technical editor that treats mathematical expressions like ordinary, editable text. Provides most of the features of generic Mac word processors plus palette-based tools for creating mathematical expressions; graphics tools to import, crop, and resize graphics; end- and footnotes with automatic numbering; sidebars, attached to paragraphs, either in the margins or as wraparounds in the text; a mathematical and scientific dictionary to supplement the general spell-checker list; italicized mathematical expressions with exception list; and a library facility for oft-used expressions. (The Educational Version, intended for students and for small machines, removes or restricts many of these special features.) The *Guide* is thorough and well-indexed; enclosed eight-page "Quick Reference" card is a useful aid. An alternative to TeX for those who require wysiwyg systems. Reads and writes in RTF (rich text format) for transfer to and from other word processors; the authors have also developed a TeX translator (available separately). LAS

Education, L. *Understanding Technology in Education.* Eds: Hughie Mackay, Michael Young, John Beynon. Falmer Pr (US Distr: Taylor & Francis), 1991, vii + 265 pp, (P). [ISBN: 1-85000-888-4]

Education, T(13-14), S. *Mathematics, A Good Beginning: Strategies for Teaching Children, Fourth Edition.* Andria P. Troutman, Betty K. Lichtenberg. Brooks/Cole, 1991, xix + 551 pp, $40.50 (P). [ISBN: 0-534-15144-2] A resource book for elementary teachers. Each chapter includes an extensive list of suggested readings, mostly from *The Arithmetic Teacher.* (*First Edition*, TR, January 1978; *Second Edition*, TR, January 1983.) BC

Education, P. *Grundlagen einer Geometriedidaktik.* Horst Struve. Bibliographisches Institut, 1990, 272 pp, (P). [ISBN: 3-411-14631-1] A monograph on the learning of geometry, based on an analysis of a specific German text. It argues that geometry is learned experimentally. JD-B

Graph Theory, P. *Graph Theory, Combinatorics, Algorithms, and Applications.* Eds: Yousef Alavi, *et al.* SIAM, 1991, xii + 635 pp, $77.50 (P). [ISBN: 0-89871-287-4] Proceedings of the second international conference on graph theory, combinatorics, algorithms, and applications held at San Francisco State University, July 1989. Fifty-eight research papers, primarily on graph theory, with many contributions from Chinese mathematicians. JPH

Combinatorics, T(17-18: 1), S, P, L. *Symmetry and Combinatorial Enumeration*

in Chemistry. Shinsaku Fujita. Springer-Verlag, 1991, ix + 368 pp, $39 (P). [ISBN: 0-387-54126-8] A text on group theory as applied to stereochemistry and combinatorial enumeration in chemistry. There's a fundamental question here: Who uses the more obscure terminology: mathematicians or chemists? BC

Discrete Mathematics, T(13-14: 1, 2). *Discrete Mathematics: An Introduction for Sofware Engineers.* Mike Piff. Cambridge Univ Pr, 1991, xi + 317 pp, $59.95; $16.95 (P). [ISBN: 0-521-38475-3; 0-521-38622-5] Topics include logic, set theory, relations, graph theory, and algorithms. Briefly treats computability, context-free languges, regular grammars, and finite state machines. Abstract algebra applied to stacks and formal languages. Expects familiarity with Pascal, Ada, or Modula-2. Sample programs in Modula-2 and solutions to selected exercises included. AD

Number Theory, P. *Lecture Notes in Mathematics-1471: Non-Archimedean L-Functions of Siegel and Hilbert Modular Forms.* Alexey A. Panchishkin. Springer-Verlag, 1991, 157 pp, $19 (P). [ISBN: 0-387-54137-3] A study of p-adic properties of special values of zeta functions of automorphic forms. The author provides background material on p-adic measures, Siegel and Hilbert modular forms, and other topics, and new results on p-adic analytic continuation of zeta functions of Siegel modular forms and of convolutions of Hilbert modular forms. SG

Number Theory, T(16-17), S, L. *Geometric and Analytic Number Theory.* Edmund Hlawka, Johannes Schoißengeier, Rudolf Taschner. Universitext. Springer-Verlag, 1991, x + 238 pp, $29 (P). [ISBN: 0-387-52016-3] An interesting blend of topics from elementary and analytic number theory. Topics include the approximation theorems of Dirichlet and Kronecker, Minkowski theorem, the Prime Number theorem, asymptotic calculations for number theoretic functions, and primes in arithmetic progressions. Ideal for undergraduate independent study. SG

Number Theory, P. *Applications of Fibonacci Numbers, Volume 4.* Eds: G.E. Bergum, A.N. Philippou, A.F. Horadam. Kluwer Academic, 1991, xxiv + 313 pp, $99. [ISBN: 0-7923-1309-7] A collection of over thirty papers presented at the Fourth International Conference on Fibonacci Numbers and Their Applications held at Wake Forest University, North Carolina, July 30-August 3, 1990. SG

Number Theory, P*, L. *Ramanujan's Notebooks, Part III.* Bruce C. Berndt. Springer-Verlag, 1991, xiii + 510 pp, $89.80. [ISBN: 0-387-96110-0] Bruce Berndt is providing a great service to twenty-first century mathematics by editing the notebooks of Ramanujan (where "editing" means proving a bundle of formulas that Ramanujan merely jotted down). This volume deals mostly with theta functions. There will be five volumes in all. Berndt merits a medal when the job's done. BC

Number Theory, P, L. *Number Theory: New York Seminar 1989-1990.* Eds: D.V. Chudnovsky, *et al.* Springer-Verlag, 1991, viii + 275 pp, $29.50 (P). [ISBN: 0-387-97670-1] Thirteen papers on analytic and algebraic number theory. Includes a long paper by David and Gregory Chudnovsky on continued fraction computations of classical constants (e.g., π). BC

Number Theory, T(15-17: 1, 2), S, P*, L*. *Logical Number Theory I: An Introduction.* Craig Smoryński. Springer-Verlag, 1991, x + 405 pp, $49 (P). [ISBN: 0-387-52236-0] An idiosyncratic introduction to logic via number theory (or is it number theory via logic?). Includes historical and philosophical digressions, one of the best of which ends: "But enough! My head is swimming with anecdotes that I'd love to tell, but I shall probably get into hot water over this last one when the parties involved recognize themselves." BC

Algebra, P. *Generators and Relations in Groups and Geometries.* Eds: A. Barlotti, *et al.* NATO ASI Ser. C, V. 333. Kluwer Academic, 1991, xv + 447 pp, $144. [ISBN: 0-7923-1161-2] The proceedings of the NATO Advanced Study Institute in Italy, April 1-14, 1990. Papers are divided into three parts: Part I is concerned with optimal factorization of matrices, and length problems; Part II with reflection geometry; Part III with applications outside geometry, especially algebra and topology. LCL

Algebra, T(15: 1), S, L. *Abstract Algebra and Famous Impossibilities.* Arthur Jones, Sidney A. Morris, Kenneth R. Pearson. Universitext. Springer-Verlag, 1991, x + 187 pp, $29.95. [ISBN: 0-387-97661-2] Self-contained development of algebraic solution of the impossibilities of squaring a circle, doubling a cube, and trisecting an angle. Assumes only linear algebra and calculus; develops symmetric functions and integration of complex-valued functions to prove π transcendental. Includes brief histories and additional reading. JPH

Calculus, S, C. *Using BestGrapher: A Computer Laboratory Guide for Calculus.* George W. Best. DC Heath, 1990, 144 pp, $15 (P); *BestGrapher Software* (Mac or IBM), $70. [ISBN: 0-669-24642-5] A workbook with typical calculus exercises adapted to use with the *BestGrapher* software. Chapters open with a few worked examples and often conclude with open-ended projects. Software (which is copy-protected) provides typical tools needed for elementary calculus (graphs, derivative graphs, tangent lines, numerical integration, zeros). Limited features enhance ease of use. Mac version includes zooming. LAS

Complex Analysis, P. *Lecture Notes in Mathematics-1468: Prospects in Complex Geometry.* Eds: J. Noguchi, T. Ohsawa. Springer-Verlag, 1991, 421 pp, $44 (P). [ISBN: 0-387-54053-9] Proceedings of the 25th Taniguchi International Symposium held during the summer of 1989. Sixteen papers on aspects of the geometry of complex structures. JO

Complex Analysis, P. *Two-Dimensional Geometric Variational Problems.* Jürgen Jost. Wiley, 1991, x + 236 pp, $87.95. [ISBN: 0-471-92839-9] Treats variational problems for mappings from a surface equipped with a conformal structure into Euclidean space or a Riemannian manifold. Develops a general theory, proving existence and regularity theorems with emphasis on geometric viewpoints, and a thorough investigation of connections with complex analysis. AWR

Differential Equations, P. *Lecture Notes in Mathematics-1455: Bifurcations of Planar Vector Fields.* Eds: J.-P. Françoise, R. Roussarie. Springer-Verlag, 1990, vi + 396 pp, $46 (P). [ISBN: 0-387-53509-8] Proceedings of the meeting held in Luminy, France in September 1988. Seventeen papers cover finiteness of number of limit cycles, numerical simulations, quadratic systems, and models of biological systems. SP

Differential Equations, S(17-18), P.** *From Gauss to Painlevé: A Modern Theory of Special Functions.* Katsunori Iwasaki, *et al.* Aspects of Math., V. E16. Friedr Vieweg, 1991, x + 347 pp, DM 78. [ISBN: 3-528-06355-6] The Painlevé functions represent the newest entry into the class of special functions. Painlevé asked in 1900 if there exist second-order nonlinear algebraic differential equations with the property that the solutions have no singularities that change with a change in initial conditions (a property enjoyed by all linear differential equations). Painlevé classified all such differential equations (six in total). These remained of mathematical interest only until recently when one of the Painlevé equations was used to describe the behavior of the correlation function for the Ising model. The authors do an excellent job of presenting both the historical and mathematical details of the subject in a form accessible to any mathematician or physicist. MPR

Partial Differential Equations, P. *Microlocal Analysis and Nonlinear Waves.* Eds: Michael Beals, Richard B. Melrose, Jeffrey Rauch. Instit. for Math. & its Applic., V. 30. Springer-Verlag, 1991, xiii + 199 pp, $29. [ISBN: 0-387-97591-8] Fourteen papers from a workshop at the IMA. Microlocal analysis is a linear technique that's being transferred, with some success, to nonlinear settings. BC

Dynamical Systems, P. *Continuum Theory and Dynamical Systems.* Ed: Morton Brown. Contemp. Math., V. 117. AMS, 1991, ix + 182 pp, $63 (P). [ISBN: 0-8218-5123-3] Seventeen papers from a joint AMS-IMS-SIAM 1989 conference. BC

Dynamical Systems, S(17), L. *Chaotic Behaviour of Deterministic Dissipative Systems.* Miloš Marek, Igor Schreiber. Cambridge Univ Pr, 1991, x + 367 pp, $79.50. [ISBN: 0-521-32167-0] Brief, non-rigorous sketch of theoretical underpinnings followed by a much more extensive and impressive survey of experimental observations of chaos in mechanical systems, electronics, lasers, semiconductors, chemical and biological systems, and hydrodynamics. SP

Dynamical Systems, P. *Instabilities and Nonequilibrium Structures III.* Eds: E. Tirapegui, W. Zeller. Math. & Its Applic., V. 64. Kluwer Academic, 1991, xi + 370 pp, $122. [ISBN: 0-7923-1153-1] Papers given at the Third International Workshop on Instabilities and Nonequilibrium Structures in Valparaíso, Chile, 1989. Organized into three major sections: dynamical systems with a finite number of variables (includes papers on statistical mechanics and cellular automata); the effect of noise on dynamical systems near bifurcation points; and experimental and phenomenological observations. AWR

Dynamical Systems, T*(16-18: 1), S, L.** *Differential Equations and Dynamical Systems.* Lawrence Perko. Texts in Appl. Math., V. 7. Springer-Verlag, 1991, xii + 403 pp, $39. [ISBN: 0-387-97443-1] A good text for a second course in differential equations, with emphasis on qualitative and geometric behavior. The four chapters

cover linear systems, local aspects of nonlinear systems, global aspects of nonlinear systems, and bifurcations of nonlinear systems. Prerequisites are linear algebra and real analysis. Many good exercises. SP

Dynamical Systems, T(16-18: 2), P, L. *An Introduction to Dynamical Systems.* D.K. Arrowsmith, C.M. Place. Cambridge Univ Pr, 1990, 423 pp, $79.50; $29.95 (P). [ISBN: 0-521-30362-1; 0-521-31650-2] A comprehensive introduction to the dynamics of flows and maps, suitable for first-year graduate and strong undergraduate students. In-depth coverage of most of the basic topics: normal forms, invariant manifolds, hyperbolicity, homoclinic phenomena, low-dimensional bifurcations, area preserving maps, and much more. Many exercises. SP

Numerical Analysis, S(17-18), P. *The Total Least Squares Problem: Computational Aspects and Analysis.* Sabine Van Huffel, Joos Vandewalle. Frontiers in Appl. Math., V. 9. SIAM, 1991, xiii + 300 pp, $28.50 (P). [ISBN: 0-89871-275-0] On a method for the numerical solution of general linear systems in which the coefficients and constants are only known approximately. Often the solution is better than that provided by least squares. Contains examples, theory, numerical improvements, sensitivity analysis, and statistical properties. RWN

Numerical Analysis, P. *Mathematical Aspects of Numerical Grid Generation.* Ed: José E. Castillo. Frontiers in Appl. Math., V. 8. SIAM, 1991, xiv + 157 pp, $24.50 (P). [ISBN: 0-89871-267-X] To study some continuous models, the continuum is first converted to a finite grid of points. This grid should conform to the geometry of the problem and the nature of the solution. Based on papers presented at a mini-symposia held at the SIAM Annual Meeting in Minneapolis in July 1988, this book discusses mathematical considerations of creation of algorithms that automatically and robustly generate these grids. Only structured grids are considered here. SP

Functional Analysis, P. *Lecture Notes in Mathematics-1469: Geometric Aspects of Functional Analysis.* Eds: J. Lindenstrauss, V.D. Milman. Springer-Verlag, 1991, ix + 191 pp, $24 (P). [ISBN: 0-387-54024-5] Surveys interspersed with original work as they were presented in the Israel seminar during academic year 1989-90. AWR

Analysis, P, L. *Inequalities Involving Functions and Their Integrals and Derivatives.* D.S. Mitrinović, J.E. Pečarić, A.M. Fink. Math. & Its Applic., V. 53. Kluwer Academic, 1991, xvi + 587 pp, $149. [ISBN: 0-7923-1330-5] A systematic and encyclopedic account based on an exhaustive search of the literature. Eighteen self-contained chapters, with large bibliographies, each related to a single "well-known classical result." LCL

Algebraic Geometry, P. *Lecture Notes in Mathematics-1462: Singularity Theory and Its Applications, Part I.* Eds: D. Mond, J. Montaldi. Springer-Verlag, 1991, viii + 408 pp, $44 (P). [ISBN: 0-387-53737-6] Part One of the proceedings of the year-long symposium on singularity theory and its applications held at the University of Warwick in 1988-89. This volume contains twenty-three papers on the geometric aspects of singularities. JO

Algebraic Geometry, P. *Topics in Noncommutative Geometry.* Yuri I. Manin. Princeton Univ Pr, 1991, vii + 164 pp, $35. [ISBN: 0-691-08588-9] A compact introduction to supergeometry and quantum groups for mathematicians and physicists at home with Lie groups and complex geometry. BC

Differential Geometry, T(18), S, P. *Discrete Groups in Space and Uniformization Problems.* Boris N. Apanasov. Math. & Its Applic., V. 40. Kluwer Academic, 1991, xvii + 482 pp, $193. [ISBN: 0-7923-0216-8] Study of discrete group actions in space and their fundamental domains. Emphasizes the geometric and algebraic properties of discrete groups of spatial domain automorphisms. Focuses on Kleinian groups. Presents theory of deformations for discrete groups, results in uniformization and the moduli problem for geometric and conformal structures. Requires knowledge of algebraic topology, differential geometry, and three-manifolds. Note price. OJ

Differential Geometry, P. *Singularities of Caustics and Wave Fronts.* V.I. Arnold. Math. & Its Applic., V. 62. Kluwer Academic, 1990, xiii + 259 pp, $99. [ISBN: 0-7923-1038-1] A caustic is a very bright curve of reflected light rays found, for example, on the bottom of a tea cup. Quoting from the series editor: "This book is about caustics—and about a large part of everything else in mathematics." This is an introduction to recent advances in the study of caustics gained by the employment of an impressive array of such diverse mathematical tools as Weyl groups of simple Lie algebras, cobordism, characteristic

classes, Dynkin diagrams, and contact geometry. SP

Geometry, S, C. *Fractal Attraction for the Macintosh, Version 1.0.* Kevin D. Lee, Yosef Cohen. Macintosh Software. Sandpiper Software (POB 8012, St. Paul, MN 55108; 612-644-7395), 1990, $49.95 (P). A multi-window design tool to generate from geometric figures (in a Design window) the fractal image (in a Fractal window) specified by the associated iterated function system (IFS) code (in a Code Window). Can transform the design (graphically) or edit the IFS equations; can crop, transform, save, import, and print graphical images. The instructional pamphlet provides a summary of the associated affine matrix transformations, as well as examples based on sample designs provided. A well-designed tool for its purpose: to illustrate the geometry of IFS-generated fractals. Bulk packs for classroom use are available from the authors. Runs on the Mac Plus and anything larger. LAS

Geometry, S*, C*, P*. *James Gleick's CHAOS: The Software, User Guide.* IBM PC Software. Autodesk, Inc. (2320 Marinship Way, Sausalito, CA 94965), 1991, $59.95 (P). "To begin to understand ...it is necessary, first of all, to play." Six playgrounds paralleling Gleick's best-selling *Chaos*: Mandelbrot sets, Magnet and Pendulum, Strange Attractors, The Chaos Game (iterated affine maps), Fractal Forgeries (artificial landscapes), and Toy Universes (cellular automata). Each playground uses similar keyboard or mouse controls to change parameters and explore variations; each provides numerous "things to try"—interesting, instructive examples available at the touch of a button. Excellent manual explains each environment and the underlying mathematics; includes a substantial bibliography of expository sources. Quick reference cards included, as are both 5" and 3.5" disks. Requires EGA or VGA display. LAS

Operations Research, S(14). *Network Reliability and Algebraic Structures.* Douglas R. Shier. Clarendon Pr, 1991, x + 144 pp, $45. [ISBN: 0-19-853386-1] Develops algebraic methods and structures (e.g., partial orders, lattices, polynomials) underlying reliability problems, modeled primarily by probabilistic, 2-terminal directed networks. Includes numeric, symbolic, and approximate solution methods. Discusses computational complexity. Builds on elementary theory of graphs, posets, probability, numerical linear algebra, and combinatorics; otherwise self-contained development with chapter notes for further references and applications. JPH

Optimization, T(17: 1), S, P. *Integer Programming.* Stanisław Walukiewicz. Math. & Its Applic., V. 46. Kluwer Academic, 1991, xvi + 182 pp, $69. [ISBN: 0-7923-0726-7] Theory and numerical methods for the general combinatorial optimization problem. Survey of some applications and standard techniques (ellipsoid, subgradient, cutting plane, near optimal methods, branch and bound, duality). Emphasis on equivalence (e.g., replacing problem by a binary or linear problem) and relaxation (e.g., replace a maximization problem by one with looser or surrogate constraints, or larger objective functions) techniques. RM

Mathematical Modelling, P. *Differential Inclusions and Optimal Control.* Michał Kisielewicz. Math. & Its Applic., V. 44. Kluwer Academic, 1991, xix + 240 pp, $124. [ISBN: 0-7923-0675-9] Functional differential equations model processes where the past dynamics of a system directly influence the future (not just through its effects in determining the present). This monograph studies theory of neutral functional differential inclusions (of the form $\dot{x}(t)\epsilon F(t, x_t, \dot{x}_t)$) of systems whose past behavior influences the present dynamics. Note price. RM

Control Theory, P. *Modeling, Estimation and Control of Systems with Uncertainty.* Eds: Giovanni B. Di Masi, Andrea Gombani, Alexander B. Kurzhansky. Progress in Systems & Control Theory, V. 10. Birkhäuser, 1991, ix + 467 pp, $98.50. [ISBN: 0-8176-3580-7] Papers from a 1990 conference in Hungary giving wide range of contributions which deal with uncertainty for control systems (e.g., arising from measurement errors or poor understanding of the underlying mechanisms) through both stochastic approaches and set-valued dynamics. RM

Systems Theory, T(17), S, L. *Linear System Theory.* Frank M. Callier, Charles A. Desoer. Texts in Elec. Engin. Springer-Verlag, 1991, xiv + 509 pp, $59.50. [ISBN: 0-387-97573-X] Growing out of notes for two courses, one on linear optimal systems for undergraduates in applied mathematics, the other a course on linear systems for first-year graduate school engineers. Covers finite-dimensional linear systems in both the continuous time and discrete time cases. Would seem to be an excellent source for a mathematician wanting to learn about applications of differential equations and lin-

ear algebra; an accessible book. AWR

Systems Theory, P. *New Trends in Systems Theory.* G. Conte, A.M. Perdon, B. Wyman. Progress in Systems & Control Theory, V. 7. Birkhäuser, 1991, xvii + 722 pp, $145. [ISBN: 0-8176-3548-3] Proceedings of a 1990 conference in Geneva on the theory and applications of systems theory. Papers cover the theory of linear and nonlinear systems, stability, control (robust, adaptive), robotics, neural net approaches. RM

Stochastic Processes, T(18: 2), P. *Stochastic Differential Equations With Applications to Physics and Engineering.* Kazimierz Sobczyk. Math. & Its Applic., V. 40. Kluwer Academic, 1991, xvi + 400 pp, $139. [ISBN: 0-7923-0339-3] A self-contained introduction to the structure and solution methods of both random and Itô stochastic differential equations. Of interest to applied mathematicians and engineers studying dynamical systems subject to random excitations. Numerous examples include responses of structures to turbulent fluids, earthquakes, and sea waves. Assumes familiarity with basic probability theory and common methods of applied mathematics. Note price. SP

Computational Statistics, S, C, P. *SuperANOVA: Accessible General Linear Modeling.* Jim Gagnon, *et al.* Macontosh Software. Abacus Concepts (1984 Bonita Ave., Berkeley, CA 94704), 1990, xvi + 322 pp, $495 (P). [ISBN: 0-944800-01-7] A versatile, intuitive point-and-click package integrating analysis of variance (including unbalanced designs, missing cells, and repeated measure designs); post-hoc tests with graphical and numerical displays; means tables with numerous options; and presentation graphics. Includes a variety of common designs (e.g., two-factor ANOVA, Latin square, regression models) with option for user-defined additions; a built-in MacDraw-like toolkit to construct and enhance presentations; and powerful spreadsheet-like data management tools. Can import data from other common Mac programs. Comprehensive user guide includes an appendix with the formulae and algorithms used in various parts of the package, and an extensive list of statistics references. LAS

Statistics, T(16-17: 1). *Bayes-Statistik.* Dieter Wickmann. Math. Texte, Band 4. Bibliographisches Institut, 1990, xiv + 226 pp, (P). [ISBN: 3-411-14671-0] An introductory text intended for prospective high school teachers. Exercises, solutions, pedagogical remarks. JD-B

Elementary Computer Science, T(12-13: 1). *Problem Solving with Pascal: An Introduction to Computer Science.* George Best. Bates Publ (129 Commonwealth Ave., Concord, MA 01742), 1989, 228 pp, $20 (P). A brief typescript text for the AP Computer Science A course. Emphasizes procedures, program structure, and modular problem solving. Can be used with many common implementations of Pascal. LAS

Applications (Fluid Dynamics), T(17-18: 1-3), S, P. *Computational Techniques for Fluid Dynamics 2: Specific Techniques for Different Flow Categories, Second Edition.* C.A.J. Fletcher. Ser. in Computat. Physics. Springer-Verlag, 1991, xiii + 493 pp, $59.50 (P). [ISBN: 0-387-53601-9] The second volume of a two-volume introduction, at graduate level, to theory and methods of computation fluid dynamics. *Volume 1* (TR, November 1991) emphasizes theory; *Volume 2* covers applications to specific sorts of flow phenomena: inviscid flow, boundary layer flow, Navier-Stokes governed flow, viscous flow. Contains a wealth of figures, computer programs, exercises. Programs are available on disk; solutions available in separate volume. PZ

Applications (Physics), S(18), P. *Monte Carlo Methods in Boundary Value Problems.* Karl K. Sabelfeld. Ser. in Computat. Physics. Springer-Verlag, 1991, xii + 283 pp, $79. [ISBN: 0-387-53001-0] Uses common methods based on local and global integral equations to investigate three different classes of boundary value problems. Presents general approaches to constructing Monte Carlo algorithms for solving integral equations. Constructs simulation formulas for scalar and vector random fields. Specifically presents applications to homogeneous and coagulative formation of aerosols and clusters, transfer of these particles in turbulent flows and inertial deposition of particles on bodies, and stochastic problems of thin plate theory. KB

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Solving The Problem Of How Students Solve Problems.

Dr. Ronald H. Stevens

Any professor will tell you that good teaching requires an understanding of their students' thinking. At the University of California Los Angeles, however, Dr. Ronald H. Stevens has found an especially innovative way to get inside the cranium of his second-year medical students.

With his award-winning "IMMEX" software program, Dr. Stevens discovers not only *if* his students can solve Immunology problems, but also *how* information was gathered and processed during the solving.

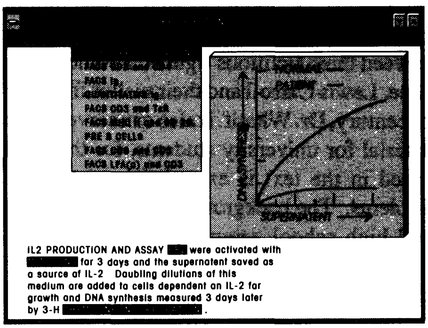
Here's how it works. Programmed in Microsoft® *Windows™ version 3.0,* the easy-to-use "IMMEX" consists of multiple cases of immune defects and a set of results from 45 laboratory tests (see figure below). Through a series of exercises and exams, students are asked to diagnose these cases by selecting the appropriate tests and examining their results.

Upon completion, graphical representations are generated by computer to show which tests were chosen and importantly, demonstrate how students searched for the solution to each problem.

It is then possible to visualize the students' thought process in a way that standard, multiple choice testing doesn't allow. For instance, Dr. Stevens can learn how organized and focused their knowledge is, how well their organization relates to critical concepts in Immunology, where major misconceptions exist and whether proper knowledge links are evident.

In turn, these insights gained through "IMMEX" can have a significant impact on teaching methods. As Dr. Stevens explains, "This approach can lead to rapid detection and remediation of individual students' problem solving difficulties, and can greatly personalize the education process."

What was Dr. Stevens' approach in developing his award-winning software? He chose to program "IMMEX" on Zenith Data Systems laptop PCs. They provided him with all the Random Access Memory and portability required to work after hours and over weekends. And that made solving the problem of how students solve problems, less of a problem.



"IMMEX" Software Program

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Napolean's Theorem (pg. 339)

NOTICE TO AUTHORS

The *Monthly* publishes articles, notes, and other features about mathematics and the profession. The readership of the *Monthly* is intended to include everybody who is mathematically inclined, including of course professional mathematicians and students of mathematics at all collegiate levels. While no single article or feature is likely to appeal to everyone, material should interest and be accessible to a large number of readers. This is the most important criterion for acceptance.

Articles may be expositions of old results or presentations of new ones. They may concern all of mathematics or one small area, a broad development or a single application, historical reminiscences or one important event. While some articles may contain the author's new research, the novelty of material and generality of the results is far less important than the clarity of exposition and general interest. Discussing one illuminating case of a well known result is far better than providing all the details of an obscure but new proposition. Articles in the *Monthly* are supposed to inform and to entertain; they are meant to be read rather than archived.

Notes are short and possibly informal articles. A note may concern a clever new proof of an old theorem, a novel way to present tired material, or a lively discussion of a philosophical (but still mathematical) issue. Also any topic is suitable, so long as it is related to mathematics. Because a note is short, the first few sentences are the most important part: They should explain the purpose and invite the reader in. Photographs or diagrams often will attract the reader's attention.

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A long time ago I made a simple observation: Theorems are more memorable when they have people's names attached. Ask students which results they remember and usually they recall the ones associated to people. Every student of Calculus knows L'Hôpital's Rule, every student of Algebra Fermat's (little) Theorem; many fewer remember the precise statement of the Mean Value Theorem or (far harder) "that result about division algebras." Mathematics is easier to appreciate when it has a human face.

Most of us call the human side of mathematics "mathematical culture." It includes the things we often think of as culture (history, philosophy, and literature), but it also includes anecdotes, fashions, and popular exposition. Great mathematics without culture is like a great symphony without an orchestra or an audience; it is beauty without soul.

This issue of the *Monthly* contains some varied examples of culture:

Great Problems of Mathematics by Laubenbacher and Pengelley (p. 313) talks about mathematical culture in the traditional sense. Their comments are based on a course that aims to show mathematics not as a collection of polished theorems but as a creative process fueled by central problems—a process carried out by people.

Zaphod Beeblebrox's Brain by Andrew Granville (p. 318) isn't history or philosophy (and *The Hitchhiker's Guide to the Galaxy* isn't great literature), but it *is* culture nonetheless. Along with some pretty mathematics, it describes the searching that went on before the final results.

Alec Norton's review of *Measure, Topology, and Fractal Geometry* (p. 378) is culture of a different kind. Indeed, the great fuss over fractals in the media is partly caused by the fact that proponents and opponents alike have made outrageous claims; outsiders stare in wonder at supposedly dispassionate mathematicians who shout at one another in print. Fashions are part of culture too.

Finally, the review of *The Man Who Knew Infinity* by Narasimhan is the kind of culture all mathematicians recognize. Few twentieth century mathematicians have received more attention than Ramanujan, yet in many ways he remains an enigmatic figure. He is the archetypical brilliant amateur, who turns out not to be an amateur at all. Secretly possessing such talent (and eventual recognition) is the quiet daydream of every young graduate student. What drove the man? What would he have accomplished in another place and time? What is his place in the mathematical hierarchy?

Clever and beautiful mathematics absorbs the mind; those are the articles readers want to study. But the human face of mathematics tickles the imagination; those are the articles readers comment on, and write letters about, and remember. Is there a lesson for the classroom here?

—John Ewing

Large Intersections of Large Sets

Paul R. Halmos

Given many large sets, can one always find many among them with a large intersection?

(1) VERTICAL LINES. The answer depends, of course, on the meaning one attaches to "many" and "large". The natural meaning of "many" involves cardinal numbers so that, for instance, a collection could be said to have "many" members if it is uncountable, or if it is just infinite, or even if it is just not empty. If "large" is interpreted to have the same (cardinal-number) meaning, then the answer to the question is no. Example: the vertical lines in the plane constitute an uncountable collection of uncountable sets such that the intersection of every subcollection with more than one element is as small as possible, namely, empty. Since set-theoretically the plane and the line are the same, it is easy to produce a similar example in the line: there exists an uncountable collection of pairwise disjoint uncountable subsets of, say, the unit interval.

(2) RADEMACHER SETS. Another possible interpretation of "large" is measure-theoretic. Sample question: does an infinite collection of measurable sets of positive measure in the unit interval always have an infinite subcollection whose intersection has positive measure? This is a trivial question to which the answer is obviously no: just consider an infinite collection of pairwise disjoint intervals (such as $(0, \frac{1}{2}), (\frac{1}{2}, \frac{3}{4}), (\frac{3}{4}, \frac{7}{8}), \ldots$). A natural way to make the question less trivial is to restrict the values of the measures that are allowed to enter. Example: does an infinite collection of measurable sets of positive measure, with measures bounded away from 0, always have an infinite subcollection whose intersection has positive measure? In the abbreviated language that is convenient to use in this context, *does an infinite collection of positive sets, bounded away from* 0, *always have an infinite subcollection with positive intersection*? This time the answer seems to depend on the underlying measure space. If the space is an infinite interval (such as $(0, \infty)$ or $(-\infty, +\infty)$), then the answer is no: look at $(0,1), (1,2), (2,3), \ldots$. What if the space is a finite interval, such as $(0, 1)$?

The answer turns out to be no again: one suitable counterexample is the collection known to students of probability as the Rademacher sets. To see them, write

$$E_1 = \left[0, \tfrac{1}{2}\right],$$

$$E_2 = \left[0, \tfrac{1}{4}\right] \cup \left[\tfrac{1}{2}, \tfrac{3}{4}\right],$$

and, in general,

$$E_n = \left[0, \frac{1}{2^n}\right] \cup \left[\frac{2}{2^n}, \frac{3}{2^n}\right] \cup \cdots \cup \left[\frac{2^n - 2}{2^n}, \frac{2^n - 1}{2^n}\right].$$

An alternative description of these Rademacher sets goes as follows: E_n is the set of those points x in $[0,1]$ in whose dyadic expansion the nth digit is 0. In this description the dyadically rational numbers (that is, the numbers of the form $\frac{k}{2^n}$) make for some annoying but unimportant ambiguity; it's unimportant because they constitute a countable set and, hence, a set of measure 0. Note, by the way, that

$$\mu(E_n) = \tfrac{1}{2}$$

for $n = 1, 2, 3, \ldots$. (Here μ is Lebesgue measure, of course.)

The sequence $\{E_1, E_2, E_3, \ldots\}$ is stochastically independent, meaning that the measure of the intersection of any k terms, $k = 1, 2, 3, \ldots$, is equal to the product of their measures (and hence equal to $\frac{1}{2^k}$). Consequence: the intersection of every infinite collection of E's has measure 0, and that concludes the proof of the negative answer.

(3) CONDENSATION POINTS. The Rademacher sets constitute a countably infinite collection—does that fact contribute to the negative answer in (2)? That is, *does an uncountable collection of positive sets always have an infinite subcollection with positive intersection*? This time the answer turns out to be yes.

The omission of the assumption of boundedness away from 0 is not a mistake: uncountability makes that assumption unnecessary. The precise statement is that every uncountable collection of positive sets contains an uncountable subcollection with measures bounded away from 0. This is a standard comment. Standard proof: every positive set has measure at least $\frac{1}{n}$ for some positive integer n, and, therefore, if a collection of positive sets is such that, for every n, only countably many of them have measure greater than $\frac{1}{n}$, then the whole collection is countable. In view of this observation, there is no loss of generality in assuming, for the rest of the proof, that $\mathfrak{C}$ is an uncountable collection of positive sets bounded away from 0.

Recall now that the measure algebra consisting of the equivalence classes of measurable sets modulo sets of measure 0 is a separable metric space, with the distance between two sets A and B being defined by the measure $\mu(A + B)$ of the symmetric difference $A + B$. (The symmetric difference of two sets is the set whose characteristic function is obtained from the given ones by addition modulo 2.) By separability, the collection $\mathfrak{C}$ has a condensation point in that space; that is, there exists a positive set P such that each ball with center P has an uncountable intersection with $\mathfrak{C}$. All that really matters is that P is a cluster point of $\mathfrak{C}$. For each $n = 1, 2, 3, \ldots$, choose A_n in $\mathfrak{C}$ so that

$$\mu(P + A_n) < \frac{1}{2^{n+1}}\mu(P).$$

It follows that

$$\begin{aligned}\mu\Big(\bigcap_{n\geqslant 1} A_n\Big) &\geqslant \mu\Big(\bigcap_{n\geqslant 1} P\cap A_n\Big) = \mu(P) - \mu\Big(\bigcup_{n\geqslant 1}(P - A_n)\Big)\\ &\geqslant \mu(P) - \sum_{n\geqslant 1}\mu(P + A_n)\\ &> \mu(P) - \sum_{n\geqslant 1}\frac{1}{2^{n+1}}\mu(P) = \frac{1}{2}\mu(P),\end{aligned}$$

and that implies the desired conclusion.

(4) LIM SUPS. Brief contemplation of the curious measure-theoretic behavior of the Rademacher sets leads in the present context to asking whether they are at least large enough to have cardinal-theoretically large intersections. That is, *does an infinite collection of positive sets bounded away from* 0 *in the unit interval always have an infinite subcollection with non-empty intersection*? The answer to that question is yes. To see the proof, observe first of all that there is no loss of generality in assuming that the prescribed collection of sets is countably infinite, say $\{E_1, E_2, E_3, \dots\}$. The question is whether there are any points that belong to infinitely many of the E's, or, in known, classical, terms, whether the lim sup of the sequence $\{E_1, E_2, E_3, \dots\}$, call it E^*, is non-empty. Since

$$E^* = \bigcap_{n \geqslant 1} \bigcup_{k \geqslant n} E_k,$$

and since the sequence $\{\bigcup_{k \geqslant n} E_k \colon k = 1, 2, 3, \dots\}$ of partial unions is decreasing, it follows that if $\mu(E_n) \geqslant \varepsilon > 0$ for all n, then

$$\mu(E^*) = \lim_{n \to \infty} \mu\left(\bigcup_{k \geqslant n} E_k\right) \geqslant \liminf_{n \to \infty} \mu(E_n) \geqslant \varepsilon.$$

Consequence: the lim sup, having positive measure, is certainly not empty.

Caution: if a sequence has an infinite subsequence with a large intersection (in any sense of the word) then the lim sup of the sequence is large (in that sense), but not conversely. Consider, for example, the sequence $\{E_1', E_2', E_3', \dots\}$ of the complements of the Rademacher sets. That complementary sequence has exactly the same measure-theoretic properties as the Rademacher sequence itself, and, in particular, it has no infinite subsequence with positive intersection. To get a new piece of information look at the lim inf, call it E_* of the original sequence $\{E_1, E_2, E_3, \dots\}$. Since

$$E_* = \liminf_{n \to \infty} E_n = \bigcup_{n \geqslant 1} \bigcap_{k \geqslant n} E_k$$

and since

$$\mu\left(\bigcap_{k \geqslant n} E_k\right) = 0$$

for all n, it follows that

$$\mu(E_*) = \lim_{n \to \infty} \mu\left(\bigcap_{k \geqslant n} E_k\right) = 0,$$

and, hence, that the lim sup E'^* of the complements is full (that is, has measure 1). In other words, the complementary sequence has a large lim sup but does not have a subsequence with large intersection.

(5) INTERVALS. Some of the preceding phenomena are known parts of set theory. Other questions about special classes of sets can be asked and have answers of some interest. What happens, for instance, if the sets to be considered are restricted not by the set-theoretic structure of the line, and not by its measure-theoretic structure, but by its order structure? That is, *does every infinite collection of subintervals of the unit interval, with lengths bounded away from* 0, *have an infinite subcollection with positive intersection*? (It doesn't matter whether the intervals are open or closed.) In other words, does the negative answer provided by the Rademacher sets change to affirmative when intervals are used?

The answer is yes. For the proof, assume that the lengths (measures) of all the intervals in the collection are bounded below by ε (> 0). If k is a positive integer such that $\frac{1}{k} < \frac{\varepsilon}{2}$, then at least one of the intervals

$$\left[0, \frac{1}{k}\right], \left[\frac{1}{k}, \frac{2}{k}\right], \ldots, \left[\frac{k-1}{k}, 1\right]$$

contains the left endpoints of an infinite subcollection of intervals in the collection. (It doesn't matter whether the endpoints in question do or do not belong to the intervals they bound.) If $\{(a_n, b_n): n = 1, 2, 3, \ldots\}$ is such an infinite subcollection, so that

$$a_n \in \left[\frac{i}{k}, \frac{i+1}{k}\right]$$

for some i and all n, then (a_n, b_n) covers the interval $[\frac{(i+1)}{k}, \frac{(i+2)}{k}]$ for all n. Conclusion: the measure of the intersection $\bigcap_{n \geqslant 1} (a_n, b_n)$ is at least $\frac{1}{k}$; q.e.d.

(6) UNCOUNTABLE INTERVALS. The proof just given made use of the finiteness of the measure of the unit interval. What happens to the same question for infinite intervals? As the question stands, the counterexample given in (2) above shows that the answer in that case is no. Does it remain no if the meaning of "many" is strengthened? That is, *does every uncountable collection of intervals in* $(-\infty, +\infty)$ *have an uncountable subcollection with a positive intersection*?

If degenerate intervals are admitted to the competition, that is, intervals that have only one point (or fewer), then the answer is clearly no and clearly uninteresting. In the proper (non-degenerate) cases the answer is yes, and the method of proof has some points of resemblance to the one just used for (4).

In view of the argument, in (3), about boundedness from below for uncountable collections, the proof will assume, with no loss of generality, that the lengths of the given intervals are bounded from below by ε (> 0). Note now that either uncountably many of the given intervals have a left endpoint, or uncountably many of them have a right endpoint (or both). (An interval such as $(-\infty, b)$ has no left endpoint.) Assume, with no loss of generality, that uncountably many have a left endpoint. If k is a positive integer such that $\frac{1}{k} < \frac{(i+1)}{k}$, then at least one of the intervals $[\frac{i}{k}, \frac{(i+1)}{k}]$ $(i = 0, \pm 1, \pm 2, \ldots)$ contains the left endpoints of an uncountable subcollection of intervals in the collection. Suppose that $\mathfrak{C}$ is an uncountable subcollection of the prescribed intervals such that for some i the left endpoint of every interval in $\mathfrak{C}$ belongs to $[\frac{i}{k}, \frac{(i+1)}{k}]$. Since the lengths of the intervals in $\mathfrak{C}$ are bounded from below by ε, it follows that each such interval covers the interval $[\frac{(i+1)}{k}, \frac{(i+2)}{k}]$. Conclusion: the measure of the intersection of all those intervals is at least $\frac{1}{k}$; q.e.d.

(7) UNCOUNTABLE POSITIVE SETS. What happens to the question raised in (6) in the general case, when intervals are replaced by measurable sets? That is, *does every uncountable collection of positive sets in* $(-\infty, +\infty)$ *have an uncountable subcollection with a positive intersection*?

The question is likely to induce a feeling of discomfort in most measure theorists. The reason is that intersections of uncountably many measurable sets enter, and such intersections are far from certain to be measurable; perhaps inner measures and outer measures ought to be considered. The main source of difficulty is the slipperiness of a measurable set. If a measurable set is altered by

adding one point to it, or omitting one from it, nothing measure-theoretically interesting is changed, and, in particular, neither the measurability of the set nor its measure is changed. If, however, each of uncountably many measurable sets is changed by the addition or omission of a point, the intersection can change radically, and, in particular, it can change from measurable to non-measurable. Since, however, the question has a definite answer that does not need to face these difficulties, perhaps the measure theorists' discomfort is unnecessary.

The answer to the question is no. Correction: the answer is no for people who regard the continuum hypothesis as a legitimate step to be used in a proof. The proof below will show that the continuum hypothesis implies the existence of an uncountable collection $\mathfrak{C}$ of positive subsets of $[0,1]$ such that every point of $[0,1]$ is contained in only countably many of them. If that is granted, then, of course, it is clear that no uncountable subcollection of $\mathfrak{C}$ can have a positive intersection.

Use the continuum hypothesis to establish a one-to-one correspondence, $\alpha \mapsto t_\alpha$ between the set of all α less than Ω and the unit interval (here Ω is the smallest uncountable ordinary number), and let

$$\{P_\alpha : \alpha < \Omega\}$$

be a similarly well-ordered collection of positive sets in the unit interval (not necessarily all of them). Since the cardinal number of Ω is less than or equal to the power of the continuum, there is no difficulty about the existence of such a well ordered set. For each α less than Ω write

$$C_\alpha = P_\alpha - \{t_\xi : \xi \leqslant \alpha\}.$$

The collection $\mathfrak{C}$ of all such C_α's is, of course, uncountable. If $t \in [0,1]$, so that $t = t_\xi$ for some ξ less than Ω, then t can belong to C_α only in case $\alpha < \xi$. Since there are only countably many such α's, it follows that every t is contained in only countably many sets of the collection $\mathfrak{C}$, and the promised construction is complete.

(8) MEASURE ALGEBRA. There is another way of proving the answer just obtained that has some measure-theoretic merit and that yields a mildly strengthened result. That other way uses a lemma of possible interest in its own right, in the subject that might be called combinatorial measure theory. Question: *does there exist a countable collection of positive sets in the unit interval such that every positive set has at least one of them as a subset*? Both yes and no can be supported by plausibility arguments—in fact (and that's the lemma) the answer is no. The positive statement of the lemma is that if $\{E_1, E_2, E_3, \ldots\}$ is a countable collection of positive sets, then there exists a positive set P such that $\mu(E_n - P) > 0$ for $n = 1, 2, 3, \ldots$. The idea of the proof is to throw away a small part of each E_n, and let P be the union of the remainders. The precise argument goes as follows. Let Q_n, for each n, be a positive subset of E_n such that

$$\mu(Q_n) < \tfrac{1}{2}\mu(E_n) \quad \text{and} \quad \mu(Q_n) < \frac{1}{3^n} \quad \text{for } n = 1, 2, 3, \ldots .$$

If $Q = \bigcup_{n \geqslant 1} Q_n$ and P is the complement (in the unit interval) of Q, then both P and Q are positive sets and

$$E_n - P = E_n \cap Q \supset E_n \cap Q_n = Q_n.$$

That's the end of the proof of the lemma, but a couple more comments are in order. (i) The proof shows that not only does there exist a set P with the stated

properties, but, in fact, there are many of them—uncountably many. (ii) The proof is not so much set-theoretic as measure-algebraic: it shows that if $\{E_1, E_2, E_3, \dots\}$ is a countable collection of non-zero elements of the measure algebra of measurable sets modulo sets of measure zero, then there exist uncountably many elements P of that measure algebra such that $\mu(E_n - P) > 0$ for $n = 1, 2, 3, \dots$.

The ground is now prepared for a second proof of the result of (7) above. This proof, too, uses the continuum hypothesis, by assuming given a one-to-one correspondence $\alpha \mapsto P_\alpha$ between the set of all α less than Ω and the set of all positive Borel sets. (Note: if c is the power of the continuum, then the cardinal number of the set of all positive sets in the unit interval is 2^c, which is too big; the cardinal number of the set of all positive Borel sets is c.) For each α less than Ω use the lemma of the preceding paragraph to exhibit a positive set C_α (which can obviously be made a Borel set) such that $\mu(P_\xi - C_\alpha) > 0$ for all ξ less than α. Assertion: the uncountable collection $\{C_\alpha\colon\ \alpha < \Omega\}$ has no uncountable subcollection with a positive intersection. Reason: each positive Borel set occurs as a P_ξ for some ξ, and the inclusion $P_\xi \subset C_\alpha$ cannot hold if $\xi < \alpha$. In other words, each positive Borel set P can be a subset of only countably many of the sets C_α. From that it follows that the intersection of uncountably many of the sets C_α cannot have any positive subsets. A re-examination of the proof thus concluded shows that it is, as foretold, measure-algebraic: it shows that there exists an uncountable collection of elements of the measure algebra such that the infimum of every uncountable subcollection is the zero element of that measure algebra.

EPILOGUE. Questions of the kind considered here were first called to my attention by a preprint of W. W. Bledsoe in 1966. The result there was the assertion (suggested by M. J. Norris) that every uncountable collection of positive sets in $(-\infty, +\infty)$ has an infinite subcollection with positive intersection—which is exactly (3) above. Bledsoe's proof was quite different from (3); so far as I know it was never published. My interest in these matters was re-aroused in the course of a recent stimulating conversation with Kevin Whyte.

There are probably many questions of the same kind still open. Might it, for instance, be profitable to ask about cardinal numbers: given a collection with cardinal κ does there always exist a subcollection with cardinal λ that has large intersection? Even the combinatorial cases of finite κ and λ might merit consideration.

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Great Problems of Mathematics: A Course Based on Original Sources

Reinhard C. Laubenbacher and David J. Pengelley

Stimulating problems are at the heart of many great advances in mathematics. In fact, whole subjects owe their existence to a single problem which resisted solution. Nevertheless, we tend to present only polished theories, devoid of both the motivating problems and the long road to their solution. As a consequence, we deprive our students of both an example of the process by which mathematics is created and of the central problems which fueled its development.

A more motivating approach could, for example, begin a discussion of infinite sets with Galileo's observation that there are as many integers as there are perfect squares. This observation seems as paradoxical to today's students as it did to Galileo. Its ingeniously simple resolution (through a better definition of "size") is a tremendous educational experience, an example of the kind of education which the German logician Heinrich Scholz characterized as "that which remains after we have forgotten everything we learned".

We have designed a lower division honors course aimed at giving students the "big picture". In the course we examine the evolution of selected great problems from five mathematical subjects. Crucial to achieving this goal is the use of original sources to demonstrate the fundamental ideas developed for solving these problems. Studying original sources allows students truly to appreciate the progress achieved through time in the clarity and sophistication of concepts and techniques, and also reveals how progress is repeatedly stifled by certain ways of thinking until some quantum leap ushers in a new era. In addition to allowing a firsthand look at the mathematical mindscape of the time, no other method would show so clearly the evolution of mathematical rigor and the conception of what constitutes an acceptable proof. Thus most homework assignments focus on gaps and difficult points in the original texts.

Since mathematics is not created in a social vacuum, we supplement the mathematical content with cultural, biographical, and mathematical history, as well as a variety of prose readings, ranging from Plato's dialogue *Socrates and the Slave Boy* to modern writings such as an excerpt on "Mathematics and the End of the World" from **[8]**. They form the basis of regular class discussions. Two good sources for such readings are **[11, 18]**. To encourage student involvement, the discussions are led by one or two students, and everybody is expected to contribute. As the finale, each student gives a short presentation of a research paper written on a topic of his or her choice.

Our course serves as an "Introduction to Mathematics," drawing good students to the subject. It attracts students from remarkably diverse disciplines, serving as a general education course for some while acting as a springboard to further mathematics for others.

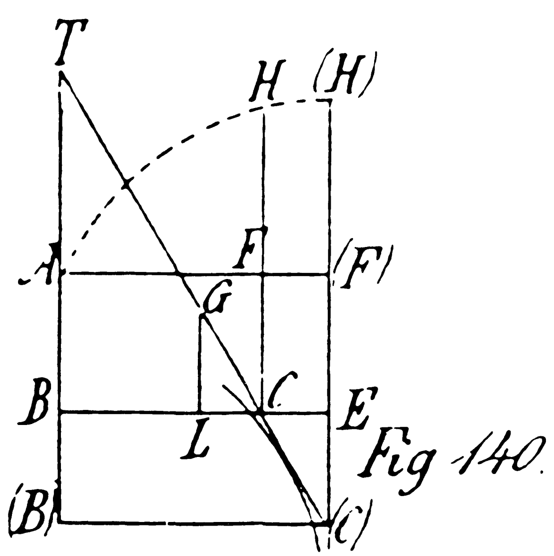
Here are our mathematical themes and original sources.

AREA AND THE DEFINITE INTEGRAL. Since ancient Greek times, mathematicians have attempted to compute areas and volumes as limits of approximations. The origins of the definite integral can be seen in Proposition 1 of Archimedes' *Measurement of the Circle* [**16**, pp. 91–93]. In his proof, Archimedes computes the area of a circle from polygonal approximations using a clever double *reductio ad absurdum* argument combined with the "method of exhaustion."

The next major advance is found in a text of Cavalieri's [**21**, pp. 214–219] illustrating his powerful "method of indivisibles" for computing the definite integral of simple polynomials. Cavalieri's book [**6**] was a very influential seventeenth century calculus text. While his method lacked rigor in part due to his cavalier attitude toward the infinite, he nevertheless succeeded in correctly computing many definite integrals.

Shortly thereafter, discovery of the inverse relationship between differentiation and integration transformed the definite integral into the most powerful computational tool in the mathematics and science of the time. Leibniz, in 1693, was the first to give a "proof" of the Fundamental Theorem of Calculus [**21**, pp. 282–284], an intuitive geometric argument based on infinitesimals (see figure below).

These ideas matured greatly in Cauchy's definition of the integral as a limit of sums in his series of calculus textbooks [**5**, vols. III and IV] [**11**, pp. 566–571], published in 1821–1823, which include his proofs of the most important theorems about the integral. Cauchy's methods are significant for two reasons: his departure from the traditional use of geometry to treat the definite integral, and his effective use of the developing concept of limit. By replacing a geometric definition by the power of algebra and the limit concept, Cauchy dispensed with the use of infinitesimals, and thus made more rigorous proofs of the basic theorems possible for the first time. Subsequently, Cauchy's work was put on a firm foundation by Weierstrass and his students, and generalized to apply to larger classes of functions via the Lebesgue integral.



An excerpt from a paper of Leibniz, "Supplementum geometriae dimensoriae, seu generalissim omnium tetragonismorum effectio per motum: similiterque multiplex constructio linae ex data tangentium conditione," published in Actorum Eruditorum Lipsiae (1693), 385–392.

THE BEGINNINGS OF SET THEORY. While the apparent paradoxes associated with infinite sets have been known since the Renaissance, they did not receive serious attention until the nineteenth century, when Bolzano made a more systematic study of them in [1]. The issue arose again when progress in the development of analysis demanded a rigorous definition of the real numbers. Increased standards of rigor and the theory of functions of several variables necessitated a complete arithmetization of the real numbers. In order to improve upon Cauchy's still partly geometric arguments for many of the central theorems in analysis, Dedekind and Cantor, both students of Weierstrass, gave two (equivalent) definitions of the real numbers not employing any geometric concepts.

Cantor's definition of the real numbers [**11**, p. 577] is based on the concept of a Cauchy sequence, a notion which Cauchy had used to give an "internal" criterion for a sequence of numbers to converge, and one which makes no reference to its limit. Once Cantor had a suitable definition for the real numbers, he was in a position to study them as an infinite set.

Bolzano had made it clear in [**1**] that he considered the property of a one-to-one correspondence between an infinite set and a proper subset fundamental to the nature of infinite sets. After Cantor realized that this property should be used as the very definition of "infinite set", it was an easy task for him to demonstrate both the countability of the rational numbers [**3**, pp. 110–111] (using a nonstandard order relation on the rationals equivalent to the usual diagonal argument) as well as the uncountability of the real numbers [**11**, pp. 579–580]. The latter proof can of course immediately be generalized to prove that the power set operation increases cardinality, thus providing the basis for Cantor's system of infinite numbers. Cantor's continuum hypothesis [**11**, pp. 580–581] (which he considered to be a theorem) became one of the important modern problems in set theory, which was solved only relatively recently.

SOLUTIONS OF ALGEBRAIC EQUATIONS. The search for algorithms to solve algebraic equations has always been one of the important problems of mathematics. Greek mathematics accomplished only the systematic solution of quadratic equations. Despite some progress by Arab mathematicians, most notably Omar Khayyam, nothing resembling a "formula" for higher degree equations emerged until the Renaissance. During that time, Greek mathematics was rediscovered and the old problems were attacked by new methods. Further progress for equations of degree three and four became possible through the introduction of algebraic techniques into Europe.

Cardano and several of his contemporaries discovered methods for solving equations such as $x^3 + ax = b$, published in his Ars Magna (The Great Art) [**20**, pp. 203–206]. In the Greek spirit, his arguments are geometric, viewing the cubic term as a volume, although the computation is easily translated into algebra.

The significance of his work (or, at least, of the publication of his book [**4**] in 1545) is twofold: it generated widespread interest in the problem of solving algebraic equations, and it raised the specter of imaginary numbers; even equations whose roots are all real may require imaginary numbers in the evaluation of Cardano's formula. (A selection of his work on imaginary roots can be found in [**20**, pp. 201–202].) Even by the time Lagrange summarized the state of the art in his lengthy 1770 memoir [**17**], no real progress had been made for equations of degree five and higher, despite much effort. Then in the early nineteenth century, Galois completely solved this two millenium old problem, using truly revolutionary methods which paved the way towards the development of abstract algebra.

FERMAT'S LAST THEOREM. The high point of Greek number theory was the determination of all Pythagorean triples by Euclid [**15**, Book X, Lemmas 1, 2; in v. 3, p. 63f] and Diophantus. The motivation was of course geometric, namely, to determine all right triangles with integer sides, via the Pythagorean Theorem. Diophantus's *Arithmetica* [**9**, **14**, **22**] inspired Fermat to conjecture in the margin of his copy what is now known as Fermat's Last Theorem, arguably the most famous open problem in all of mathematics. (Fermat's annotation can be found in [**10**, p. 2] [**11**, p. 218] [**20**, p. 213].)

Fermat probably could prove the conjecture for $n < 5$, but it was left to Euler to publish the first explicit proofs (which contained a gap for $n = 3$). Euler's proof for $n = 4$ [**21**, pp. 36–37] is quite accessible, using Fermat's method of "infinite descent" to reduce the problem to the determination of Pythagorean triples. (See e.g. [**10**, pp. 5–7] for a rigorous classification of all Pythagorean triples.)

The problem subsequently has had immense impact on the development of algebraic number theory and algebraic geometry. Examples of modern approaches are the use of complex roots of unity to factor the equation in various subfields of the complex numbers, and a reformulation in terms of algebraic geometry by considering rational points of curves. A good reference for modern developments is [**10**].

THE PARALLEL POSTULATE. Since the time Euclid included his parallel postulate as a "self-evident truth", it has been the subject of controversy, and for two thousand years geometers attempted to prove it. It was not until the nineteenth century that these attempts were shown to be futile through the simultaneous development of non-Euclidean geometry by Bolyai, Lobachevsky, and Gauss. Their work demonstrated that geometrical axiomatic systems exist independent of the physical world.

Euclid's *Elements* was the first attempt at an axiomatized mathematical theory, with rigorous proofs based on his definitions, postulates and common notions [**15**, Book I; in v. 1, pp. 153–155]. A good illustration of their use is the proof of the Pythagorean Theorem [**15**, Book I, Proposition 47; in v. 1, pp. 349–350], which of course requires the parallel postulate.

Lobachevsky published his exploration of a non-Euclidean geometry in his *Geometrical Researches on the Theory of Parallels*, translated in [**2**], and his *Pangeometry* [**20**, pp. 360–374]. The first work presents Lobachevsky's development of the basic theorems of his non-Euclidean geometry and their proofs. The second, written near the end of his life, is more expository, giving a condensed presentation of the final development of his ideas. The consistency, and thus the acceptability, of this non-Euclidean geometry was made beautifully clear later in the century when Euclidean models for it were constructed, such as Poincaré's conformal model in the disk [**19**, pp. 241–242] [**24**, p. 2.3f] [**7**, **12**, **13**, **23**].

These revolutionary ideas were popularized and developed further by Riemann, evolving into differential geometry and forming the mathematical basis for the physical theory of relativity. The shock waves of this revolution also affected the humanities, demolishing Kant's philosophy of space, and raising many fundamental questions in epistemology.

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Zaphod Beeblebrox's Brain and the Fifty-ninth Row of Pascal's Triangle

Andrew Granville

1. INTRODUCTION. A popular problem for an introductory combinatorics course is to prove that

The number of odd integers in any row of Pascal's triangle is always a power of 2. (1)

There seem to be two approaches to this question. The first uses the following remarkable observation of Kummer (which was made in 1855):

For any prime p and positive integers $n \geq m \geq 0$, the exact power of p that divides the binomial coefficient $\binom{n}{m}$ is given by the number of 'carries' when adding m and $n - m$ in base p. (*)

Thus the binomial coefficient $\binom{n}{m}$ is odd if and only if we have no carries when adding m and $n - m$ in base 2. A moment's thought and we see that this is equivalent to the statement that the set of 1's in the binary expansion of m is a subset of the set of 1's in the binary expansion of n. Therefore the number of odd binomial coefficients $\binom{n}{m}$ with $n \geq m \geq 0$ is given by the number of distinct subsets of the set of 1's in the binary expansion of n, which is precisely $2^{\#_2(n)}$, where $\#_2(n)$ is the number of 1's in the binary expansion of n. (This was first proved by Glaisher in 1899.)

The second, more elegant, approach is significant in the area of cellular automata (see [7]):

We start by replacing each entry of Pascal's triangle with an asterisk ("*") if it is odd, a blank (" ") if it is even. The problem above begins to count the number of asterisks in each row. Moreover, the normal rule of construction of Pascal's triangle (an entry equals the sum of the two immediately above) becomes a very simple binary rule:

An entry is an asterisk if and only if one of the entries immediately above is "*" and the other is blank. In FIGURE 1 we show this graphically:

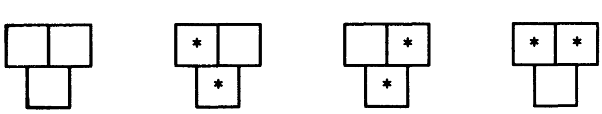


FIG. 1. The rules for addition (mod 2).

Thus Pascal's triangle itself looks like

```
               *
              * *
             *   *
            * * * *
           *       *
          * *     * *
         *   *   *   *
        * * * * * * * *
       *               *
      * *             * *
     *   *           *   *
    * * * *         * * * *
   *       *       *       *
  * *     * *     * *     * *
 *   *   *   *   *   *   *   *
* * * * * * * * * * * * * * * *
```

FIG. 2. The odd elements of Pascal's triangle (mod 2).

Continue FIGURE 2 for a few lines, stare at it, and a clear pattern begins to emerge: For every fixed $k \geq 0$, a triangle, T_k, is formed by the first 2^k rows (that is the coefficients $\binom{n}{m}$ (mod 2), for $0 \leq m \leq n \leq 2^k - 1$). T_{k+1} is then constructed by putting three copies of T_k in a triangle, with all blanks in the middle:

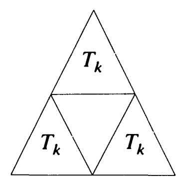
$T_{k+1} =$ 

FIG. 3. The construction of the first 2^{k+1} rows of Pascal's triangle (mod 2) from the first 2^k.

The proof that row n has precisely $2^{\#_2(n)}$ odd entries, follows easily from induction on k: For given n, there exists a k such that row n belongs to T_{k+1} but not T_k. Now, as in FIGURE 3, we see that row n (modulo 2) is composed of two copies of row $m(:= n - 2^k)$ with some blanks in the middle. Therefore, row n contains twice the number of asterisks of row m, namely, $2.2^{\#_2(m)}$ (by induction) $= 2^{\#_2(n)}$.

Many authors have worked on the corresponding problem of counting the entries of a given row of Pascal's triangle, that are not divisible by some fixed prime p. The first approach above works easily to give an exact count (see **[5]**); however the pictures generated by the second approach above are much more interesting, and are really rather pretty (see **[7]**).

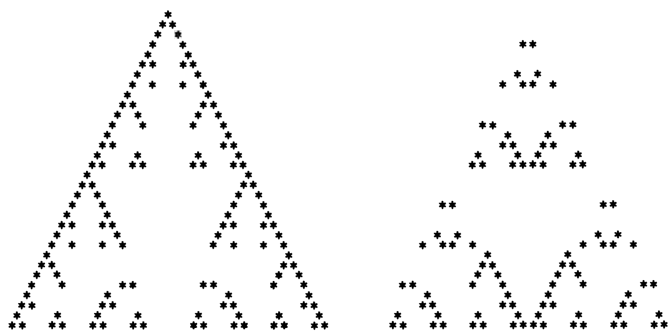
In the autumn of 1988, I presented these ideas as part of an introductory combinatorics course at the University of Toronto. One student asked whether a similar result holds when one counts the number of entries that belong to the congruence class $1 (\text{mod } 4)$, in a given row of Pascal's triangle. As I didn't know the answer, I suggested that the class compute the first few lines of Pascal's triangle (mod 4) to see if any pattern emerged. When they did so it transpired that the student had asked a very good question: We observed that the odd entries of row n of Pascal's triangle are either all $\equiv 1 (\text{mod } 4)$ or are split equally between the arithmetic progressions 1 (mod 4) and -1 (mod 4). Thus it seemed that the number of entries $\equiv 1 (\text{mod } 4)$ in row n is either $2^{\#_2(n)}$ or $2^{\#_2(n)-1}$, and the number $\equiv -1 (\text{mod } 4)$ is either 0 or $2^{\#_2(n)-1}$, respectively.

After class, I went to the library to find out whether this had previously been observed and how to prove it (it didn't seem to follow from any straightforward

modification of either of the two methods above). Rather surprisingly this pattern had not been noticed, and as it seemed unlikely that such an attractive result would be unknown, I started to think that perhaps the pattern eventually ended. However, after computing the first 60 or 70 lines of Pascal's triangle (mod 4), I found that, not only did this pattern continue to emerge, but I could even guess how to distinguish between the two cases above:

The number of entries $\equiv 1 (\mathrm{mod}\ 4)$ *equals the number of entries* $\equiv -1 (\mathrm{mod}\ 4)$ *in row n if and only if there are two consecutive* 1*'s in the binary expansion of n; otherwise there are no entries* $\equiv -1 (\mathrm{mod}\ 4)$ *in row n.* (2)

At the next class Rajesh Goyal, one of the computer science students attending, volunteered to draw two diagrams similar to FIGURE 2; the first with an asterisk only for entries $\equiv 1 (\mathrm{mod}\ 4)$; the second with an asterisk only for entries $\equiv -1 (\mathrm{mod}\ 4)$. We present these diagrams in FIGURE 4.



1 (mod 4) $\qquad$ −1 (mod 4)

FIG. 4. The odd elements of Pascal's triangle (mod 4).

As you can see, no predictable pattern leaps out, though certain members of the class were convinced that they could distinguish a maple leaf insignia! I suggested that Rajesh now draw Pascal's triangle again, this time placing all the odd entries in the same picture, but assigning different colours to the entries that were 1 (mod 4) and −1 (mod 4). Unfortunately no recognizable pattern evolved, and so the class returned to the course syllabus.

A few weeks later, still frustrated by this question, I came across a passage in Douglas Adams' science fiction/comedy novel *The Hitchhiker's Guide to the Galaxy*. There, Zaphod Beeblebrox, who has been acting unaccountably (even to himself), decides to run a series of tests on his two brains to see what is wrong. Having tried all the "standard" tests and having found nothing wrong, he proceeds to superimpose the X-rays of his two brains and look at the image through a green filter, which exposes, to his astonishment, the cauterized initials of the culprit who has been tampering with his heads!

It occurred to me to try a similar approach to our problem with Pascal's triangle. The idea was to colour those entries that are 1 (mod 4) *blue*, those that are −1 (mod 4) *yellow*, and leave the rest *blank*. Then, by superimposing different subtriangles of Pascal's triangle, to observe whether any pattern emerges (using the natural rules *blank* + *blank* = *blank*, any *colour* + *blank* = that *colour*, 2 times a particular *colour* = that *colour*, and *blue* + *yellow* = *green*). To my delight, this worked! To explain what happened, define U_k to be the triangle made up of the first 2^k rows of Pascal's triangle, coloured *blue*, *yellow* and *blank* as above (note

that by altering the *blue* and *yellow* squares of U_k to asterisks, we get T_k). By FIGURE 3, and the fact that Pascal's triangle is symmetric about a vertical line drawn down its centre, we see that

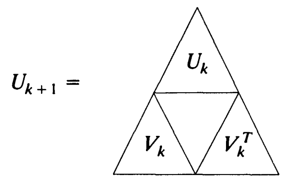


FIG. 5. The structure of Pascal's triangle (mod 4).

for some triangle V_k, where V_k^T is defined to be the reflection of V_k about a vertical line down its centre. So, if we wish to determine U_{k+1} then, by Figure 5, we must address the problem of determining V_k. By looking at a few such triangles V_k, it is easy to spot the pattern given in FIGURE 6.

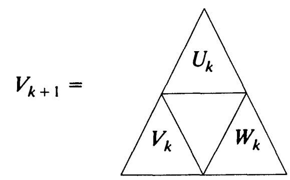


FIG. 6. The structure of V_{k+1} (mod 4).

Here W_k is some unknown pattern of *yellow* and *blue*. To try to find a simple way to derive W_k, I then used the *Beeblebrox* method to compare W_k with various other matrices and surprisingly found the important fact needed:

When we superimpose W_k onto V_k^T, every entry is either *green* or *blank*. In other words, the entry of W_k corresponding to a given entry e of V_k^T is *blank* if e is *blank*, *yellow* if e is *blue*, and *blue* if e is *yellow*. We represent this in FIGURE 7 for $k = 0, 1, 2, 3$, using ⊠ for *blue*, ⧇ for *yellow*, and ⊠ for *green* (since this journal is monochromatic!):

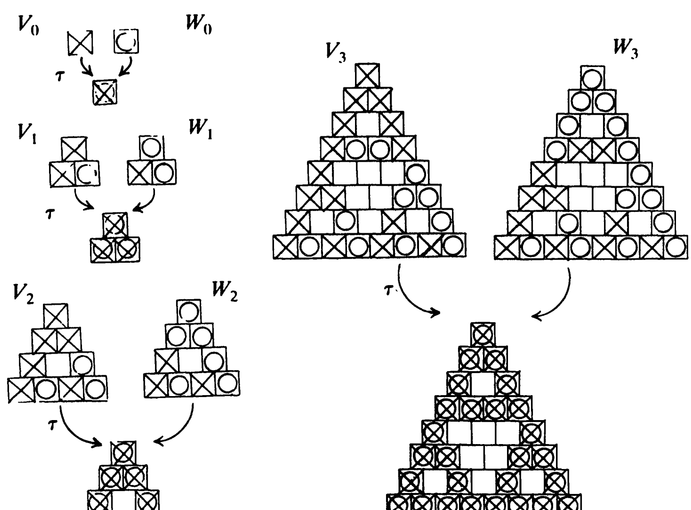


FIG. 7. *The Beeblebrox method*—Superimposing the transpose of V_k onto W_k.

Given the observation in FIGURE 7, which gives a complete description of the 'growth' of Pascal's triangle (mod 4), it is relatively simple to confirm (2). We again proceed by induction on n: Choose k so that $2^k \leq n < 2^{k+1}$. Using FIGURE 5 we see that row n of Pascal's triangle is given (from left to right) by row $n_1(:= n - 2^k)$ of V_k, some zeroes and then row n_1 of V_k^T. Thus the number of elements of row n, congruent to $j \pmod 4$ (for $j = 1$ or -1), is twice the number of such elements in row n_1 of V_k.

Now, if $n_1 < 2^{k-1}$ then using FIGURE 6 we see that row n_1 of V_k is precisely row n_1 of U_{k-1}, and so of Pascal's triangle itself. (2) then follows from the induction hypothesis, by noting that $(n)_2$ contains consecutive digits 11 if and only if $(n_1)_2$ does.

If $n_1 \geq 2^{k-1}$ then using FIGURE 6 we see that row n_1 of V_k is row $n_2(:= n_1 - 2^{k-1})$ of V_{k-1}, then some zeroes, followed by row n_2 of W_{k-1}. Now by the observation in FIGURE 7, the number of elements of row n_2 of W_{k-1} that are congruent to $j \pmod 4$ (for $j = 1$ or -1) is precisely the number of elements of row n_2 of V_{k-1} that are congruent to $-j \pmod 4$, and so we see that row n_1 of V_k contains the same number of elements $\equiv 1 \pmod 4$ and $\equiv -1 \pmod 4$. Of course, as 11 are the left most digits of $(n)_2$ (as $n_1 \geq 2^{k-1}$), the equation (2) follows immediately.

It remains only to prove the truth of the observations explained by FIGURES 6 and 7, which we do in the next section, a fairly straightforward task.

Having established that the number of odd integers in any given row of Pascal's triangle is a power of 2, and that the number $\equiv 1 \pmod 4$ (or $\equiv -1 \pmod 4$)) is, likewise, either 0 or a power of 2, it now seems reasonable to investigate the numbers of integers in each row that are congruent to 1, 3, 5 or 7 (mod 8). Preliminary computations indicate what we might expect from the results that we have already obtained:

In each row of Pascal's triangle, the number of integers in each of the arithmetic progressions 1, 3, 5 *and* 7 (mod 8) *is either* 0 *or a power of* 2. (3)

Having computed that (3) holds true in the first 50 or so rows of Pascal's triangle, we will now try to prove (3) using the same sort of approach that we used to prove (2).

First, though, let's incorporate FIGURES 6 and 7 into one diagram, into the form we shall actually prove in Section 2: Define, for a subtriangle A of Pascal's triangle (mod 4), $-A$ to be the triangle A with *blanks* the same and the colours *blue* and *yellow* swapped around. Then we have

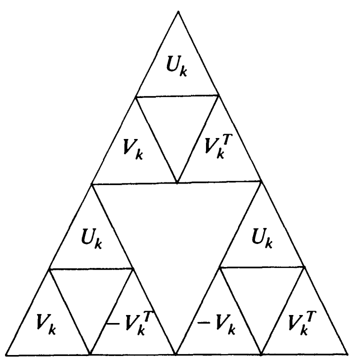


FIG. 8. The structure of U_{k+2} (mod 4).

Thus we may think of U_{k+2} being formed from U_{k+1} as follows: Cut U_{k+1} into the four triangles of FIGURE 5. Then multiply, element by element, the entries of these triangles by the triangle M (which is given in FIGURE 9 below) which gives V_{k+1}; similarly multiplying by M^T gives V_{k+1}^T. This is illustrated in FIGURE 9:

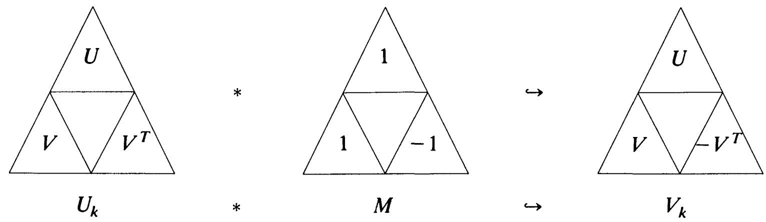


FIG. 9. The action of the growth triangle (mod 4).

This 'triangle', $M(= M_2)$, we will call the *growth triangle* (mod 4). So, to begin to prove (3), we attempt to find a growth triangle (mod 8); and later to start a proof of the corresponding statement (mod 16), we will find a growth triangle (mod 16).

For fixed positive integer b, define D_k to be the first $2^k - 1$ rows of Pascal's triangle (mod 2^b), with all even entries replaced by 0. Define E_k as in FIGURE 10.

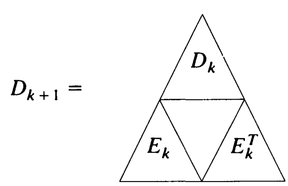


FIG. 10. Structure of D_{k+1} (mod 2^b).

In order to determine D_{k+1} from D_k we must be able to find E_k from D_k. For $b = 1$ and 2 this is done as above by noting that $E_k = D_k * M_b$ for each $k \geq b - 1$ where $M_1 = \triangle 1$ and M_2 is as in FIGURE 9. More generally, we shall prove, in section 2, that

$$E_k = D_k * M_b \quad \text{for each } k \geq b - 1, \tag{4}$$

where the triangle M_b (containing 3^{b-1} non-zero subtriangles) remains to be specified. We give the examples for $b = 3$ and 4 in FIGURE 11 (which you can test!).

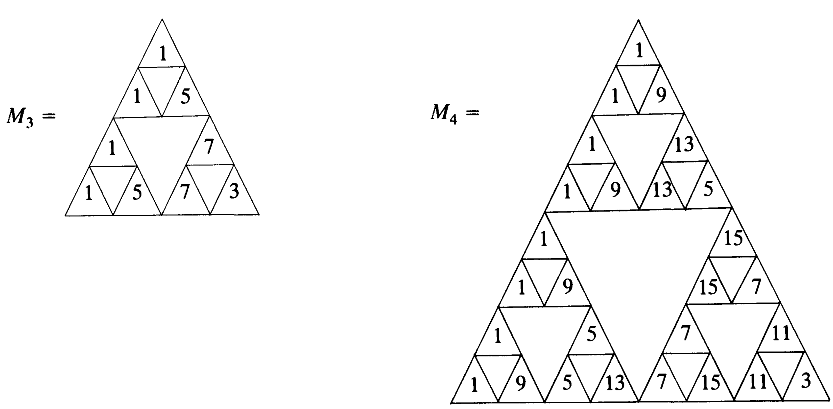


FIG. 11. The growth triangles (mod 8) and (mod 16).

Note that we could have formulated (2) in a similar way to (3):

In each row of Pascal's triangle the number of integers in each of the arithmetic progressions 1 *and* $-1 \pmod 4$ *is either* 0 *or a power of* 2.

The reason that we gave the rather more precise statement (2) is that it fit easily into our induction hypothesis. Similarly, we will reformulate (3) so that the statement fits easily into an induction hypothesis. Note first, though, that in order to use the last row of M_3 it is necessary to 'cut up' the row of Pascal's triangle into four quadrants. Of course, it isn't really necessary for the second and third rows, as there are only two halves, and Pascal's triangle is symmetric. The last row of M_3 is only used when transforming row m into row $n = 2^{k+1} + m$ where m is of the form $m = 2^k + 2^{k-1} + r$ and $0 \le r \le 2^{k-1} - 1$. For such rows m we will describe only the first two quadrants as the other two may be deduced from symmetry.

If $(n)_2$ *contains no* 11 *and no* 101 *then all entries are* $\equiv 1 \pmod 8$.

If $(n)_2$ *contains no* 11 *but has a* 101 *then there are an equal number of entries* $\equiv 1$ *and* $5 \pmod 8$.

If $(n)_2$ *contains both a* 11 *and a* 101, *or it contains a* 1111, *then there are an equal number of entries in each of* 1, 3, 5 *and* $7 \pmod 8$, *and similarly in each quadrant (when relevant).* (3)

If n does not belong to any of the cases above then, in binary, it has the form given in FIGURE 12.

$$(n)_2 = \underbrace{1\ \ 1 \ldots 1}_{t_1\ \text{1's}}\ \underbrace{0\ \ 0 \ldots 0}_{u_1\ \text{0's}}\ \underbrace{1\ \ 1 \ldots 1}_{t_2\ \text{1's}} \cdots \underbrace{0\ \ 0 \ldots 0}_{u_m\ \text{0's}}\ \underbrace{1\ \ 1 \ldots 1}_{t_{m+1}\ \text{1's}} \cdots \quad \cdots$$

Fig. 12. The binary structure of n in the remaining cases. (Here each $u_j \ge 2$ and each $t_j = 1, 2$ or 3.)

If $t_1 = 2$ *and all other* $t_j = 1$, *then all entries of the first quadrant are* $\equiv 1 \pmod 8$ *and all entries of the second quadrant are* $\equiv 7 \pmod 8$.

If $t_1 = 2$ *and each other* $t_j = 1$ *or* 3 *(and at least one* $t_j = 3$*), then there are equal numbers* $\equiv 1$ *and* $3 \pmod 8$ *in the first quadrant and there are equal numbers* $\equiv 5$ *and* $7 \pmod 8$ *in the second quadrant.* (3)

If not as above and if each $t_j = 1$ *or* 2 *then there are equal numbers* $\equiv 1$ *and* $7 \pmod 8$.

If not as above and if each $t_j = 1$ *or* 3 *then there are equal numbers* $\equiv 1$ *and* $3 \pmod 8$.

Otherwise there are equal number of entries (in each quadrant, when relevant) $\equiv 1, 3, 5$ *and* $7 \pmod 8$.

The proof of this statement is straightforward, though lengthy. The advantage of (3)′ is that it is easy to prove and (3) can be deduced immediately; we leave checking the details to the reader!

After proving (2) and (3) one now wishes to generalize our result to the odd residue classes (mod 16), then (mod 32), etc. An obvious problem is that the statement corresponding to (2) and (3)′ for Pascal's triangle (mod 16) promises to be extraordinarily long. However, as such a statement might provide the clues necessary to guess at the correct statement in the general case—an odd arithmetic progression (mod 2 to an arbitrary power)—it seems worth finding. I worked on this problem for several days, but the statement just seemed to be getting ever longer!

Wishing to reduce the amount of work necessary, I asked my colleague, Yiliang Zhu, to run some programs on his computer to test a few ideas. The results that we got were unexpected—it seemed that most of our ideas failed. Nonetheless, certain that such a proof must exist, we did a number of other computations. Our efforts were not rewarded; nothing seemed to work. Finally, we simply printed out the first 128 rows of Pascal's triangle, (mod 16), and made a visual inspection to see if we could deduce any new patterns. And there it was, the reason that things didn't seem to work—Row 59. We give the first half in FIGURE 13:

1, 11, 15, 13, 0, 0, 0, 0, 7, 13, 1, 3, 0, 0, 0, 0, 1, 11, 15, 13, 0, 0, 0, 0, 15, 5, 9, 11, 0, 0

Fig. 13. Half of Row 59 (mod 16) (with 0 (mod 2) denoted by 0).

Unbelievably, there are exactly six entries of Row 59 in each of the congruence classes 1, 11, 13 and 15 (mod 16)! Our pattern has come to an end, but not before providing us with some interesting mathematics, as well as a couple of pleasant surprises.

2. THE GROWTH TRIANGLES. In the previous section all the assertions that we used in the proofs of (2) and (3)′ were justified there, except for the existence of the growth triangle, i.e. the formula (4). That is, we need to show that if

$$2^k \leq n < 2^{k+1}, j < n/2, \quad \text{and} \quad \binom{n}{j} \text{ is odd,} \tag{5}$$

then the ratio

$$\binom{n}{j} \Big/ \binom{m}{j} \pmod{2^b}, \quad \text{where } m = n - 2^k,$$

is fixed according to the position of $\binom{m}{j}$ in a similar triangle with 2^{b-1} rows. More precisely, we must prove

Proposition 1. *Let b be a positive integer and suppose j, k and n are integers satisfying* (5), *with $k \geq b-1$. Define*

$$m' = [m/2^{k+1-b}], n' = [n/2^{k+1-b}]$$

and

$$j' = [j/2^{k+1-b}], \quad \text{where } m = n - 2^k. \tag{6}$$

Then

$$\binom{n}{j} \Big/ \binom{m}{j} \equiv \binom{n'}{j'} \Big/ \binom{m'}{j'} \pmod{2^b}. \tag{7}$$

M_b is therefore a triangle with 2^{b-1} rows. The nth row has $2n - 1$ entries and the (n, k)th entry is given by

$$\begin{cases} \binom{n + 2^{b-1} - 1}{\frac{k-1}{2}} \Big/ \binom{n-1}{\frac{k-1}{2}} \pmod{2^b} & \text{if both } k \text{ and } \binom{n-1}{\frac{k-1}{2}} \text{ are odd;} \\ 0 & \text{otherwise.} \end{cases}$$

In order to prove Proposition 1 we shall prove a result that allows us to compute binomial coefficients in modular arithmetic: In 1878 Lucas gave a simple formula for any binomial coefficient $\binom{n}{m} \pmod p$, when p is prime, in terms of the digits of m and n when they are written in base p. When $\binom{n}{m}$ is not divisible by p, this formula can be rewritten as

$$\binom{n}{m} \equiv \binom{[n/p]}{[m/p]} \binom{n_1}{m_1} \pmod{p}, \tag{8}$$

where n_1 (and similarly, m_1) is defined as the least non-negative residue of $n \pmod p$ (note that (*) provides an easy way to determine whether p divides $\binom{n}{m}$). By iterating (8) it is very easy to compute $\binom{n}{m} \pmod p$. We will prove a generalization of this formula for binomial coefficients $\pmod{p^b}$ for arbitrary positive integers b.

Proposition 2. *Suppose that prime p is given. For each positive integer j, define n_j to be the least non-negative residue of $n \pmod{p^j}$. If p does not divide $\binom{n}{m}$ then*

$$\binom{n}{m} \equiv \binom{[n/p]}{[m/p]} \binom{n_b}{m_b} \Big/ \binom{[n_b/p]}{[m_b/p]} \pmod{p^b}, \tag{9}$$

for any positive integer b.

We notice two immediate consequences:

Corollary 1. *If p does not divide $\binom{n}{m}$ and $m \equiv n \pmod{p^b}$ then $\binom{n}{m} \equiv \binom{[n/p]}{[m/p]}$* $\pmod{p^b}$.

By iterating this we get

Corollary 2. *If p does not divide $\binom{n}{m}$ and $m \equiv n \pmod{p^k}$ where $k \geq b - 1$ then* $\binom{n}{m} \equiv \binom{[n/p^{k+1-b}]}{[m/p^{k+1-b}]} \pmod{p^b}$.

From this we can easily give the

Proof of Proposition 1: By (*), 2 does not divide $\binom{n}{m}$ (as there are no carries when adding m and 2^k in base 2), and so $\binom{n}{m} \equiv \binom{n'}{m'} \pmod{2^b}$ by Corollary 2. By a similar argument $\binom{n-j}{m-j} \equiv \binom{n'-j'}{m'-j'} \pmod{2^b}$, and so the result follows as

$$\begin{aligned} \binom{n}{j} \Big/ \binom{m}{j} &= \binom{n}{m} \Big/ \binom{n-j}{m-j} \\ &\equiv \binom{n'}{m'} \Big/ \binom{n'-j'}{m'-j'} = \binom{n'}{j'} \Big/ \binom{m'}{j'} \pmod{2^b}. \end{aligned}$$

Finally we must prove Proposition 2:

Proof of Proposition 2: As p does not divide $\binom{n}{m}$ (by hypothesis), we know that each digit of n is at least as large as the corresponding digit of m when written in base p, by (*). Therefore each digit of $[n/p]$, $[n/p^b]$, n_b and $[n_b/p]$ is at least as large as the corresponding digit of $[m/p]$, $[m/p^b]$, m_b and $[m_b/p]$, respectively, (in base p), and so, by (*), none of the binomial coefficients in (9) are divisible by p. This also implies that

$$[n/p^b] - [m/p^b] - [(n-m)/p^b] = 0. \tag{10}$$

Now, for any positive integer n,

$$\begin{aligned} n! &= \prod_{j\geq 0} \prod_{\substack{r=1 \\ p^j \| r}}^{n} r \\ &= \left(\prod_{j \geq 0} \prod_{\substack{r \leq n/p^j \\ p \nmid r}} r \right) p^{\sum_{i\geq 1}[n/p^i]}, \end{aligned}$$

where $p^j \| n$ means that p^j is the highest power of p that divides n. Dividing this formula by the similar formula for $[n/p]!$ we get

$$n!/[n/p]! = p^{[n/p]} \prod_{r \leq n,\, p \nmid r} r. \tag{11}$$

Now, as $r \equiv r_b \pmod{p^b}$ for any integer r, we see that the product of those integers, coprime to p, between any two consecutive multiples of p^b, is congruent to $\prod_{r \leq p^b,\, p \nmid r} r \pmod{p^b}$. Similarly, the product of those integers, coprime to p, between cp^b and $cp^b + d$, (for any positive integers c and d), is congruent $\pmod{p^b}$ to the product of those integers, coprime to p, less than or equal to d. Therefore

$$\left(\prod_{r \leq n,\, p \nmid r} r \right) \equiv \left(\prod_{r \leq p^b,\, p \nmid r} r \right)^{[n/p^b]} \left(\prod_{r \leq n_b,\, p \nmid r} r \right) \pmod{p^b}.$$

The result then follows from combining this equation with (10) and (11) to evaluate

$$\frac{\binom{n}{m}}{\binom{[n/p]}{[m/p]}} \Bigg/ \frac{\binom{n_b}{m_b}}{\binom{[n_b/p]}{[m_b/p]}} \pmod{p^b},$$

and by using the fact (established above) that none of the binomial coefficients in (9) are divisible by p.

Actually Proposition 2 also provides another proof that if row n contains entries that are $-1 \pmod 4$ then n contains a '11' in its binary digit pattern: If row n contains an entry that is $-1 \pmod 4$, then choose k as large as possible so that row $q := [n/2^k]$ also contains an entry that is $-1 \pmod 4$. Suppose that the entry is $\binom{q}{r}$. By our choice of k, we see that

$$\binom{[q/2]}{[r/2]} \equiv 1 \pmod 4,$$

and so, by (9),

$$\binom{q_2}{r_2} \equiv -\binom{[q_2/2]}{[r_2/2]} \pmod 4.$$

By trying all possibilities for q_2 and r_2 (note that q_2 can only take the values 0, 1, 2 and 3), we see that this can only occur for $q_2 = 3$ and $r_2 = 1$ or 2. Therefore q_2 has a '11' in its binary digit pattern, and thus so does q and hence n.

3. GROWTH TRIANGLES FOR ARBITRARY PRIME POWERS. The idea of the growth triangles, used here for powers of 2, may be generalized to arbitrary prime power moduli. To prove this we need simply prove the following generalization of Proposition 1:

Proposition 3. *Let b be any positive integer, p be a given prime, and j, k and n be integers satisfying*

$$p^k \le n < p^{k+1}, \qquad k \ge b-1 \quad \text{and} \quad p \nmid \binom{n}{j}.$$

Then

$$\binom{n}{j} \Big/ \binom{n_k}{j_k} \equiv \binom{n'}{j'} \Big/ \binom{n'_k}{j'_k} \pmod{p^b}$$

where u_k is the least non-negative residue of $u(\mathrm{mod}\, p^k)$, for $u = j$ and n, and $u' = [u/p^{k+1-b}]$ for $u = j$, j_k, n and n_k.

Proof: The proof is an almost immediate generalization of that of Proposition 1; the only real difference is that we re-express the binomial coefficients here in a slightly more complicated way:

$$\binom{n}{j} \Big/ \binom{n_k}{j_k} = \binom{n}{n_k}\binom{n-n_k}{j-j_k} \Big/ \binom{j}{j_k}\binom{n-j}{n_k-j_k}.$$

We leave it to the reader to complete the details of the proof.

The action of the growth triangle T_q, for $q = p^b$, is rather different than before, as we now create $(p^2+p)/2$ new non-zero triangles from each original one—see FIGURE 14.

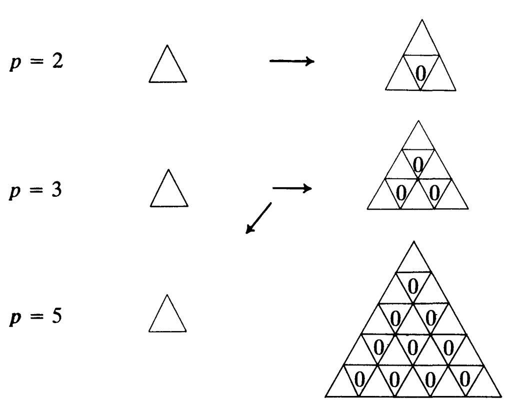


FIG. 14. The action of the growth triangle T_q, where q is a prime power of p.

T_q is composed of $(p^2 + p)/2$ large subtriangles, arranged so that there is one such subtriangle on the top row, two on the second row, ..., and p on the pth row, with zeroes in between (see FIGURE 14 for this structure). Each of these subtriangles has p^{b-1} rows, indexed by $0, 1, \ldots, p^{b-1} - 1$, and the ith row contains $2i + 1$ columns indexed by $j = 0, 1, \ldots, 2i$. The value of the (i, j)th entry in the $(m + 1)$st subtriangle of the $(n + 1)$st row (of subtriangles) $(0 \le m \le n \le p - 1)$, reading left to right, is

$$\begin{cases} \dbinom{np^{b-1} + i}{mp^{b-1} + j/2} \Big/ \dbinom{i}{j/2} \pmod{p^b} & \text{if } j \text{ is even and } p \nmid \dbinom{i}{j/2} \\ 0 & \text{otherwise.} \end{cases}$$

Notice that if $q = p$ is prime (that is $b = 1$) then i and j can only take the values $i = j = 0$. Thus the $(m + 1)$st subtriangle of the $(n + 1)$st row of subtriangles has only one entry, $\binom{n}{m} \pmod{p^b}$. Therefore T_p is just the first p rows of Pascal's triangle (mod p), with a zero between each pair of consecutive entries on each row. (This result is, essentially, given in **[5]**.)

We give some examples of T_q, where q is a power of 3, in FIGURE 15:

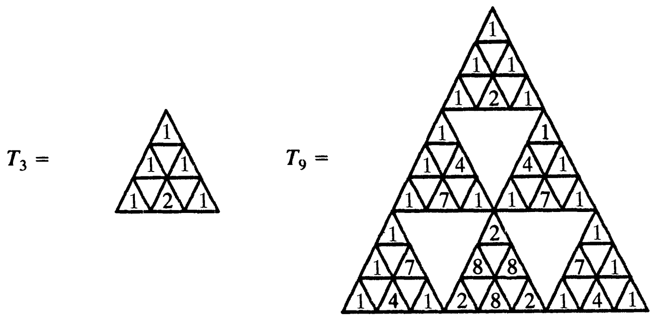


FIG. 15. Some examples of T_q when q is a power of 3.

The reader should note that if $q = 2^b$ then T_q is formed from M_b as in FIGURE 16 below.

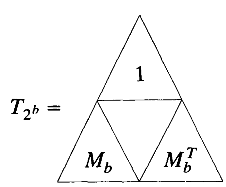


FIG. 16. Constructing T_{2^b} from M_b.

4. SELF SIMILARITY MODULO p. A beautiful aspect of the picture of Pascal's triangle modulo 2 (FIGURE 2) is that the 'pattern' inside any triangle of black squares is similar in design to that of any subtriangle, though larger in size. If we extend Pascal's triangle to infinitely many rows, and reduce the scale of our picture in half each time that we double the number of rows, then the resulting design is

called *self-similar*—that is, our picture can be reproduced by taking any subtriangle and magnifying it.

Many examples of self-similarity have been investigated by Mandelbrot [6]. Such pictures provide simple mathematical models for natural processes which are *self-organizing* (such as the growth of frost on a windowpane).

The process used to generate Pascal's triangle modulo 2 (FIGURE 1) may be modified to give further interesting, and sometimes self-similar, configurations: The patterns given by altering the 'rules' of FIGURE 1 and the number of dimensions in the picture, are known as *cellular automata*. Perhaps the most interesting example of these is Conway's *Game of Life* (see [3]).

Using the final remarks of the previous section we will obtain an interesting generalization, but in a different direction: In cellular automata, the 'cells' have two possible states—0 or 1 (*off* or *on*, *blank* or *asterisk*, *dead* or *alive*). However the entries of Pascal's triangle modulo k can be in any of k possible states—$0, 1, 2, \dots$ or $k-1$. As we shall see below, the idea of self-similarity has an interesting analogue when we allow many states. As a representation of natural processes, such cells may be thought of as containing more complex information than simply whether they are alive or dead; for instance colour, texture or even gender. Human cells are known to contain complex information, which is passed on (and sometimes modified) when they replicate: it may be that this process can be described by automata with a large number of possible states.

We start by reviewing the notion of self-similarity in terms of our growth triangles: The triangle formed by the first 2^{k+1} rows of Pascal's triangle (mod 2), is constructed from three copies of the first 2^k rows, positioned as in FIGURE 3. An easy proof of this may be given using induction: Include in the induction hypothesis the fact that the 2^kth row of Pascal's triangle (mod 2) is made up entirely of 1's. Then the 2^k+1th row has 1's on either end, with 0's all the way in between. Directly underneath each of these 1's we get a new triangle, which is formed in exactly the same way as the initial triangle of Pascal's triangle; these two new triangles are independent until they meet. Their meeting occurs at the 2^{k+1}th row, that is the 2^kth row of each of these new triangles and thus, by the induction hypothesis, this row is all 1's. This completes the induction hypothesis.

We now give a similar easy proof for the existence and structure of the growth triangles T_p, for each prime p: Start by noting that the $p+1$th row of Pascal's triangle (mod p) has 1's on either end with 0's all the way in between (this is a consequence of the elementary fact that p divides $\binom{p}{i}$ for each i, $1 \le i \le p-1$, which may be deduced from (*)). Directly underneath each of these 1's we get a new triangle, which is formed in exactly the same way as the initial triangle of Pascal's triangle and these two new triangles are independent until they meet (which happens in the $2p$th row). Thus the $2p+1$th row has 1 on either end, 2 in the middle, and 0's all the way in between. Again we find that directly underneath each of these 1's we get a new triangle, which is formed in exactly the same way as the initial triangle of Pascal's triangle, but the values underneath the 2 are twice the values in the initial triangle of Pascal's triangle. These three triangles meet in the $3p$th row, and thus the $3p+1$th row has 1's on either end, 3's at one-third and two-thirds of the way across and 0's everywhere else. We get the same triangle forming underneath the 1's, but this time 3 times the initial triangle under the 3's. We continue this process, and we see that (by an easily constructed induction hypothesis) the $kp+1$th row of Pascal's triangle (mod p) is a copy of the $k+1$th row, with $p-1$ 0's placed between consecutive entries. Finally, when we do this p times we will have constructed the first p^2 rows of Pascal's triangle (mod p) and

we can start the whole process again, this time with the larger triangle formed by the first p^2 rows, as the $p^2 + 1$th row ($= kp + 1$th row with $k = p$) has 1's on either end with 0's all the way in between. We now see how FIGURE 14 explains the growth of Pascal's triangle (mod p).

The pattern just proved has a delightful consequence noted by Long [**5**]: Cut Pascal's triangle up into subtriangles of p^k rows, where these subtriangles have 1 entry in the top row, 2 entries in the second row, ... and p^k entries in the p^kth row. The first p^k rows of Pascal's triangle give the only entry in the first row of a triangle of these subtriangles. Rows $p^k + 1$ to $2p^k$ of Pascal's triangle provide the two subtriangles of the second row of this new triangle, after missing out the large inverted triangle of 0's in between. Similarly, rows $(r - 1)p^k + 1$ to rp^k of Pascal's triangle provide the r subtriangles of the rth row of our triangle of subtriangles, after missing out the $r - 1$ large inverted triangles of 0's in between. This resulting triangle of subtriangles has the most extraordinary property—it still obeys the binary rule of FIGURE 1. That is that any two consecutive subtriangles on a row of this triangle add together, componentwise (mod p), to give the triangle immediately underneath.

ACKNOWLEDGMENTS. I would like to thank Rajesh Goyal and Yiliang Zhu for their contributions described herein, Douglas Adams for inspiration, and Nicole Magnuson for help in preparing this paper.

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On Devaney's Definition of Chaos

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Chaotic dynamical systems have received a great deal of attention in recent years (see for instance [2], [3]). Although there has been no universally accepted mathematical definition of chaos, the popular text by Devaney [1] isolates three components as being the essential features of chaos. They are formulated for a continuous map f: $X \to X$ on some metric space X (to avoid degenerate cases we will assume in this note that X is not a finite set). The first of Devaney's three conditions is that f is *transitive*; that is, for all non-empty open subsets U and V of X there exists a natural number k such that $f^k(U) \cap V$ is nonempty. In a certain sense, transitivity is an irreducibility condition. The second of Devaney's conditions is that the periodic points of f form a dense subset of X. Devaney refers to this condition as an "element of regularity" ([1], p. 50). The final condition is called *sensitive dependence on initial conditions*; f verifies this property if there is a positive real number δ (a *sensitivity constant*) such that for every point x in X and every neighborhood N of x there exists a point y in N and a nonnegative integer n such that the n^{th} iterates $f^n(x)$ and $f^n(y)$ of x and y respectively, are more than distance δ apart. This sensitivity condition captures the idea that in chaotic systems minute errors in experimental readings eventually lead to large scale divergence. Sensitive dependence on initial conditions is thus widely understood as being the central idea in chaos.

Devaney's Definition of Chaos. Let X be a metric space. A continuous map f: $X \to X$ is said to be chaotic on X if

1. f is transitive,
2. the periodic points of f are dense in X,
3. f has sensitive dependence on initial conditions.

The aim of this note is to prove the following elementary but somewhat surprising result.

Theorem. *If f: $X \to X$ is transitive and has dense periodic points then f has sensitive dependence on initial conditions.*

Before proving this Theorem, let us discuss some of the ideas that motivated it. First of all, any definition of chaos must face the obvious question: Is it preserved under topological conjugation? That is to say, if f is chaotic and if we have a

commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ h\big\downarrow & & \big\downarrow h \\ Y & \xrightarrow{g} & Y \end{array}$$

where Y is another metric space and h is a homeomorphism, then is g necessarily chaotic? Certainly transitivity and the existence of dense periodic points are preserved as they are purely topological conditions. However, sensitivity is a metric property and in general it is not preserved under topological conjugation, as the following simple example shows. Let X be the subset $(1, \infty)$ of the real line, equipped with the standard metric, let f be multiplication by 2, let Y be the set $\mathbb{R}^+$ of positive reals and let h be *log*. Clearly f has sensitive dependence on initial conditions but g is just a translation and hence is not sensitive for the standard metric on $\mathbb{R}^+$. In fact, as we leave to the reader to verify, it is not difficult to find transitive examples for which sensitivity is not preserved under conjugation. Nevertheless, as the above Theorem shows, transitivity and dense periodic points together (trivially) assure that sensitivity is preserved. Before closing this paragraph on conjugation, let us remark that sensitivity can be regarded as a topological concept if one restricts one's attention to compact spaces X (which is often the case in practice). Indeed, suppose that X is compact and that f is conjugate to g as in the above diagram. Suppose as well that f has sensitive dependence on initial conditions, with sensitivity constant δ. Let D_δ denote the set of pairs (x_1, x_2) of points in X which are separated by distance at least δ. Then D_δ is a compact subset of the Cartesian product $X \times X$ and so its image E_δ in $Y \times Y$ under the map $(x_1, x_2) \mapsto (h(x_1), h(x_2))$ is also compact. Consequently the minimum distance $\delta_Y > 0$ exists between E_δ and the diagonal in $Y \times Y$. It is easy to verify that g has sensitive dependence on initial conditions with sensitivity constant δ_Y.

Proof of Theorem: We suppose that $f\colon X \to X$ is transitive and has dense periodic points.

First observe that there is a number $\delta_0 > 0$ such that for all $x \in X$ there exists a periodic point $q \in X$ whose orbit $O(q)$ is of distance at least $\delta_0/2$ from x. Indeed, choose two arbitrary periodic points q_1 and q_2 with disjoint orbits $O(q_1)$ and $O(q_2)$. Let δ_0 denote the distance between $O(q_1)$ and $O(q_2)$. Then by the triangle inequality, every point $x \in X$ is at distance at least $\delta_0/2$ from one of the chosen two periodic orbits. We will show that f has sensitive dependence on initial conditions with sensitivity constant $\delta = \delta_0/8$.

Now let x be an arbitrary point in X and let N be some neighborhood of x. Since the periodic points of f are dense, there exists a periodic point p in the intersection $U = N \cap B_\delta(x)$ of N with the ball $B_\delta(x)$ of radius δ centered at x. Let n denote the period of p. As we showed above, there exists a periodic point $q \in X$ whose orbit $O(q)$ is of distance at least 4δ from x. Set

$$V = \bigcap_{i=0}^{n} f^{-i}\big(B_\delta(f^i(q))\big).$$

Clearly V is open and it is non-empty since $q \in V$. Consequently, since f is transitive, there exists y in U and a natural number k such that $f^k(y) \in V$.

Now let j be the integer part of $k/n + 1$. So $1 \leqslant nj - k \leqslant n$. By construction, one has

$$f^{nj}(y) = f^{nj-k}(f^k(y)) \in f^{nj-k}(V) \subseteq B_\delta(f^{nj-k}(q)).$$

Now $f^{nj}(p) = p$, and so by the triangle inequality,

$$\begin{aligned} d(f^{nj}(p), f^{nj}(y)) &= d(p, f^{nj}(y)) \\ &\geqslant d(x, f^{nj-k}(q)) - d(f^{nj-k}(q), f^{nj}(y)) - d(p, x), \end{aligned}$$

where d is the distance function on X. Consequently, since $p \in B_\delta(x)$ and $f^{nj}(y) \in B_\delta(f^{nj-k}(q))$, one has

$$d(f^{nj}(p), f^{nj}(y)) > 4\delta - \delta - \delta = 2\delta.$$

Thus, using the triangle inequality again, either $d(f^{nj}(x), f^{nj}(y)) > \delta$ or $d(f^{nj}(x), f^{nj}(p)) > \delta$. In either case, we have found a point in N whose nj^{th} iterate is more than distance δ from $f^{nj}(x)$. This completes the proof.

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Dilemma of the Sleeping Stockbroker

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The Dow Jones Industrial Average, a real number, is published daily. The stock market has a complicated time-dependent probabilistic structure. Although we do not know how it works,

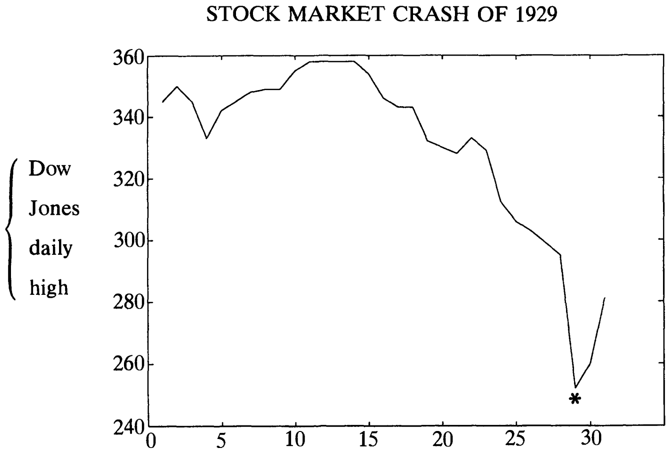

FIGURE: The asterisk indicates the Dow Jones Average on "Black Tuesday", October 29, 1929.

let us assume that the structure does not change with time. Then the Dow Jones average, as a function of time, is the output of a stationary stochastic process: A sequence

$$\mathbf{X} = \langle \ldots X_{-2} X_{-1} X_0 X_1 \ldots \rangle$$

of real-valued random variables, each a map[1] from some underlying probability space to $\mathbb{R}$. Write $P(X_0 \in [341, 367])$ to indicate the probability that today (time zero) the Dow Jones average is between 341 and 367; the function $A \mapsto P(X_0 \in A)$ is the *distribution* of X_0. The process is *stationary* if the joint distributions are independent of time:

For any finite list $A_1, \ldots, A_K$ of subsets of $\mathbb{R}$, the probability

$$P((X_{n+1} \in A_1) \,\&\, (X_{n+2} \in A_2) \,\&\ldots\&\, (X_{n+K} \in A_K))$$

is independent of n.

Henceforth, all processes are assumed to be stationary.

An example of a stationary process is a roulette wheel: The next spin produces a number according to the same distribution as the last spin. And each spin is independent of all previous (and all succeeding) spins. Such a process $\mathbf{X}$ is called a *Bernoulli process*: Each random variable X_n has the same distribution as X_0, and

[1]All sets are tacitly Borel sets and all maps are Borel functions.

the $\{X_n\}_{-\infty}^{\infty}$ are mutually independent; that is, the above joint distribution equals the product $\prod_{k=1}^{K} P(X_k \in A_k)$.

Prediction. The Dow Jones average would be a *deterministic process* if: Given exact knowledge of the infinite past

$$\langle \dots X_{-3} X_{-2} X_{-1} \rangle$$

perfect prediction of today's value X_0 is possible. That is, there is a prediction function $\pi\colon \mathbb{R}^{\mathbb{Z}} \dashrightarrow \mathbb{R}$ such that

$$P\big(\pi\langle \dots X_{-3} X_{-2} X_{-1} \rangle = X_0\big) = 1.$$

This leads to a question of Hillel Furstenberg, which we formulate in a fanciful setting.

DILEMMA OF THE SLEEPING STOCKBROKER. Suppose that —fortunately— the Dow Jones Average *is* predictable but that —unfortunately— our overworked stockbroker likes to sleep every second day, thus missing that day's value. Can he nonetheless predict the Dow on those days when he *does* come in?

If X_0 is a function of $\{X_{-m}\}_{m=1}^{\infty}$, is it a function of $\{X_{-2n}\}_{n=1}^{\infty}$?

Furstenberg's query is motivated by noting that if the process takes on only finitely many values (Range(X_0) is a finite set) then the answer is "Yes." This standard fact follows from entropy considerations: A finite-valued process is deterministic if and only if its entropy is zero. Speeding through a process twice as fast will double the entropy of the process. Sampling every second term is a factor of the speeded-up process. So this factor also has entropy zero and is therefore deterministic. Entropy theory is treated in many books, such as **[1]**.

The goal of this article is to answer Furstenberg's question negatively in a strong sense.

LEMMA OF THE SLEEPING STOCKBROKER. *There exists a real-valued stationary process* **V** *such that*

(a) *The process is deterministic; indeed, it is a function of any two consecutive terms. Each V_n is predictable from knowledge of the pair $V_0 V_1$.*

(b) *Suppose $\{n_i\}_{i=-\infty}^{\infty}$ is a sequence of indices with no consecutive pair: $n_{i+1} > 1 + n_i$ for all i. Then*

$$\dots V_{n_{-2}} V_{n_{-1}} V_{n_0} V_{n_1} \dots$$

is a Bernoulli process.

CONSTRUCTION. All random variables will take values in the half-open interval $[0,1)$. View $[0,1)$ as the unit circle and let $\oplus$ and $\ominus$ denote addition and subtraction mod 1. Say that X is *uniformly distributed* if for each $A \subset [0,1)$ the probability $P(X \in A)$ is the Lebesgue measure of A.

Encoding into pairs. Suppose Y and X are independent random variables, where Y is arbitrary and X is uniformly distributed. Then

$$Z \stackrel{\triangleleft}{=} Y \ominus X$$

(use "$\stackrel{\triangleleft}{=}$" to mean "is defined to be") is also uniformly distributed. Indeed, conditioning that Y is a specific value $a \in [0,1)$, variable X is still uniformly distributed and hence $a \ominus X$ is also, since Lebesgue measure is invariant under translation. So Z is uniformly distributed. Moreover, Z is independent of Y, since its conditioned distribution $Z|_{Y=a}$ does not depend on a.

Consequently, the relation $Y = X \oplus Z$ is a symmetric relation between X and Z. That is, these ordered triples have equal joint distributions:

$$\operatorname{JointDistr}(X,Y,Z) = \operatorname{JointDistr}(Z,Y,X).$$

This leads to a useful relation between processes. Suppose $\mathbf{Y}$ is an arbitrary process and $\mathbf{X}$ is a uniformly distributed Bernoulli process which is independent of $\mathbf{Y}$. Define $\mathbf{Z}$ by

$$\mathbf{X} \oplus \mathbf{Z} = \mathbf{Y};$$

i.e., by the relation $X_n \oplus Z_n = Y_n$. Then $\mathbf{Z}$ is uniformly distributed Bernoulli and independent of $\mathbf{Y}$. Furthermore, the process of triples

$$\begin{array}{cccccc} \dots X_{-2} & X_{-1} & X_0 & X_1 & X_2 & X_3 \dots \\ \dots Y_{-2} & Y_{-1} & Y_0 & Y_1 & Y_2 & Y_3 \dots \\ \dots Z_{-2} & Z_{-1} & Z_0 & Z_1 & Z_2 & Z_3 \dots \end{array}$$

has its finite joint distributions invariant under two operations. Under the *shift*, replacing n everywhere by $n + 1$. And under the *flip*, exchanging the top and bottom rows, that is, switching each X_n with Z_n. Alternating these operations shows that the process of ordered pairs

$$\dots (Z_{-2}, Z_{-1})(X_{-1}, X_0)(Z_0, Z_1)(X_1, X_2)(Z_2, Z_3)(X_3, X_4) \dots$$

is stationary.

It is convenient to encode these pairs into a real-valued process. Let $\Phi\colon [0,1)^2 \to [0,1)$ be a bijection. Let $\mathbf{W} \stackrel{\triangleleft}{=} \operatorname{Twist}[\mathbf{X}, \mathbf{Z}]$ mean that $\mathbf{W}$ is the stationary process of encoded pairs

$$W_n \stackrel{\triangleleft}{=} \begin{cases} \Phi(X_{n-1}, X_n) & \text{if } n \text{ is even;} \\ \Phi(Z_{n-1}, Z_n) & \text{if } n \text{ is odd.} \end{cases}$$

The Induction. Let $\Phi_{\mathbb{Z}}\colon [0,1)^{\mathbb{Z}} \to [0,1)$ be a bijection. Given a process $\mathbf{W}$, let Compress[$\mathbf{W}$] be the process whose nth term is

$$\Phi_{\mathbb{Z}}(j \mapsto W_{n+j}).$$

Compress [$\mathbf{W}$] is stationary, since $\mathbf{W}$ is.

Pick some arbitrary initial process $\mathbf{W}^{(0)}$ and create a sequence $\mathbf{W}^{(1)}, \mathbf{W}^{(2)}, \dots$ of processes. At stage k, define $\mathbf{W}^{(k)}$ as follows:

(i) Set $\mathbf{Y}^{(k-1)} \stackrel{\triangleleft}{=} \operatorname{Compress}[\mathbf{W}^{(k-1)}]$.

(ii) Pick a uniformly distributed Bernoulli process $\mathbf{X}^{(k)}$ which is independent of all previously chosen processes and define $\mathbf{Z}^{(k)}$ by

$$\mathbf{X}^{(k)} \oplus \mathbf{Z}^{(k)} = \mathbf{Y}^{(k-1)}.$$

(iii) Let $\mathbf{W}^{(k)} \stackrel{\triangleleft}{=} \operatorname{Twist}[\mathbf{X}^{(k)}, \mathbf{Z}^{(k)}]$.

Let $\Phi_{\mathbb{N}}\colon [0,1)^{\mathbb{N}} \to [0,1)$ be a bijection. Defining

$$V_n \stackrel{\triangleleft}{=} \Phi_{\mathbb{N}}\big(W_n^{(1)}, W_n^{(2)}, W_n^{(3)}, \dots\big)$$

will produce the process $\mathbf{V}$ claimed in the theorem.

Proof of Determinism (a): Fix k and suppose the values of $W_0^{(k)}$ and $W_1^{(k)}$ are known. By decoding via Φ^{-1} one obtains $X_0^{(k)}$ and $Z_0^{(k)}$. Adding them together yields $Y_0^{(k-1)}$ which, when decoded by $\Phi_{\mathbb{Z}}^{-1}$, reveals $W_n^{(k-1)}$ for all n.

Since the value of V_0, V_1 gives all pairs $W_0^{(k)}, W_1^{(k)}$, for each n we know the list of values $\{W_n^{(k-1)}\}_{k=1}^{\infty}$ and hence V_n.

Proof of Independence (b): Fix k. If we condition on the outcome of processes $\mathbf{W}^{(1)}, \mathbf{W}^{(2)}, \ldots, \mathbf{W}^{(k-1)}$ then process $\mathbf{X}^{(k)}$ is still uniformly distributed Bernoulli, since it was chosen independently of all foregoing processes. Since $\mathbf{Y}^{(k-1)}$ is now determined, random variables $X_n^{(k)}$ and $Z_n^{(k)}$ are functions of each other. Thus, insofar as questions of independence are concerned we may replace Z's by X's and regard each $W_{n_i}^{(k)}$ as the pair

$$X_{n_i-1}^{(k)} X_{n_i}^{(k)}. \tag{$*$}$$

As i ranges over $\mathbb{Z}$, no subscript in $(*)$ occurs more than once. Thus

$$\left\{W_{n_i}^{(k)} \middle| i \in \mathbb{Z}\right\}$$

is Bernoulli. Since its distribution is independent of the joint process $\mathbf{W}^{(1)}, \ldots, \mathbf{W}^{(k-1)}$, induction on k yields that the random variables

$$\left\{W_{n_i}^{(k)} \middle| i \in \mathbb{Z},\, k \in \mathbb{N}\right\} \tag{$**$}$$

are mutually independent. So the $\{V_{n_i} | i \in \mathbb{Z}\}$ certainly are. $\square$

Three closing remarks. Process $\mathbf{V}$ has an arbitrarily chosen process $\mathbf{W}^{(0)}$ as a factor. Yet for $k = 1, 2, \ldots$, variable $W_1^{(k)}$ is just an encoding of two independent uniformly distributed random variables; hence its distribution is unaffected by $\mathbf{W}^{(0)}$. From $(**)$, the $W_1^{(k)}$ are mutually independent. Consequently, the distribution of V_1 (or any individual V_n) in no way depends upon the process $\mathbf{W}^{(0)}$ which got the construction going.

The property that process $\mathbf{V}$ satisfies can be restated. For any $S \subset \mathbb{Z}$, the set

$$\{V_n | n \in S\}$$

will be Bernoulli, as long as S contains no translate of the pattern $\{0, 1\}$. What other patterns permit such a process?

To the accuracy that stockbrokers need, the Dow Jones Average is a number to two decimal places. So one can view the Dow as a countable-valued process (not finite-valued; it is unclear that the Dow is bounded ...) and it is natural to wonder whether DILEMMA OF THE SLEEPING STOCKBROKER can arise in the countable case.

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Converses of Napoleon's Theorem

John E. Wetzel

Interesting converse results in elementary geometry can often be found by taking certain parts of a figure as given "in position" and investigating the extent to which various other parts of the figure are determined. In this article we use this tactic to obtain some apparently new converses of the well-known theorem of Napoleon. Geometry is more a point of view than a methodology, and we employ a variety of different arguments (synthetic, coordinate, transformational, complex analytic) to establish our results. To set the stage, we begin with an overview of Napoleon's theorem and a glimpse of its long history.

1. NAPOLEON'S THEOREM AND TORRICELLI'S CONFIGURATION. The familiar but curious theorem attributed to Napoleon Bonaparte asserts that the centers L, M, N of the three equilateral triangles $\triangle BXC$, $\triangle CYA$, $\triangle AZB$ built outwards on the sides $\overline{BC}, \overline{CA}, \overline{AB}$ of an arbitrary triangle $\triangle ABC$ are the vertices of an equilateral triangle, and the same is true of the centers L', M', N' of the three inward equilateral triangles $\triangle CX'B$, $\triangle AY'C$, $\triangle BZ'A$.

The configuration formed by a triangle, the equilateral triangles on its sides, the "Napoleon" triangles, and various connecting lines and circles (commonly called "Torricelli's configuration" a century ago), has many elegant and unexpected properties.

The outward case. Suppose (FIGURE 1) that $\triangle ABC$ is a positively oriented triangle (so that $A \to B \to C \to A$ is counterclockwise). The outer Napoleon triangle $\triangle LMN$ is also positively oriented, and its center coincides with the centroid G of $\triangle ABC$. Lines $\overleftrightarrow{AX}, \overleftrightarrow{BY}, \overleftrightarrow{CZ}$ are concurrent at a point F, called the *outward Fermat point* of $\triangle ABC$, and F lies on the circumcircle of each outward equilateral triangle $\triangle BXC$, $\triangle CYA$, $\triangle AZB$ and also on the circumcircle of the inner Napoleon triangle $\triangle L'M'N'$. Lines $\overleftrightarrow{AX}, \overleftrightarrow{BY}, \overleftrightarrow{CZ}$ make acute angles of 60° with each other at F, and $AX = BY = CZ = \pm AF \pm BF \pm CF$, a minus sign being taken if the angle of $\triangle ABC$ at that vertex exceeds 120°. The vertices A, B, C are symmetric to F in the sidelines $\overleftrightarrow{MN}, \overleftrightarrow{NL}, \overleftrightarrow{LM}$ of the outer Napoleon triangle $\triangle LMN$. Lines $\overleftrightarrow{AL}, \overleftrightarrow{BM}, \overleftrightarrow{CN}$ are concurrent. When $\triangle ABC$ has a 120° angle, F is the vertex of that angle; when $\triangle ABC$ has an angle larger than 120°, F lies in the angle vertical to that angle; and when every angle of $\triangle ABC$ is smaller than 120°, F lies inside $\triangle ABC$ and is the point P that solves the problem Fermat posed to Torricelli: minimize $f(P) = PA + PB + PC$. When the largest angle of $\triangle ABC$ exceeds 120°, the solution of Fermat's problem is the vertex of that largest angle.

The inward case. Analogous properties hold for the inward case. Suppose (FIGURE 2) that $\triangle ABC$ is a positively oriented scalene triangle. The inner Napoleon triangle $\triangle L'M'N'$ is negatively oriented, and its centroid coincides with the centroid G of $\triangle ABC$. Lines $\overleftrightarrow{AX'}, \overleftrightarrow{BY'}, \overleftrightarrow{CZ'}$ are concurrent at a point F', called

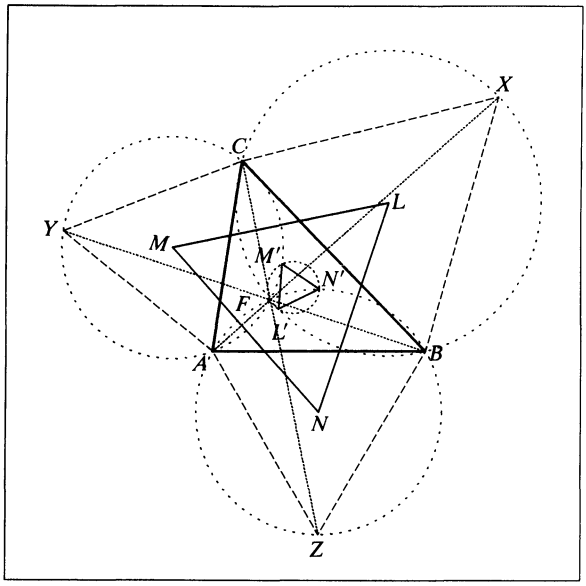

FIG. 1. The outward case.

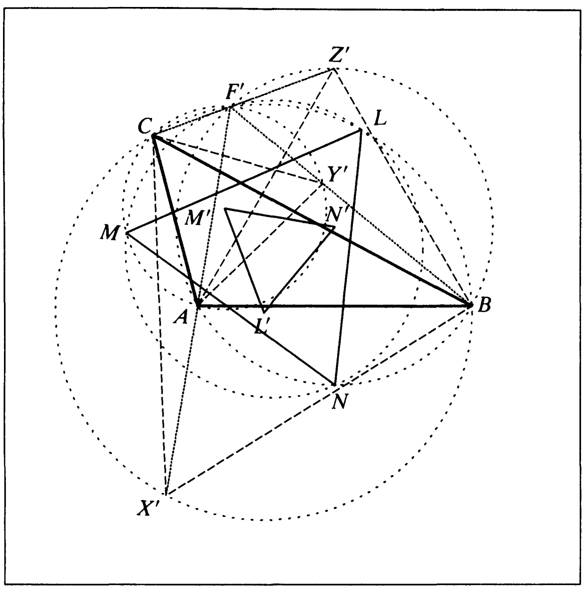

FIG. 2. The inward case.

the *inward Fermat point* of $\triangle ABC$, and F' lies on the circumcircle of each inward equilateral triangle $\triangle CX'B$, $\triangle AY'C$, $\triangle BZ'A$ and on the circumcircle of the outer Napoleon triangle $\triangle LMN$. Lines $\overleftrightarrow{AX'}, \overleftrightarrow{BY'}, \overleftrightarrow{CZ'}$ make acute angles of 60° with each other at F', and $AX' = BY' = CZ' = \pm AF' \pm BF' \pm CF'$, a minus sign being taken at each vertex where the angle of $\triangle ABC$ is larger than 60°. The vertices A, B, C are symmetric to F' in the sidelines $\overleftrightarrow{M'N'}, \overleftrightarrow{N'L'}, \overleftrightarrow{L'M'}$ of the inner Napoleon triangle $\triangle L'M'N'$, and lines $\overleftrightarrow{AL'}, \overleftrightarrow{BM'}, \overleftrightarrow{CN'}$ are concurrent. The point F' is never inside $\triangle ABC$. When $\triangle ABC$ has exactly one 60° angle, F' is that vertex; and when two angles of $\triangle ABC$ are both larger or both smaller than 60°, F' lies

outside $\triangle ABC$ inside the angle at the third vertex. Refining a claim of Courant and Robbins [**4**; pp. 354–359], Brownawell and Goodman [**2**] have shown that if $\angle A \geqslant 60°$ and $\angle B \geqslant 60°$, for example, then F' is the point P that maximizes $g(P) = PC - PA - PB$. When $\triangle ABC$ has two angles less than $60°$, the solution of this maximum problem is the vertex of the smallest angle.

The collinear case. Most of these properties, suitably phrased, are correct when A, B, C are collinear (FIGURE 3) and the distinction between "inner" and "outer" is lost. In this case the inner and outer pictures are symmetric in the line of collinearity.

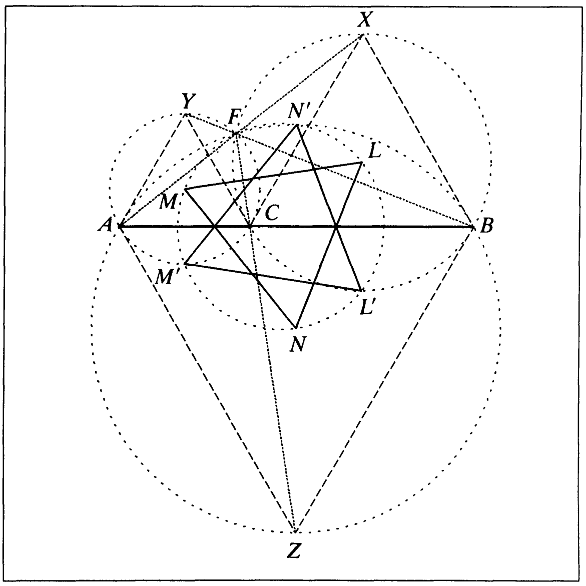

FIG. 3. The collinear case.

Some further properties. Here are a few of the many additional properties of the Torricelli configuration that appear in the early literature. Triangles $\triangle AYZ'$, $\triangle AY'Z$, etc., are congruent to $\triangle ABC$, and their circumcenters lie on the circumcircle of $\triangle ABC$. The sum of the areas of the Napoleon triangles $\triangle LMN$ and $\triangle L'M'N'$ is the average of the areas of the three outward equilateral triangles on the sides of $\triangle ABC$, and the difference of these areas is the area of $\triangle ABC$. The line $\overleftrightarrow{FF'}$ through the two Fermat points bisects the segment $\overline{HG}$ that joins the orthocenter H and the centroid G of $\triangle ABC$. The point Q so that the figure $F'HFQ$ is a parallelogram lies on the circumcircle of $\triangle ABC$. And the triangle formed by the lines through A, B, C perpendicular to $\overleftrightarrow{AF}, \overleftrightarrow{BF}, \overleftrightarrow{CF}$ is the largest equilateral triangle that can be circumscribed about $\triangle ABC$, and its area is $4(ABC)$. (These results and more can be found in Mackay [**21**].)

Finally we mention one particularly elegant recent observation (Garfunkel and Stahl [**15**]). Let A_1, A_2 be the trisection points of the side $\overline{BC}$ of $\triangle ABC$ with A_1 nearer B, and define B_1, B_2 and C_1, C_2 similarly on $\overline{CA}$ and $\overline{AB}$. Then the summits of the six outward and six inward equilateral triangles on the sides of the irregular hexagon $A_1A_2B_1B_2C_1C_2$ form concentric regular hexagons.

Sources. Napoleon's theorem is surely one of the most-often rediscovered results in mathematics. The literature is extensive and offers almost a plethora of related results, extensions, and generalizations, supported by divers arguments. Many

writers have used it as a kind of touchstone to establish the efficacy of their favorite approaches to geometry. An assortment of proofs can be found in the following readily available sources: Court [**5**, pp. 105–107], Coxeter and Greitzer [**6**, pp. 60–65, 82–83], Demir [**7**], Fettis [**10**], Finney [**11**], Forder [**14**, p. 40], Garfunkel and Stahl [**15**], Honsberger [**17**, pp. 24–36, 40, 147–152], Johnson [**18**, pp. 218–224], Mauldon [**22**], Rabinowitz [**25**], Yaglom [**32**, pp. 38–40, 93–97]. Most of these references discuss related results and some properties of the full configuration. Generalizations of various kinds can be found in many of these references, and especially in, for example, Berkhan and Meyer [**1**, pp. 1216–1219], Douglas [**8**], Finsler and Hadwiger [**12**], Fisher, Ruoff, and Shilleto [**13**], Gerber [**16**], Neumann [**23**], [**24**], Rigby [**26**], and Schütte [**28**], most of which list numerous additional sources.

Why Napoleon? The early history of Napoleon's theorem and the Fermat points F, F' (which are also called the *isogonic centers* of $\triangle ABC$) is summarized in Mackey [**21**], who traces the fact that $\triangle LMN$ and $\triangle L'M'N'$ are equilateral to 1825 to one Dr. W. Rutherford [**27**] and remarks that the result is probably older. The attribution of the result to Napoleon (1769–1821) has itself been the object of study (Cavallaro [**3**], Scriba [**29**]). Mackay does not mention Napoleon, nor does any other nineteenth century reference with which I am familiar. The earliest attribution I have seen appeared in 1911 in Faifofer [**9**, p. 186], where the result, posed as Problem 494, is accompanied by the parenthetical comment, "Teorema proposto per la dimostrazione da Napoleone a Lagrange." It would be of historical interest to trace the result back to Napoleon, although as Coxeter and Greitzer [**6**, p. 63] remark, "the possibility of his knowing enough geometry for this feat is as questionable as the possibility of his knowing enough English to compose the famous palindrome, ABLE WAS I ERE I SAW ELBA."

2. CONVERSES OF NAPOLEON'S THEOREM. Interesting converse problems arise from taking parts of the Torricelli configuration as given and trying to determine the range of variability of the remaining parts of the figure. For example, one can consider existence and uniqueness questions concerning the "progenitor" triangle $\triangle ABC$ when some of the derived points $X, Y, Z, X', Y', Z', L, M, N, L', M', N', F, F'$ are prescribed. There are many possibilities, ranging from trivial to quite involved. In the following sections we consider several such converse questions.

The earliest result of this kind of which I am aware is a construction problem posed in 1868 by E. Lemoine [**20**]: Construct the triangle, given the summits of the equilateral triangles built on its sides.

An elegant construction for Lemoine's problem was provided the following year by L. Kiepert [**19**]. Points X, Y, Z are given (FIGURE 4), to be the summits of equilateral triangles $\triangle XBC$, $\triangle AYC$, $\triangle ABZ$. Let P, Q, R be the summits of the outward equilateral triangles on the sides of $\triangle XYZ$. Then A, B, C are the midpoints of $\overline{XP}, \overline{YQ}, \overline{ZR}$, respectively. Kiepert's argument (repeated in Wetzel [**31**]) uses Ptolemy's theorem and properties of the Fermat point $F = \overleftrightarrow{XP} \cap \overleftrightarrow{YQ} \cap \overleftrightarrow{ZR}$. A perspicuous motion proof can also be given. Write W_θ for the rotation about a point W through the (trigonometric) angle θ, write arguments on the left, and compose motions from the left. Then (FIGURE 4) the motion $Y_{60}X_{60}Z_{60}$ fixes A and consequently is halfturn about A. But $PY_{60}X_{60}Z_{60} = ZX_{60}Z_{60} = QZ_{60} = X$. Thus A is the midpoint of $\overline{PX}$.

In 1956, in an article whose principal objective was to promote the use of motions in the teaching of geometry, H. G. Steiner [**30**] used motions to show that

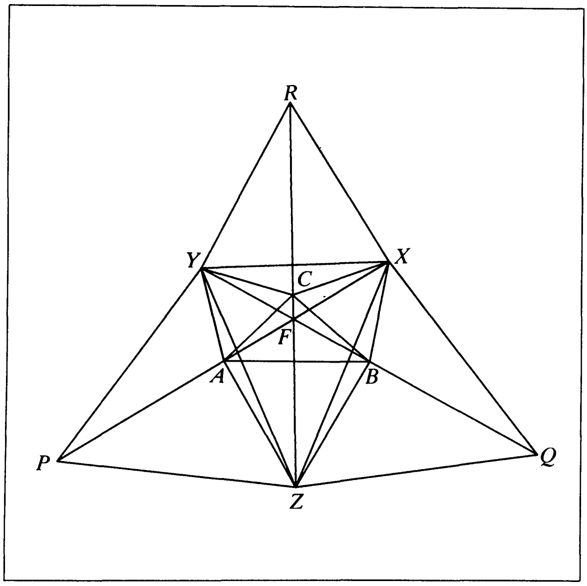

FIG. 4. Kiepert's construction.

if three distinct points X, Y, Z are given, there are, in general, *eight* triangles $\triangle ABC$ so that the triangles $\triangle ABX$, $\triangle BCY$, $\triangle CAZ$ are equilateral.

Steiner's elegant motion argument is as follows. If each of α, β, γ is $\pm 60°$, then each of the eight motions $Z_\gamma Y_\beta X_\alpha$ is a halfturn or a rotation through $\pm 60°$, and in every case it has a unique fixed point A. Defining $C = AZ_\gamma$ and $B = CY_\beta$, we see that $A = AZ_\gamma Y_\beta X_\alpha = CY_\beta X_\alpha = BX_\alpha$; and consequently triangles $\triangle CAZ$, $\triangle BCY$, $\triangle ABX$ are all equilateral. On the other hand, if $\triangle ABC$ is such a triangle, it is clear from a sketch that A is the fixed point of one of these eight motions, the signs depending on the relative orientations of $\triangle ABC$, $\triangle ABX$, $\triangle BCY$, $\triangle CAZ$. (The complicated question of whether $\triangle ABX$, $\triangle BCY$, $\triangle CAZ$ turn out to be outward or inward on the sides of $\triangle ABC$ is considered in Wetzel **[31]**.)

3. A CONVERSE OF NAPOLEON'S THEOREM. Suppose the two Napoleon triangles $\triangle LMN$ and $\triangle L'M'N'$ are given "in position." Is $\triangle ABC$ determined? We show that it is, provided that $\triangle LMN$ and $\triangle L'M'N'$ have the same center.

Obviously there is *at most one* generating triangle $\triangle ABC$, because the sidelines a, b, c of $\triangle ABC$ must be the mediators (i.e., the perpendicular bisectors) of the segments $\overline{LL'}, \overline{MM'}, \overline{NN'}$. The existence is a little more trouble. The core of the argument is the following lemma, for which we give first a traditional synthetic proof and then an argument that uses coordinates.

Lemma 1. *Five points X, X', Y, Y', T are arranged so that $\angle X'TY' = 120°$, $\angle YTX = 120°$, $XT = YT$, $X'T = Y'T$, and $XT \neq X'T$* (FIGURE 5). *Then the mediators of the segments $\overline{XX'}$ and $\overline{YY'}$ meet at a point S, and $\triangle SXX'$ and $\triangle SYY'$ are both equilateral.*

Proof: A rotation of 120° about T carries X' to Y' and Y to X, so $\overleftrightarrow{X'Y}$ and $\overleftrightarrow{XY'}$ meet at 60° at a point W. The circles through X, X', W and Y, Y', W meet at W and again at a second point S. Then $\angle XSX' = \angle XWX' = 60° = \angle Y'WY =$

$\angle Y'SY$, and so $\angle XSY' = \angle X'SY$. Since $\angle SY'X = \angle SYX'$ and $XY' = YX'$, $\triangle XSY' \cong \triangle X'SY$. Hence $SX = SX'$ and $SY = SY'$. ■

This synthetic proof, in the classical tradition, assumes that the points are positioned as in the figure. Similar arguments can, of course, be given in the various other cases, but we prefer instead to rely on a short computational proof using coordinates that is unexceptionable.

Second Proof. Introduce coordinates so that T is the origin and X' and Y' have coordinates $(2,0)$ and $(-1,\sqrt{3})$. If X has coordinates $(2s,2t)$ with $s^2+t^2 \neq 1$, then Y has coordinates $(-s+\sqrt{3}t, -\sqrt{3}s-t)$. Consequently the mediators of $\overline{XX'}$ and $\overline{YY'}$ have equations

$$(1-s)x - ty = 1 - s^2 - t^2$$

$$(s-\sqrt{3}t-1)x + (\sqrt{3}s+t+\sqrt{3})y = 2(1-s^2-t^2);$$

and these two lines meet at a point S with coordinates $(s+\sqrt{3}t+1, -\sqrt{3}s+t+\sqrt{3})$. A calculation confirms that $SX = XX' = X'S$ and $SY = YY' = Y'S$ irrespective of the values of s and t. ■

A point S so that both $\triangle SXX'$ and $\triangle SYY'$ are equilateral exists even when $XT = X'T$, but then the mediators of $\overline{XX'}$ and $\overline{YY'}$ coincide. Note that according to Napoleon's theorem the circumcenters of $\triangle SXX'$, $\triangle SYY'$, $\triangle XWY$, $\triangle X'WY'$ (marked in FIGURE 5) form a 60° rhombus.

Finally, here is our first converse of Napoleon's theorem.

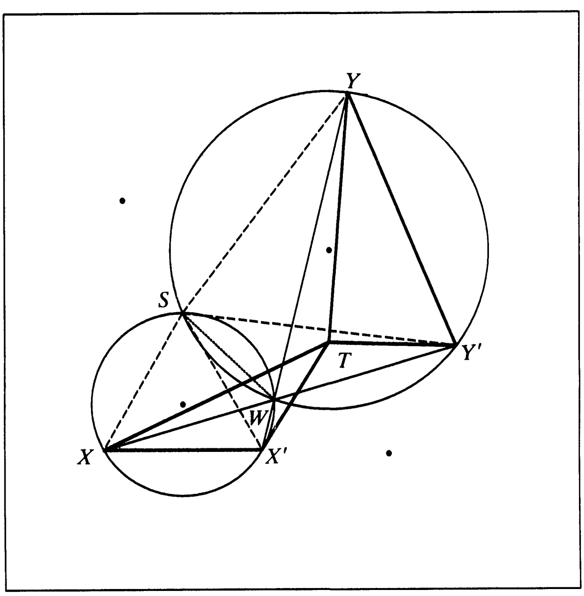

FIG. 5. An essential lemma.

Theorem 2. *Two equilateral triangles $\triangle LMN$ and $\triangle L'M'N'$ are given in position so that*

(a) *$\triangle LMN$, $\triangle L'M'N'$ are oppositely oriented and have the same center G, and*
(b) *$LM > L'M'$.*

Then there is exactly one triangle $\triangle ABC$ having $\triangle LMN$ as its outer Napoleon triangle and $\triangle L'M'N'$ as its inner Napoleon triangle, and its sides are the mediators of $\overline{LL'}, \overline{MM'}, \overline{NN'}$.

Proof: Suppose without loss of generality that $\triangle LMN$ is positively oriented. Taking the points X, X', Y, Y', T in Lemma 1 to be N, N', M, M', G, we conclude that the mediators b and c of $\overline{MM'}$ and $\overline{NN'}$ meet in a point A so that $\triangle AMM'$ and $\triangle ANN'$ are both equilateral. Similarly, if a is the mediator of $\overline{LL'}$, $B = c \cap a$, and $C = a \cap b$, then $\triangle BNN'$, $\triangle BLL'$, $\triangle CLL'$, $\triangle CMM'$ are all equilateral. The points A, B, C are different and non-collinear. (If two of the three points A, B, C were coincident, then the three mediators a, b, c would be concurrent and all three points would coincide. Then the $\pm 60°$ rotation about C that carries L to L' would carry M' to M (otherwise M would go to M' and $LM = L'M'$, contrary to (b)). This same rotation carries N to N' or N' to N, and correspondingly $LN = L'N'$ or $M'N' = MN$, contrary to (b) again. It follows that B and C, C and A, A and B lie on opposite sides of $\overleftrightarrow{LL'}, \overleftrightarrow{MM'}, \overleftrightarrow{NN'}$. Consequently, L, L', M, M', N, N' are the centers of equilateral triangles on sides $\overline{BC}, \overline{CA}, \overline{AB}$ of $\triangle ABC$. Thus $\triangle LMN$ and $\triangle L'M'N'$ are Napoleon triangles of $\triangle ABC$, and (according to (b)) $\triangle LMN$ is outer and $\triangle L'M'N'$ is inner. ■

The conclusion of the theorem is true when the inner Napoleon triangle $\triangle L'M'N'$ collapses to a point. It is also worth mentioning that any two vertices of one Napoleon triangle with any vertex of the other completely determine both Napoleon triangles "in position," so that according to the theorem they determine $\triangle ABC$ uniquely provided only that $\triangle L'M'N'$ is smaller than $\triangle LMN$.

Some consequences. The figure formed by two oppositely oriented concentric equilateral triangles has many nice properties that seem not so easy to prove without the superstructure provided by Theorem 2. Indeed, all the properties described in Section 1 have counterparts in this figure. Here are a few specific examples.

Corollary 3. *Two oppositely oriented concentric equilateral triangles $\triangle LMN$ and $\triangle L'M'N'$ are given, with $L \neq L'$, $M \neq M'$, and $N \neq N'$. Then*

(a) *Lines $\overleftrightarrow{LL'}, \overleftrightarrow{MM'}, \overleftrightarrow{NN'}$ lie in a pencil. They are parallel if $LM = L'M'$ and concurrent otherwise.*
(b) *The points $\overleftrightarrow{LM} \cap \overleftrightarrow{L'M'}, \overleftrightarrow{MN} \cap \overleftrightarrow{M'N'}, \overleftrightarrow{NL} \cap \overleftrightarrow{N'L'}$ are collinear (one might be at infinity).*
(c) *The centroid of the triangle $\triangle A_0B_0C_0$ of midpoints A_0, B_0, C_0 of $\overline{LL'}, \overline{MM'}, \overline{NN'}$ is at the common center G of the two given triangles.*
(d) *Suppose that $LM \neq L'M'$, and let A be the point in which the mediators b, c of $\overline{MM'}, \overline{NN'}$ intersect. Then the points S, S' symmetric to A in $\overleftrightarrow{MN}, \overleftrightarrow{M'N'}$ lie on the circumcircles of $\triangle L'M'N'$, $\triangle LMN$.*

Proof: (a) It is easy to verify directly that the lines are parallel if $\triangle LMN$ and $\triangle L'M'N'$ have the same circumcircle. If $LM > L'M'$, then $\overleftrightarrow{LL'}, \overleftrightarrow{MM'}, \overleftrightarrow{NN'}$ are the mediators of the sides of $\triangle ABC$ and hence are concurrent at the circumcenter of that triangle.

(b) This follows from (a) by Desargues' theorem.

(c) When $LM > L'M'$, the common center G of $\triangle LMN$ and $\triangle L'M'N'$ is the centroid of $\triangle ABC$ and consequently also the centroid of its medial triangle $A_0B_0C_0$. The result when $LM = L'M'$ follows by continuity, for example.

(d) Points S, S' are the Fermat points of $\triangle ABC$. ■

4. ANOTHER CONVERSE. To what extent is the progenitor triangle $\triangle ABC$ determined if only one Napoleon triangle is given in position? The answer to this question is a little more complicated.

Suppose $\triangle PQR$ is a given equilateral triangle, to play the role of $\triangle LMN$ *or* $\triangle L'M'N'$. Taking our cue from FIGURES 1, 2, and 3, we generate the vertices A, B, C by reflecting a point S in the lines $\overleftrightarrow{QR}, \overleftrightarrow{RP}, \overleftrightarrow{PQ}$ (FIGURE 6). Since $PB = PS = PC$, $\triangle BPC$ is isosceles, and it is easy to see that $\angle CPB = 2\angle QPR = \pm 120°$ by summing angles at P. Consequently P is the center of an equilateral triangle built on $\overline{BC}$. Similarly for Q and R, of course, and since $\triangle PQR$ is equilateral it follows that it is a Napoleon triangle of $\triangle ABC$. The problem is to determine when $\triangle PQR$ is inner and when it is outer. Here is our second converse.

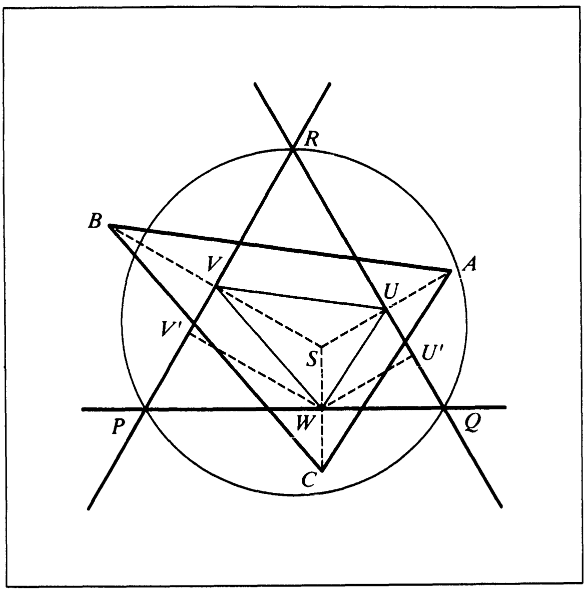

FIG. 6. The case of one given Napoleon triangle.

Theorem 4. *Let $\triangle PQR$ be a positively oriented equilateral triangle with circumcircle Γ, and for any point S let A, B, C be the points symmetric to S in $\overleftrightarrow{QR}, \overleftrightarrow{RP}, \overleftrightarrow{PQ}$. Then:*

(a) *When S lies on Γ, points A, B, C are collinear.*

(b) *When S lies inside Γ, then $\triangle PQR$ is the outer Napoleon triangle of $\triangle ABC$, and S is its outward Fermat point. The largest angle of $\triangle ABC$ is greater than, equal to, or less than $120°$ according to whether S lies outside, on, or inside $\triangle PQR$.*

(c) *When S lies outside Γ, then $\triangle PQR$ is the inner Napoleon triangle of $\triangle ABC$, and S is its inward Fermat point. One angle of $\triangle ABC$ is a $60°$ angle precisely when S lies on a sideline of $\triangle PQR$, and $\triangle ABC$ has two angles larger than $60°$ when S lies in one of the regions off a vertex of $\triangle PQR$ and two angles smaller than $60°$ when S lies in one of the regions off an edge of $\triangle PQR$.*

Proof: Let U, V, W be the feet of the perpendiculars from S to $\overleftrightarrow{QR}, \overleftrightarrow{RP}, \overleftrightarrow{PQ}$. Since a dilatation with center S and ratio 2 carries U, V, W to A, B, C, it is enough to study triangle $\triangle UVW$.

The fact that U, V, W are collinear if and only if S lies on the circumcircle Γ is well known, and the line on which they lie is the Simson line of S. (See Coxeter and Greitzer [**6**] for an exposition of this classical theory.)

When S moves, the orientation of its pedal triangle $\triangle UVW$ remains unchanged unless the points U, V, W become collinear, which occurs only when S lies on Γ. It follows that the orientation of $\triangle UVW$, and so of $\triangle ABC$, agrees with that of $\triangle PQR$ for S inside Γ and is opposite that of $\triangle PQR$ for S outside Γ. Consequently $\triangle PQR$ is the outer Napoleon triangle of $\triangle ABC$ when S lies inside Γ and the inner Napoleon triangle of $\triangle ABC$ when S lies outside Γ.

Now suppose S lies inside Γ, and suppose (with no loss of generality) that $\angle UWV$ is a maximal angle of $\triangle UVW$. (Since the sides of $\triangle UVW$ are proportional to the distances PS, QS, RS (see Coxeter and Greitzer [**6**, p. 23]), this requires only that S lie in the sector PGQ, where G is the center of $\triangle PQR$.) Let U', V' be the feet of the perpendiculars from W to $\overleftrightarrow{QR}, \overleftrightarrow{RP}$. Then $\angle U'WV' = 120°$, and it is apparent that $\angle UWV$ is less than, equal to, or greater than $120°$ according to whether S lies inside, on, or outside $\triangle PQR$.

A similar argument can be given in the inward case (c); we omit the minutiæ. ∎

The assertion can be phrased more symmetrically. If $\Gamma_P, \Gamma_Q, \Gamma_R$ are the images of Γ under reflection in $\overleftrightarrow{QR}, \overleftrightarrow{RP}, \overleftrightarrow{PQ}$, then $\triangle PQR$ is the outer Napoleon triangle of $\triangle ABC$ when A, B, C lie inside $\Gamma_P, \Gamma_Q, \Gamma_R$ (respectively) and the inner Napoleon triangle of $\triangle ABC$ when A, B, C are outside $\Gamma_P, \Gamma_Q, \Gamma_R$.

I am indebted to G. D. Chakerian for much of this elegant geometric argument, contained in a letter dated November 28, 1979. My original argument employed coordinates. In another letter dated January 2, 1980, Chakerian observed that parts of the theorem follow immediately from the formula

$$(ABC) = \frac{3\sqrt{3}}{4}(r^2 - \rho^2)$$

for the signed area of $\triangle ABC$ in terms of the radius r of Γ and $\rho = GS$ (see, for example, Johnson [**18**, p. 139]).

In summary, we have the following:

Corollary 5. *A progenitor triangle exists for a given equilateral triangle* $\triangle$LMN *and Fermat point F precisely when F lies inside the circumcircle of* $\triangle$LMN, *and then it is unique. A progenitor triangle exists for a given equilateral triangle* $\triangle$L′M′N′ *and Fermat point F′ precisely when F′ is outside the circumcircle of triangle* $\triangle$L′M′N′, *and then it is unique.*

Exercise: What is the story if L, M, N, and F' (or L', M', N', and F) are given?

5. MIXED CONVERSES. Finally we consider two similar-looking converse situations for which the results turn out to be surprisingly different.

The case X, Y, N. When X, Y, N are prescribed, we shall see that again there is a significant circle. An analysis using motions gets us started. In FIGURE 1 it is plain that the motion $Y_{60}X_{60}N_{120}$ fixes A, and consequently it must be A_{240}. Suppose conversely that points P, Q, R are given (to play the role of X, Y, N). The motion

$Q_{60}P_{60}R_{120}$, being a 240° rotation, has a unique fixed point A. Let $C = AQ_{60}$ and $B = CP_{60}$. Then $AQ_{60}P_{60}R_{120} = CP_{60}R_{120} = BR_{120}$, so $\triangle ACQ$ and $\triangle CBP$ are equilateral with $\angle AQC = \angle BPC = 60°$ and $\triangle BAR$ is isosceles with $\angle BRA = 120°$.

Under what circumstances are A, B, C collinear? And if they are not collinear, when are P, Q, R the points X, Y, N of $\triangle ABC$ and when are they X', Y', N'? In other words, when is $\triangle ABC$ positively oriented and when negatively? Here is the result.

Theorem 6. *Distinct points P, Q, R are given (to play the role of X, Y, N). Let S be the point so that $\triangle PQS$ is a positively oriented equilateral triangle, and let T be the center of $\triangle PQS$. Let Γ be the circle through T with center S. Then there exists a unique triple ABC so that $\triangle BCP$ and $\triangle CAQ$ are equilateral and $\triangle CAR$ is isosceles with $\angle R = \pm 120°$; and (see* FIGURE 7):

(a) *If R lies on Γ, points A, B, C are collinear;*
(b) *If R lies inside Γ, $\triangle ABC$ is positively oriented, and its points X, Y, N are the given points P, Q, R;*
(c) *If R lies outside Γ, $\triangle ABC$ is negatively oriented, and its points X', Y', N' are the given points P, Q, R.*

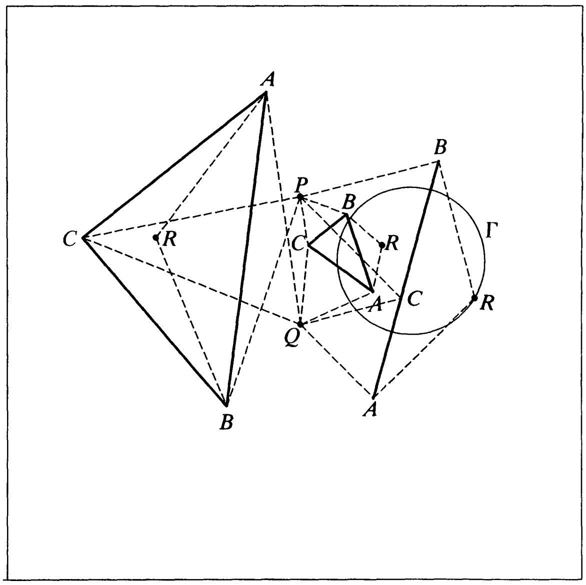

FIG. 7. The case X, Y, N.

Proof: Matters such as these are easily handled in the complex plane. Introduce coordinates so that the points P, Q, and T have complex coordinates 1, 0, and $d = 1/2 - (\sqrt{3}/6)i$, respectively, and write $h = e^{i\pi/6}$. Recall that in the complex plane, rotation through the (trigonometric) angle θ about a point z' is given by the linear mapping $w = z' + e^{i\theta}(z - z')$. If R has complex coordinate z_0, we see by composing the mappings that the key motion $Q_{60}P_{60}R_{120}$ is given by the transformation $w = h + (1 + \bar{h})z_0 - hz$. The coordinates a, c, b of the fixed point A of

this transformation and of the points $C = AQ_{60}$ and $B = CP_{60}$ are easy to compute: $a = d + \bar{h}z_0$, $b = hd + z_0$, and $c = -hd + hz_0$. The signed area (ABC) of $\triangle ABC$ in the complex plane whose vertices have coordinates a, b, c is given by the determinant

$$(ABC) = \frac{i}{4}\begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix} = -\frac{1}{2}\Im m\left(a\bar{b} + b\bar{c} + c\bar{a}\right),$$

and for the case at hand a short calculation shows that $(ABC) = -\frac{1}{4}\sqrt{3}\left(|z_0 - \bar{h}|^2 - \frac{1}{3}\right)$. Now the claims of the theorem are easily checked. ■

To summarize, if two different points U, V are given, let $\triangle UVS$ be a positively oriented equilateral triangle and let $\Gamma(U, V)$ be the circle with center S that passes through the center T of $\triangle UVS$. Then we have the following Napoleon converse.

Corollary 7. *A progenitor triangle exists for given X, Y, N precisely when N lies inside the circle $\Gamma(X, Y)$, and it is unique; and a progenitor triangle exists for given X', Y', N' precisely when N' lies outside the circle $\Gamma(X', Y')$, and it is unique.*

The case X, Y, N'. The situation if the points X, Y, N' are prescribed is quite different. Again we begin by examining an appropriate motion. It is plain in FIGURE 1 that the motion $Y_{60}X_{60}N'_{-120}$ is the identity, because it is a translation that fixes the point A. Consequently $N'_{120} = Y_{60}X_{60}$, and $\triangle XYN'$ is a positively oriented isosceles triangle with $\angle XN'Y = 120°$. A similar argument using the motion $Y'_{-60}X'_{-60}N_{120}$ shows that $\triangle X'Y'N$ is a negatively oriented isosceles triangle with $\angle X'NY' = -120°$.

Conversely, suppose that points P, Q, R are given (to play the role of X, Y, N'), and let A be any fixed point of the motion $Q_{60}P_{60}R_{-120}$ and $C = AQ_{60}$ and $B = CP_{60}$. Then $\triangle ACQ$ and $\triangle CBP$ are equilateral with $\angle AQC = \angle BPC = 60°$, and $\triangle BAR$ is isosceles with $\angle BRA = -120°$. But the motion $Q_{60}P_{60}R_{-120}$ is a translation, so it has a fixed point precisely when it is the identity; and it is easy to see that this occurs precisely when $\triangle PQR$ is a positively oriented isosceles triangle with $\angle PQR = 120°$. Then A can be chosen arbitrarily, and B, C determined.

Similarly, the translation $Q_{-60}P_{-60}R_{120}$ has a fixed point just when it is the identity, and this occurs precisely when $\triangle PQR$ is a negatively oriented isosceles triangle with $\angle PQR = -120°$. Again A can be chosen arbitrarily, and B, C determined: $C = AQ_{-60}$, $B = CP_{-60}$.

In either case, $\triangle ACQ$ and $\triangle CBP$ are equilateral with $\angle AQC = \angle BPC = \pm 60°$, and $\triangle BAR$ is isosceles with $\angle BRA = \pm 120°$.

When are A, B, C collinear? If A, B, C are not collinear, when are the given points P, Q, R the points X, Y, N' of $\triangle ABC$ and when are they the points X', Y', N? In other words, how is $\triangle ABC$ oriented? Here is our final converse.

Theorem 8. *Distinct points P, Q, R are given (to play the role of X, Y, N'). Let S be the point so that $\triangle PQS$ is positively oriented and equilateral, and let R_1 be the center of $\triangle PQS$ and R_2 the points symmetric to R_1 in $\overleftrightarrow{PQ}$. Let Γ_1 be the circle determined by the points Q, R_1, S and Γ_2 the circle symmetric to Γ_1 in $\overleftrightarrow{PQ}$. Then there are three points A, B, C so that $\triangle CPB$ and $\triangle AQC$ are equilateral with $\angle CPB = \angle AQC = \pm 60°$ and $\triangle RBA$ is isosceles with $\angle BRA = \pm 120°$ if and only if R is R_1 or R_2; and in either case, one of A, B, C can be chosen arbitrarily and the other two*

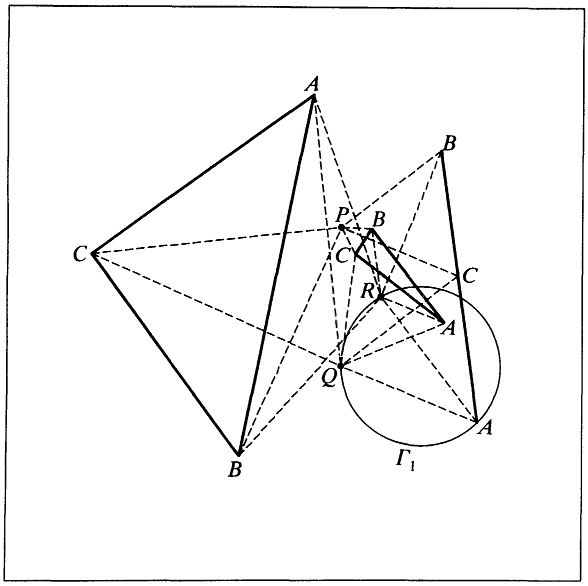

FIG. 8. The case X, Y, N'.

determined. Suppose $R = R_1$. Then (FIGURE 8):

(a) *If A lies on Γ_1, then A, B, C are collinear;*
(b) *If A lies inside Γ_1, then $\triangle ABC$ is positively oriented, and its points X, Y, N' are the given points P, Q, R;*
(c) *If A lies outside Γ_1, then $\triangle ABC$ is negatively oriented, and its points X', Y', N are the given points P, Q, R.*

If $R = R_2$, the orientation of $\triangle ABC$ is reversed, but the other assertions are unchanged.

Proof: If there are points A, B, C with the property described, the remarks prior to the statement of the theorem imply that $R = R_1$ or $R = R_2$. Suppose the former. In the complex coordinate system employed in the proof of Theorem 5 above, if A has coordinate a, then the coordinates of B and C are $b = \bar{h} - \bar{h}a$ and $c = ha$. A calculation shows that

$$(ABC) = -\frac{\sqrt{3}}{4}\left(\left|a + \frac{\sqrt{3}}{3}i\right|^2 - \frac{1}{3}\right).$$

The various claims are now immediate, and the assertions in the case $R = R_2$ follow from the symmetry in $\overleftrightarrow{PQ}$. ■

To summarize, if two different points U, V are given, let $W_1 = f_1(U, V)$ be the vertex of the positively oriented isosceles triangle with base $\overline{UV}$ and base angle 30° and $W_2 = f_2(U, V)$ the vertex of the negatively oriented isosceles triangle with base $\overline{UV}$ and base angle 30°; and let $\Gamma_j = \Gamma_j(U, V)$ be the circle tangent to $\overleftrightarrow{UV}$ at V through W_j. Then we have the following Napoleon converse:

Corollary 9. *A progenitor triangle $\triangle ABC$ for given points X, Y, N' exists precisely when $N' = f_1(X, Y)$ or $N' = f_2(X, Y)$. In the former case, A can be chosen*

arbitrarily inside the circle $\Gamma_1(X, Y)$, and $C = AY_{60}$ and $B = CX_{60}$. In the latter case, A can be chosen arbitrarily inside the circle $\Gamma_2(X, Y)$, and $C = AY_{-60}$ and $B = CX_{-60}$.

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On a Theorem of Frobenius: Solutions of $x^n = 1$ in Finite Groups

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Given a finite group G and a positive integer n, we write $f_n(G)$ to denote the number of solutions in G to the equation $x^n = 1$. A celebrated theorem of Frobenius asserts that if n divides $|G|$, then n divides $f_n(G)$. There are numerous proofs of Frobenius' theorem in the literature (see, for example, [1] or Corollary 41.11 of [2] or Theorem 9.1.1 of [3]); some of these are representation theoretic and others are "elementary". The authors believe, however, that the proof presented below is the easiest yet to appear.

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In the following, the symbol "p" will always represent a prime integer, and "G" a finite group. If $n > 0$ is an integer, we shall write n_p to denote the largest power of p which divides n. For example $24_2 = 8$.

If $g \in G$ is any element, it is well known that there is a unique decomposition $g = xy$ with x and y commuting, such that x has p-power order and y has order not divisible by p. This establishes a bijection between the elements of G and the ordered pairs (x, y) for which y has order prime to p and $x \in \mathbf{C}_G(y)$ has p-power order. We use this correspondence to prove the following lemma, which is the key to our argument.

Lemma 1. *Given any group G, integer n and prime p, write $q = n_p$ and let T be a set of representatives for those conjugacy classes of elements $y \in G$ such that $y^{n/q} = 1$. Then*

$$f_n(G) = \sum_{t \in T} |G : \mathbf{C}_G(t)| f_q(\mathbf{C}_G(t)).$$

Proof: By the remarks preceding the statement of the lemma, each group element $g \in G$ corresponds to a certain pair (x, y) with $x \in \mathbf{C}_G(y)$. Furthermore, the orders of these elements satisfy $o(g) = o(x)o(y)$ and hence $g^n = 1$ iff $x^q = 1$ and $y^{n/q} = 1$. It follows that $f_n(G)$ is the sum of the quantities $f_q(\mathbf{C}_G(y))$ as y runs over all solutions to $y^{n/q} = 1$ in G. Since $f_q(\mathbf{C}_G(y))$ remains constant as y runs over the $|G : \mathbf{C}_G(t)|$ elements in the conjugacy class represented by $t \in T$, the stated formula follows. ■

We shall prove Frobenius' theorem first in the case where n is a prime power. To expedite the discussion, we say that a group G has the *p-Frobenius* property if q divides $f_q(G)$ for every power q of p such that q divides $|G|$.

Lemma 2. *Let q be a power of p with q dividing $|G|$. Suppose $H \subseteq G$ is a subgroup having the p-Frobenius property. Then q divides $|G:H|f_q(H)$.*

Proof: If $q \mid |H|$, this is trivial, and so we write $q_0 = |H|_p$ and assume $q_0 < q$. Then $f_q(H) = f_{q_0}(H)$ is divisible by q_0 and $|G:H|$ is divisible by $|G|_p/q_0$. It follows that $|G|_p$ divides $|G:H|f_q(H)$ and thus q does too. ■

We shall prove that the p-Frobenius property always holds. We need the following observation.

Lemma 3. *The number of elements of order exactly m in any group G is a multiple of $\varphi(m)$, where φ is Euler's function.*

Proof: Define an equivalence relation on G by setting $x \equiv y$ if $\langle x \rangle = \langle y \rangle$. All of the elements in any given equivalence class have equal order and if that order is equal to m, the class has cardinality $\varphi(m)$. The result follows. ■

Theorem 4. *Let G be any finite group. Then G has the p-Frobenius property.*

We remark that Theorem 4 can be viewed as a considerably strengthened and generalized version of Cauchy's theorem: if p divides $|G|$, then G has an element of order p. Since our argument appeals to Cauchy's theorem (at least for abelian groups), it does not provide an independent proof of Cauchy's result.

Proof of Theorem 4: We use induction on $|G|$. Let q be a p-power with $q \mid |G|$. We show that $q | f_q(G)$ by considering first the case where $q = |G|_p$. Applying Lemma 1 with $n = |G|$ and sorting the terms according to whether or not $t \in \mathbf{Z}(G)$, we obtain

$$|G| = f_n(G) = |T \cap \mathbf{Z}(G)| f_q(G) + \sum_{t \in T - Z(G)} |G : \mathbf{C}_G(t)| f_q(\mathbf{C}_G(t)).$$

Each term in the sum on the right has the form $|G:H|f_q(H)$ where $H < G$. By the inductive hypothesis, H satisfies the p-Frobenius property and so by Lemma 2, q divides $|G:H|f_q(H)$. Since q also divides $|G|$, we conclude that q divides $|T \cap \mathbf{Z}(G)|f_q(G)$ and it suffices to show that p does not divide $|T \cap \mathbf{Z}(G)|$.

We have

$$T \cap \mathbf{Z}(G) = \{y \in \mathbf{Z}(G) | y^{n/q} = 1\}.$$

this is a subgroup of $\mathbf{Z}(G)$ which contains no element of order p. It follows by Cauchy's theorem that $|T \cap \mathbf{Z}(G)|$ is not divisible by p.

Now suppose $q < |G|_p$. As $f_{|G|_p}(G)$ is divisible by $|G|_p$, it is divisible by q and it suffices to show that $f_{|G|_p}(G) - f_q(G)$ is divisible by q. This quantity is the number of elements of G having p-power order exceeding q. By Lemma 3, however, the number of elements of G with order p^e is a multiple of $\varphi(p^e) = (p-1)p^{e-1}$. If $p^e > q$, this is divisible by q, and the result follows. ■

Theorem 5. (Frobenius). *Suppose n divides $|G|$. Then n divides $f_n(G)$.*

Proof: It suffices to show for each prime p that q divides $f_n(G)$, where $q = n_p$. By Lemma 1, $f_n(G)$ can be expressed as a sum of terms of the form $|G:H|f_q(H)$,

where $H \subseteq G$ satisfies the p-Frobenius property by Theorem 4. By Lemma 2, each term is divisible by q and the result follows. ■

We close by mentioning that Frobenius' theorem is often stated in a form more general than our Theorem 5. If $a \in G$ is an arbitrary element, we write $f_n(G, a)$ to denote the number of solutions in G to the equation $x^n = a$. The following result is essentially 9.1.1 of [3].

Theorem 6. *For every choice of $a \in G$ and positive integer n, the number $f_n(G, a)$ is divisible by g.c.d.$(n, |\mathbf{C}_G(a)|)$.*

In fact, Theorem 6 can be proved by methods similar to those in this paper. We have chosen to present only the proof of Theorem 5, since that is surely the most interesting and important case, and we wished to give the easiest possible proof for that result.

Nevertheless, it seems appropriate to sketch briefly the ideas involved in proving Theorem 6. Reduction to the case where $a \in \mathbf{Z}(G)$ is immediate. We can then reduce to the case where n is a power of p by using the equation

$$f_n(G, a) = \sum_{t \in T} |G : \mathbf{C}_G(t)| f_q(G, t)$$

where $q = n_p$ and T is a set of representatives for the conjugacy classes of elements $y \in G$ such that $y^{n/q} = a$. The next step is to note that if n is a power of p and a is central, then $f_n(G, a) = f_n(G, x)$, where $a = xy = yx$ and x has p-power order while y has order not divisible by p.

What remains at this point is the case where n is a p-power and a is a central p-element. If $a = 1$, we are done by Theorem 4 and otherwise we note that each solution to the equation $x^n = a$ lies in a unique cyclic subgroup of order $no(a)$. Also, every such subgroup contains precisely n solutions, and the result follows.

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On a Problem of Stein Concerning Infinite Covers

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Let $(m:a)$ denote the arithmetic progression of all integers congruent to the integer a modulo m; m is called its **modulus**. In [1] the system

$$\{(3^i : 2\cdot 3^{i-1}) : i \geq 1\} \cup \{(3^i 2^j : 3^{i-1} + 2^{j-1}3^i) : i \geq 1, j \geq 1\} \tag{1}$$

is claimed to settle a problem of Stein [2] by possessing various properties, one of which is that every integer is included in at least one arithmetic progression in the system. Actually it is not hard to see that none of the arithmetic progressions in (1) contain 0 or any power of 3. In the present paper the system (1) will be modified so as to provide a correct example of the type requested by Stein.

A system $\{(m_i : a_i) : i \in S\}$ is called a **cover** in case its union is the set of all integers, and **exact** if its sets are pairwise disjoint. The system is **incongruent** if $m_i \neq m_j$ whenever $i \neq j$, and **infinite** if S is infinite. Stein's problem is to find an infinite incongruent exact cover $\{(m_i : a_i) : i \in S\}$ such that

$$\sum_{i \in S} m_i^{-1} = 1$$

and $\{m_i : i \in S\} \neq \{2^j : j \geq 1\}$.

The following theorem is proved in [1].

Theorem 1. *Let $\{(m_i : a_i) : i \in S\}$ and $\{(n_j : b_j) : j \in T\}$ be exact covers, and suppose $s \in S$. Then*

$$\{(m_i : a_i) : i \in S, i \neq s\} \cup \{(n_j m_s : a_s + b_j m_s) : j \in T\}$$

is an exact cover.

Let p be an integer greater than 1. The error in [1] comes in the assumption that if we "imagine k to be infinite" in the finite exact cover

$$\{(p^i : tp^{i-1}) : 1 \leq i \leq k, 0 < t < p\} \cup \{(p^k : p^k)\},$$

then the resulting system

$$\{(p^i : tp^{i-1}) : i \geq 1, 0 < t < p\} \tag{2}$$

is an infinite exact cover. Actually 0 is in no arithmetic progression of (2). We can modify (2) to get an exact cover with $p-1$ arithmetic progressions with exact modulus p^i by using Stein's trick of recursively covering an integer with smallest absolute value among those still uncovered. (It should be pointed out that Stein never claims that $\{(2^i : 2^{i-1}) : i \geq 1\}$ is a cover.)

Theorem 2. *Suppose p is an integer greater than* 1. *Define*

$$a_{it} = \begin{cases} \dfrac{1 + (2t - p)p^{i-1}}{2} & \textit{if } p \textit{ is odd}, \\[2ex] \dfrac{p + (p^2 - 2(p+1)t)(-p)^{i-1}}{2(p+1)} & \textit{if } p \textit{ is even}. \end{cases}$$

Then $\{(p^i : a_{it}): i \geq 1, 0 < t < p\}$ *is an exact cover with*

$$\sum \frac{1}{m_i} = \sum_{i \geq 1} \frac{p-1}{p^i} = 1.$$

Proof: It is easily checked that the numbers a_{it} are integers, using the fact that $p \equiv -1 (\operatorname{mod} p + 1)$ in the case when p is even. To see that the sets $(p^i : a_{it})$ do not overlap, suppose that $(p^i : a_{it}) \cap (p^j : a_{ju})$ is nonempty. If $i \leq j$, this leads to

$$tp^{i-1} \equiv up^{j-1} (\operatorname{mod} p^i),$$
$$t \equiv up^{j-i} (\operatorname{mod} p),$$

for p odd, and

$$-t(-p)^{i-1} \equiv -u(-p)^{j-1} (\operatorname{mod} p^i),$$
$$t \equiv u(-p)^{j-i} (\operatorname{mod} p),$$

for p even. In either case we get a contradiction unless $i = j$, which implies that $t = u$ because $0 < t, u < p$.

Finally we show that the sets $(p^i : a_{it})$, $i \geq 1$, $0 < t < p$, cover the integers. Let x be any integer. First suppose that p is odd. We let $2x - 1 = p^{i-1}y$, where p does not divide y, and note that y is odd. Choose an integer t, $0 < t < p$, such that

$$2t \equiv y (\operatorname{mod} p),$$

so that $y = pk + 2t$, where k is an odd integer. Then we find that

$$a_{it} = x - \frac{p^i(k+1)}{2},$$

and so x is in $(p^i : a_{it})$.

If p is even we define i and y by $(p+1)x - p/2 = p^{i-1}y$, where p does not divide y, and let $(-1)^i y = pk + t$, with $0 < t < p$. Then we compute a_{it}, which we know is an integer, to be

$$x - \frac{\left(\frac{p}{2} - t + k\right)(-p)^i}{p+1},$$

and so x is again in $(p^i : a_{it})$. ■

Taking $p = 2$ and $p = 3$ in Theorem 2 gives the infinite exact covers

$$\left\{\left(2^i : \frac{1 - (-2)^{i-1}}{3}\right) : i \geq 1\right\} \tag{3}$$

and

$$\left\{\left(3^i : \frac{1 - 3^{i-1}}{2}\right) : i \geq 1\right\} \cup \left\{\left(3^i : \frac{1 + 3^{i-1}}{2}\right) : i \geq 1\right\}. \tag{4}$$

If, following [1], we use Theorem 1 to break up each arithmetic progression

$$(m_i : a_i) = \left(3^i : \frac{1 + 3^{i-1}}{2}\right)$$

in the second part of (4) with the system

$$(n_j : b_j) = \left(2^j : \frac{1 - (-2)^{j-1}}{3}\right), j \geq 1,$$

then we get the infinite exact incongruent cover

$$\left\{\left(3^i : \frac{1 - 3^{i-1}}{2}\right) : i \geq 1\right\} \cup \left\{\left(3^i 2^j : \frac{1 + 3^i + (-2)^j 3^{i-1}}{2}\right) : i \geq 1, j \geq 1\right\}.$$

This satisfies the conditions of Stein's problem, since

$$\sum_{i \geq 1, j \geq 0} 3^{-i} 2^{-j} = 1.$$

A question that arises is whether numbers a_{it} simpler than those given in Theorem 2 could have been used by multiplying by a factor of the denominator relatively prime to all the moduli. An example would be to replace system (3) with

$$\left\{\left(2^i : 1 - (-2)^{i-1}\right) : i \geq 1\right\}. \tag{5}$$

In fact no arithmetic progression in system (5) contains the integer 1. This example shows that the following theorem may fail if S is infinite.

Theorem 3. *Suppose $\{(m_i : a_i) : i \in S\}$ is a cover, where S is a finite set, and suppose that $(r, m_i) = 1$ for all $i \in S$. Then $\{(m_i : ra_i) : i \in S\}$ is also a cover.*

Proof: Suppose that $T \subseteq S$. Then by the Chinese remainder theorem (see [**3**, p. 62]) or [**4**, p. 59]) we have that

$$\bigcap_{i \in T} (m_i : a_i) \neq \varnothing \tag{6}$$

if and only if (m_i, m_j) divides $a_i - a_j$ whenever $i, j \in T$ and $i \neq j$, and in the latter case the left side of (6) is an arithmetic progression with modulus $LCM\{m_i : i \in T\}$.

Let $M = LCM\{m_i : i \in S\}$, and for any set Q let $Q^* = Q \cap \{1, 2, \ldots, M\}$, and denote the cardinality of Q by $|Q|$. Since (m_i, m_j) divides $a_i - a_j$ if and only if (m_i, m_j) divides $ra_i - ra_j$, we see that

$$\left|\bigcap_{i \in T} (m_i : a_i)^*\right| = \left|\bigcap_{i \in T} (m_i : ra_i)^*\right|$$

for any subset T of S. Then by the inclusion-exclusion principle

$$\left|\bigcup_{i\in S}(m_i:ra_i)^*\right| = \sum_{k=1}^{\infty}(-1)^{k-1}\sum_{\substack{T\subseteq S\\|T|=k}}\left|\bigcap_{i\in T}(m_i:ra_i)^*\right|$$

$$= \sum_{k=1}^{\infty}(-1)^{k-1}\sum_{\substack{T\subseteq S\\|T|=k}}\left|\bigcap_{i\in T}(m_i:a_i)^*\right| = \left|\bigcup_{i\in S}(m_i:a_i)^*\right| = M,$$

since $\{(m_i:a_i): i\in S\}$ is a cover. Thus the arithmetic progressions in $\{(m_i:ra_i): i\in S\}$ also form a cover. ■

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Remember this, the rule for giving an extempore lecture is—let the mind rest from the subject entirely for an interval preceding the lecture, after the notes are prepared; the thoughts will ferment without your knowing it, and enter into new combinations; but if you keep the mind active upon the subject up to the moment, the subject will not ferment but stupefy.

—*A. De Morgan*

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John Banks is a La Trobe graduate and graduate student while **Jeff Brooks, Grant Cairns, Gary Davis** and **Peter Stacey** are La Trobe staff members with undergraduate degrees from La Trobe, Queensland, Monash and Cambridge and doctorates from La Trobe, Montpellier, Monash and Oxford. Their collective mathematical interests span differential and classical geometry, combinatorial group theory, mathematical education, operator algebras and dynamical systems. The present paper arose from a departmental seminar on Chaotic Dynamical Systems, based on the book by R. L. Devaney.

Jonathan L. King received his Ph.D. in 1984 from Stanford, working in Ergodic Theory with Don Ornstein. After a Fellowship at SUNY at Albany, he spent a year at College Park and two at Berkeley courtesy of an NSF Postdoc. The present article uses a simple combinatorial idea to construct a process with an unusual independence.

John E. Wetzel received his Ph.D. at Stanford University under Professor Halsey L. Royden. He has been on the faculty of the University of Illinois for nearly 30 years. Professor Wetzel's recent research interests and publications have been in the area of classical combinatorial geometry.

Alec Norton was raised in California in the Silicon Valley, near San Francisco Bay. He obtained a B.S. at Harvey Mudd College, an M.A. at Oxford University on a Marshall Scholarship, and a Ph.D. at the University of California at Berkeley in 1987. He is currently working in geometric dynamical systems with support from a National Science Foundation Postdoctoral Research Fellowship.

Raghavan Narasimhan did his undergraduate studies in Madras, India, and received his Ph.D. from the University of Bombay while he was a member of the Tata Institute of Fundamental Research. He was on the faculty at the University of Geneva, Switzerland, and is currently at the University of Chicago. His main research interests are several complex variables and number theory.

The Latest Mersenne Prime

Will they ever stop coming? David Slowinski and Paul Gage, of Cray Research, recently announced the discovery of the latest (and largest) Mersenne prime. The 32nd known Mersenne prime is $2^{756839} - 1$, a number with 227,831 digits. The number was shown prime using a program written by Slowinski and Gage on a Cray-2 computer at Harwell Laboratory in Didcot, England.

Proving a random number of this size is prime would be impossible. (Trial division, for example, would be futile—there are about 10^{113910} primes to divide.) For Mersenne primes, there is the famous Lucas-Lehmer test: $M_p = 2^p - 1$ is prime if and only if M_p divides U_p where $\{U_n\}$ is the sequence of numbers starting with $U_2 = 4$ and defined recursively by $U_n = U_{n-1}^2 - 2$. Raising numbers to powers—even such large powers—is possible with some clever work. Squaring a number with over 200,000 digits is not easy, however. Slowinski and Gage used an algorithm of Schonhage and Strassen that employs the Fast Fourier Transform (in a clever implementation by Dennis Kuba, also of Cray Research). Checking the M_{756839} for primality the first time still required many hours of computer time; rechecking it on a machine with 16 processors required 20 minutes.

Before this, the largest known Mersenne prime was M_{216092}; the next before that is M_{110503} (discovered by Colquitt and Welsh, *Mathematics of Computation*, 56:194, April 1991, pp. 867–870). Are there others in between? No one is sure. The computer at Harwell discovered the new prime after checking only 85 exponents. Slowinski, an old hand at finding Mersenne Primes, says, "We were incredibly lucky." Slowinski seems to have more than his share.

PROBLEMS AND SOLUTIONS

Edited by:
Richard T. Bumby, Fred Kochman and Douglas B. West

Proposed problems should be sent to the MONTHLY PROBLEMS address given on the inside front cover. Please include solutions, relevant references, etc. Three copies are requested.

Solutions of published problems should arrive before September 30, 1992 at the MONTHLY PROBLEMS address given on the inside front cover. Solutions should be typed with double spacing, including the problem number and the solver's name and mailing address. Two copies suffice. A self-addressed postcard or label should be included if an acknowledgement is desired.

*An asterisk (*) after the number of a problem, or part of a problem, indicates that no solution is currently available. Partial solutions will be useful in such cases. Otherwise, the published solution is likely to be based on a solution which is complete and correct. Of course, an elegant partial solution or a method leading to a more general result is always useful and welcome. In addition, references to other appearances of MONTHLY problems or to solutions of these problems in the literature are also solicited.*

PROBLEMS

10211. *Proposed by Herbert S. Wilf, University of Pennsylvania, Philadelphia, PA.*

Choose integers a, b, c, d, and let $K = 2(b^2 + a^2d - abc)$. Show that every member of the sequence defined by $y_0 = a^2$, $y_1 = b^2$ and

$$y_{n+1} = (c^2 - 2d)y_n - d^2y_{n-1} + Kd^n \qquad (n \geqslant 1)$$

is the square of an integer.

10212. *Proposed by Seung-Jin Bang, Seoul, Korea.*

Let $a(n)$ be the integer closest to $\sqrt[3]{n}$. Evaluate $\Sigma_{n=1}^{\infty} a(n)^{-4}$.

10213. *Proposed by P. G. Walsh, University of Waterloo, Waterloo, Ontario, Canada.*

Suppose that x and y are positive integers such that $x + xy$ and $y + xy$ are both squares.

(a) Prove that exactly one of x or y is a square.

(b) Characterize all such pairs of integers x, y.

10214. *Proposed by Stephen Penrice, Emory University, Atlanta, GA.*

For all integers $n > 1$, let $f(n)$ denote the largest real number such that, for any set of non-negative real numbers satisfying $a_1 + \cdots + a_n < f(n)$, the n by n matrix with $a_1, \ldots, a_n$ along the main diagonal and -1 in all other positions is invertible. Show that $f(n)$ is well defined, and obtain an explicit formula for it.

10215. *Proposed by Michael Barr, McGill University, Montreal, Quebec, Canada.*

Let R be an associative ring (not necessarily commutative or possessing a unit element) with no non-zero nilpotent elements. Suppose that r and s are two elements of R such that $r^d = s^d$ and $r^e = s^e$, where d and e are relatively prime positive integers. Show that $r = s$.

10216. *Proposed by G. Bennett, Indiana University, Bloomington, IN.*

Let $A = (a_{i,j})$ be an m by n matrix with integer entries. A set of locations, H, in A is called an "echelon" if, whenever $(k, l) \in H$, $i \leqslant k$ and $j \leqslant l$, one has $(i, j) \in H$. Consider the family of operations

$$\begin{array}{ll} \mathbf{S}_{i,j}: & \text{subtract 1 from } a_{i,j}\text{; add 1 to } a_{i+1,j} \\ \mathbf{E}_{i,j}: & \text{subtract 1 from } a_{i,j}\text{; add 1 to } a_{i,j+1} \\ \mathbf{X}_{i,j}: & \text{subtract 1 from } a_{i,j} \end{array}$$

(for all values of i and j for which the operations can be defined). Show that there is a sequence of these operations reducing A to the zero matrix if and only if $\Sigma\{a_{i,j}: (i, j) \in H\} \geqslant 0$ for every echelon H.

10217. *Proposed by Brian J. Philp, University of Birmingham, Birmington, UK.*

Suppose $\{\alpha_j\}_{j=1}^{\infty}$ is a sequence of complex numbers.
(a) Prove that if $n^{-1}\Sigma_{j=n}^{2n}\alpha_j \to \lambda$ and $n^{-1}\Sigma_{j=n}^{4n}\alpha_j \to 3\lambda$, then $n^{-1}\alpha_n \to 0$.
(b) Is it true that if $n^{-1}\Sigma_{j=n}^{3n}\alpha_j \to 2\lambda$ and $n^{-1}\Sigma_{j=n}^{9n}\alpha_j \to 8\lambda$, then $n^{-1}\alpha_n \to 0$.

10218. *Proposed by David Dwyer, University of Evansville, Evansville, IN.*

For positive real numbers r and positive integers n, put

$$\phi(n, r) = (nr) + (\lfloor nr \rfloor r),$$

where $(x) = x - \lfloor x \rfloor$ denotes the "fractional part" of x.
Find $\{r \in \mathbf{R}^+: \phi(n, r) > 1 \text{ for all } n \in \mathbf{Z}^+\}$.

10219. *Proposed by Alan Horwitz, Penn State University, Media, PA.*

(a) Suppose that the function f is positive on $\mathbf{R}$ and that $f''(x)$ exists for all $x \in \mathbf{R}$. Prove that there exists $x_0 \in \mathbf{R}$ such that the second order Taylor polynomial of f centered at x_0 is also positive on $\mathbf{R}$.
(b)* Let n be an arbitrary even integer, and suppose that f is positive on $\mathbf{R}$ and that $f^{(n)}(x)$ exists for all $x \in \mathbf{R}$. Does there exist $x_0 \in \mathbf{R}$ such that the n-th order Taylor polynomial of f centered at x_0 is also positive on $\mathbf{R}$?

NOTES

(10214) Other occurrences of the determinant of the matrix in this problem have been found in the literature. **(10215)** A element r of a ring is called "nilpotent" if $r^d = 0$ for some positive integer d. **(10216)** The author has provided the following descriptive commentary on the problem: "The reader may find it helpful to think in terms of a rectangular board whose squares are occupied by beans (possibly *alien*). Any bean may move South or East or be eXcised. A negative number of beans may be thought of as *has-beans*, while a bean lying to the North/West of another bean may be viewed as *superior*."

SOLUTIONS

A Very Special Function

E3393 [1990, 528]. *Proposed by Bruce C. Berndt, University of Illinois, Urbana, IL.*

Define a function on $(-1/e, \infty)$ as follows. If $-1/e < x < \infty$, determine the unique number t in $(1/e, \infty)$ such that $x = t \log t$ and then put $\phi(x) = 1/(1 + \log t)$.

Show that $\phi^{(k)}(0) = (-k)^k$ for $k = 1, 2, 3, \ldots$.

Solution I independently by Timothy S. Norfolk, University of Akron, Akron, OH, and John Henry Steelman, Indiana University of Pennsylvania, Indiana, PA. We use the following well-known result for integers $m \geq n \geq 0$, easily proved by induction on n:

$$\sum_{k=0}^{m} (-1)^k k^n \binom{m}{k} = \begin{cases} 0 & \text{if } m > n \\ (-1)^n n! & \text{if } m = n. \end{cases} \tag{1}$$

Now set $u = \log t$ so that $x = ue^u$. Thus,

$$x^k = u^k e^{ku} = \sum_{j=0}^{\infty} \frac{k^j u^{j+k}}{j!} = \sum_{n=k}^{\infty} \frac{k^{n-k} u^n}{(n-k)!}. \tag{2}$$

By setting $0^0 = 1$, employing (2), inverting the order of summation, and using (1), we find for $|x| < 1/e$ that

$$\sum_{k=0}^{\infty} \frac{(-k)^k}{k!} x^k = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{(-k)^k k^{n-k} u^n}{k!(n-k)!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k k^n \binom{n}{k} \frac{u^n}{n!}$$
$$= \sum_{n=0}^{\infty} (-1)^n u^n = \frac{1}{1+u} = \phi(x).$$

By Taylor's Theorem, we have $\phi^{(k)}(0) = (-k)^k$, as desired.

Solution II by N. J. Fine, Deerfield Beach, FL. Set $0^0 = 1$, and let $f(x) = \sum_{k=0}^{\infty}((-k)^k/k!)x^k$. For x sufficiently small, we have

$$f(x) = \sum_{k,n=0}^{\infty} \frac{(-n)^k}{k!} x^k \frac{1}{2\pi i} \int_{|z|=2} \frac{dz}{z^{n-k+1}}$$

$$= \frac{1}{2\pi i} \int_{|z|=2} \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \sum_{k=0}^{\infty} \frac{(-nzx)^k}{k!} dz$$

$$= \frac{1}{2\pi i} \int_{|z|=2} \sum_{n=0}^{\infty} \frac{e^{-nzx}}{z^{n+1}} dz = \frac{1}{2\pi i} \int_{|z|=2} \frac{dz}{z - e^{-xz}}.$$

For $|x|$ sufficiently small, the integrand has a simple pole at $z = z_0$, where $z_0 = e^{-xz_0}$, $|z_0| < 2$, and z_0 is unique. The residue at z_0 is $1/(1 + xe^{-xz_0})$. If we set $t' = 1/z_0$, we find $1/t' = e^{-x/t'}$, which means that in fact $t' = t$, and then the residue theorem implies $f(x) = \phi(x)$. By the definition of f and Taylor's Theorem, we have $\phi^{(k)}(0) = (-k)^k$.

Solution III independently by S. L. Paveri-Fontana, Università di Milano, Milan, Italy, and Richard Stong, University of California, Los Angeles, CA. Recall the Lagrange inversion formula (e.g., E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (4th ed., Cambridge University Press, 1962), p. 133).

If f and F are holomorphic in a neighborhood of the point a, $F(a) \neq 0$, and x is given by $x = (\zeta - a)/F(\zeta)$ for ζ near a, then for $|x|$ small we have

$$f(\zeta) = f(a) = \sum_{k=1}^{\infty} \frac{x^k}{k!} \frac{d^{k-1}}{da^{k-1}} \left[f'(a)\{F(a)\}^k \right].$$

If we set $\zeta = \log t$, then $\zeta = xe^{-\zeta}$, which implies that ζ satisfies the required condition when $F(\zeta) = e^{-\zeta}$ and $a = 0$. Applying the inversion formula with $f(\zeta) = e^{\zeta} = t$, we find that for $|x|$ small

$$t = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}(-(k-1))^{k-1} = 1 + x + \sum_{k=1}^{\infty} \frac{(-k)^k}{(k+1)!} x^{k+1}.$$

Since $\phi(x) = 1/(1 + \log t) = dt/dx$, differentiating this sum yields $\phi(x) = 1 + \sum_{k=1}^{\infty}(-k)^k x^k/k!$, from which we conclude $\phi^{(k)}(0) = (-k)^k$.

Editorial comment. Solvers exhibited a wide variety of solutions. Formulas for the derivatives of composite functions, further variants of the Lagrange inversion formula, other applications of complex analysis, and differential equations satisfied by $\phi(x)$ were employed by readers to obtain other solutions. The University of South Alabama Problem Group noted that this problem is a variant of former *Monthly* problems E2828 [1980, 203; 1981, 445] and 6387 [1982, 338; 1983, 711]. Kee-Wai Lau observed that the result follows from theorems in the proposer's book, *Ramanujan's Notebooks, Part I* (Springer-Verlag, 1985), p. 81 (equation 17.4) and p. 83 (example 1).

Solved also by U. Abel (Germany), J. Anglesio (France), J. Braselton, E. A. Herman, J. Huntley & D. Tepper, H. Kappus (Switzerland), I. Kastanas, K.-W. Lau (Hong Kong), O. P. Lossers (Netherlands), J. McHugh, J. S. Sumner, University of South Alabama Problem Group, and the proposer.

Polygons inscribed in the Unit circle

6635 [1990, 535]. *Proposed by J. Michael Steele, Princeton University, Princeton, NJ.*

Suppose $\theta_1, \theta_2, \theta_3, \ldots$ are independent random variables each uniformly distributed on $[0, 2\pi]$. Let $\theta_{n:1}, \theta_{n:2}, \ldots, \theta_{n:n}$ denote the order statistics for $\{\theta_i\colon 1 \le i \le n\}$, i.e., $\theta_{n:1}, \theta_{n:2}, \ldots, \theta_{n:n}$ are the numbers $\theta_1, \theta_2, \ldots, \theta_n$ arranged so that $\theta_{n:1} \le \theta_{n:2} \le \cdots \le \theta_{n:n}$. Adopting the convention that $\theta_{n:n+1} = \theta_{n:1}$, we put

$$S_n = \sum_{i=1}^{n} \{\cos\theta_{n:i+1} + \cos\theta_{n:i}\}\{\sin\theta_{n:i+1} - \sin\theta_{n:i}\}$$
$$= \sum_{i=1}^{n} \sin(\theta_{n:i+1} - \theta_{n:i})$$

so that S_n is twice the area of the polygon with vertices

$$(\cos\theta_{n:i}, \sin\theta_{n:i}), \qquad 1 \le i \le n.$$

Show that

(i) $0 \le S_1 \le S_2 \le S_3 \le S_4 \le \cdots \le 2\pi$, and

(ii) $S_n \to 2\pi$ as $n \to \infty$, with probability one.

Solution by Richard Stong, Department of Mathematics, University of California at Los Angeles. Part (i) is clear since the polygon whose area is given by $S_n/2$ is contained in the polygon whose area is given by $S_{n+1}/2$ and all are contained in the unit disc.

To obtain part (ii), for any positive integers n and N let $A_{n,N}$ be the set of all sequences $\theta = \{\theta_1, \theta_2, \theta_3, \ldots\}$ for which one of the intervals $[2\pi k/N, 2\pi(k+1)/N]$ contains none of $\theta_1, \theta_2, \ldots, \theta_n$. Clearly $\text{Prob}\{A_{n,N}\} \le N(1 - 1/N)^n$. The upper bound goes to zero as n grows; therefore, with probability one, for any given N there is a sufficiently large n with θ not belonging to $A_{n,N}$. However, if θ is not in $A_{n,N}$, where $N \ge 4$, then no two consecutive $\theta_{n:i}$ differ by more than $4\pi/N$. In this case, since $(\sin x)/x$ is decreasing on $(0, \pi)$,

$$\sin(\theta_{n:i+1} - \theta_{n:i}) \ge (\theta_{n:i+1} - \theta_{n:i})(4\pi/N)^{-1}\sin(4\pi/N)$$

and, by addition,

$$S_n \ge (N/2)\sin(4\pi/N).$$

Thus, with probability one, S_n eventually exceeds $(N/2)\sin(4\pi/N)$ for each $N \ge 4$, and the conclusion follows, since $\lim_{N\to\infty}(N/2)\sin(4\pi/N) = 2\pi$.

Editorial comment. Several solvers observed that the uniform distribution of the θ_i can be replaced by any distribution whose support is the entire interval $[0, 2\pi]$. Others noted that the two assertions of the problem hold for any (deterministic) sequence $\{\theta_i\}$ that is dense in $[0, 2\pi]$.

Solved also by André Adler, Wolfgang J. Bühler (Germany), David Callan, R. J. Chapman (England), Ellen Hertz, John H. Lindsey II, O. P. Lossers (The Netherlands), Kenneth Schilling, and the proposer.

An Application of Philip Hall's Marriage Theorem

E3399 [1990, 611]. *Proposed by Robert W. Floyd, Stanford University, CA.*

On a certain island with n married couples, every couple consists of a hunter and a farmer. The Ministry of Hunting has divided the island into n hunting ranges of equal size A. Working independently, the Ministry of Agriculture has divided the island into n farming ranges of equal size A. The Ministry of Marriage insists that the hunting range and the farming range assigned to each couple be close together. To everyone's surprise the Ministry of Assignments is able to allot the hunting ranges and the farming ranges to the various couples in such a way that each couple's two ranges overlap. The Ministry of Religion declares this to be a miracle.

(a) Show that in fact no miracle is involved by proving the existence of a positive number δ_n depending only on n with the following property: No matter how the Ministries of Hunting and Agriculture have made their divisions, it is possible for the Ministry of Assignments to make its choices in such a way that each couple's two ranges overlap by an area at least $\delta_n A$.

(b) Determine the best possible value of δ_n.

Solution by Daniel Velleman, Amherst College, Amherst, MA.

(a) Let $\delta_n = 4/(n+1)^2$ if n is odd and $4/(n(n+2))$ if n is even. Note that δ_n is the minimum value of $1/(k(n-k+1))$ for $k \in \{1, 2, \ldots, n\}$. To see that this choice of δ_n works, consider any set of hunting ranges $V_H = \{H_1, H_2, \ldots, H_n\}$ and farming ranges $V_F = \{F_1, F_2, \ldots, F_n\}$. Define a bipartite graph with vertex set $V_H \cup V_F$ with an edge between H_i and F_j if and only if the area of $H_i \cap F_j$ is at least $\delta_n A$. The task of the Ministry of Assignments is to find a perfect matching in this graph.

If there were no such matching, then, by Hall's well-known marriage theorem, there would be a set $X \subseteq V_H$ containing more elements than its neighborhood $N(X) = \{F_j | F_j \text{ is adjacent to some } H_i \in X\}$. Assume without loss of generality that $X = \{H_1, H_2, \ldots, H_k\}$ and $N(X) \subseteq \{F_1, F_2, \ldots, F_{k-1}\}$ for some $k \leq n$. Then $\bigcup_{i=1}^{k} H_i \setminus \bigcup_{j=1}^{k-1} F_j$ has area at least $kA - (k-1)A = A$, so $H_i \setminus \bigcup_{j=1}^{k-1} F_j$ has area at least A/k for some $i \in \{1, 2, \ldots, k\}$. This region is completely covered by $F_k, F_{k+1}, \ldots, F_n$, so $H_i \cap F_j$ has area at least $A/(k(n-k+1)) \geq \delta_n A$ for some $j \in \{k, k+1, \ldots, n\}$. But then there would be an edge between H_i and F_j, contradicting the fact that $F_j \notin N(X)$. Thus there must be a perfect matching and the Ministry of Assignments can do its job.

(b) The value given for δ_n in part a) is best possible. To see this, suppose δ_n were greater than $1/(k(n-k+1))$ for some $k \in \{1, 2, \ldots, n\}$. Choose the hunting ranges $H_1, H_2, \ldots, H_n$ however you please. Choose the farming ranges $F_1, F_2, \ldots, F_{k-1}$ so that they are completely contained in $\bigcup_{i=1}^{k} H_i$ and $H_i \setminus \bigcup_{j=1}^{k-1} F_j$ has area A/k for $1 \leq i \leq k$. Now choose the remaining farming ranges so that $H_i \cap F_j$ has area $A/(k(n-k+1))$ for $1 \leq i \leq k$ and $k \leq j \leq n$. Then in the associated bipartite graph, there is no edge from H_i to F_j for $1 \leq i \leq k$ and $k \leq j \leq n$. Applying the marriage theorem to the set $X = \{H_1, H_2, \ldots, H_k\}$, we see that the Ministry of Assignments cannot do its job.

Editorial note. Several solvers quoted a result of Marcus and Ree (*Quart. J. Math Oxford* (2) 10 (1959) 295–302), who showed that any $n \times n$ doubly stochastic matrix has a diagonal all of whose entries are at least $\delta_n = 1/\lfloor (n+1)^2/4 \rfloor$. This result is equivalent to the result in part a) above.

Solved also by J. Balogh (student, Hungary), K. Bozeman, D. Callan, R. J. Chapman (England), S. Degenhardt (student), B. Heiligers & 0. Krafft (Germany), R. High, A. Kresch (student), J. H. Lindsey II, O. P. Lossers (The Netherlands), G. Rote (Austria), E. Schmeichel, J. Schiermeyer (Germany), R. Stong, J. S. Sumner & K. L. Dove, J. T. Ward, J. M. Weinstein, National Security

Agency Problems Group, Western Maryland College Problems Group, and the proposer. Two incorrect solutions were received.

The Ballot Problem in Disguise

E3402 [1990, 612]. *Proposed by Joseph Kupka, Monash University, Clayton, Victoria, Austrailia.*

A population consisting of particles of various types evolves in time according to the following rule: Each particle is deemed to belong to a unique generation $n = 1, 2, 3, \ldots$. Each particle produces a certain number of "offspring" particles, and, for each n, generation $n + 1$ comprises the totality of offspring of the particles in generation n. A particle of type $i = 0, 1, 2, \ldots$ produces exactly $i + 2$ offspring, one each of types $0, 1, 2, \ldots, i + 1$. Let $N(n, k)$ denote the number of particles in the nth generation when the first generation consists of a single particle of type k. Find a formula for $N(n, k)$.

Solution by José Heber Nieto, Universidad del Zulia, Maracaibo, Venezuela. The answer is

$$N(n,k) = \frac{k+2}{2n+k}\binom{2n+k}{n-1} = \binom{2n+k-1}{n-1} - \binom{2n+k-1}{n-2}.$$

This formula, found heuristically, is easily proved by induction on n. Obviously it holds for $n = 1$, since $N(1, k) = 1$. Assume that the formula holds for n and all $k \geq 0$. Since the numbers N clearly satisfy the recurrence $N(n + 1, k) = \sum_{i=0}^{k+1} N(n, i)$, we have

$$N(n+1,k) = \sum_{i=0}^{k+1}\left\{\binom{2n+i-1}{n-1} - \binom{2n+i-1}{n-2}\right\}.$$

Using the well-known identity

$$\sum_{j=a}^{b}\binom{j}{m} = \binom{b+1}{m+1} - \binom{a}{m+1},$$

we finally obtain

$$\begin{aligned} N(n+1,k) &= \binom{2n+k+1}{n} - \binom{2n-1}{n} - \binom{2n+k+1}{n-1} + \binom{2n-1}{n-1} \\ &= \binom{2n+k+1}{n} - \binom{2n+k+1}{n-1} \\ &= \binom{2(n+1)+k-1}{(n+1)-1} - \binom{2(n+1)+k-1}{(n+1)-2}. \end{aligned}$$

Editorial comment. Almost every solver proceeded by induction ("guessing" the solution) or by employing binomial coefficient identities. Several solvers noted the occurrence of the Catalan numbers when $k = 0$, and there is indeed a direct transformation to the famous "ballot problem." There is one particle in generation n for each sequence of integers $a_1, \ldots, a_n$ such that $a_1 = k$ and $0 \leq a_i \leq a_{i-1} + 1$ for $i \geq 2$. If we set $b_i = k + i - 1 - a_i$, then we have $b_1 = 0$ and $b_{i-1} \leq b_i \leq k + i - 1$ for $i \geq 2$. Each such sequence $b_1, \ldots, b_n$ corresponds to an up/right lattice path from $(1, 0)$ to $(n, n + k)$ that does not go above the line $y = x + k$, given by taking the step from $x = i - 1$ to $x = i$ at $y = b_i$ for $i \geq 2$. Without the con-

straint, there are $\binom{2n+k-1}{n-1}$ such paths. Each bad path reaches $y = x + k + 1$. If we reflect the initial portion of the path (up to where this first happens) about the line $y = x + k + 1$, we obtain a path from $(-k-1, k+2)$ to $(n, n+k)$, and each such path corresponds to a unique bad path. Hence there are $\binom{2n+k-1}{n-2}$ bad paths in the original set, yielding the desired formula. The ballot problem (in which the winning candidate never leads by more than the final amount) was solved by essentially this method in D. André, "Solution directe du problème résolu par M. Bertrand", *C. R. Acad. Sci. Paris* 105 (1887), 436–437. J. C. Binz suggested a generalization in which a particle of type i produces particle of types $1, \dots, i+m$; the particles in the nth generation correspond to paths from $(1,0)$ to $(n, nm+k)$ that do not go above the line $y = mx + k$, but these seem to be difficult to count, because the reflection method does not provide a simple formula.

Solved also by B. D. Beasley, J. C. Binz (Switzerland), D. Callan, J. L. Drost, K. Ford (student), J. W. Grossman, J. H. Lindsey II, O. P. Lossers (The Netherlands), A. Raws III, J. H. Steelman, R. Stong, J. S. Sumner & K. L. Dove, J. T. Ward, Anchorage Math Solutions Group, National Security Agency Problems Group, Western Maryland College Problems Group, and the proposer.

Building Hollow Cubes from m-Bricks

E3405 [1990, 847]. *Proposed by Charles Vanden Eynden, Illinois State University, Normal, IL.*

Suppose m, n are integers with $1 < m \le n$. Let S_n be the thick-walled box obtained by removing the cube

$$\{(x, y, z) : 1 < x, y, z < n-1\}$$

from the cube

$$\{(x, y, z) : 0 \le x, y, z \le n\}.$$

For which pairs of integers m, n can S_n be constructed with $m \times 1 \times 1$ blocks?

Solution by Robin J. Chapman, University of Exeter, United Kingdom. We can construct S_n with $m \times 1 \times 1$ blocks if and only if $m = 2$ and n is even.

Suppose first that $m = 2$ and n is even. We construct S_n with $2 \times 1 \times 1$ blocks as follows. The top and bottom "layers" of S_n are $n \times n \times 1$ blocks and can be constructed by building $n \times 1 \times 1$ blocks from $2 \times 1 \times 1$ blocks and then pasting them together. Each remaining layer is an $n \times n \times 1$ block with the central $(n-2) \times (n-2) \times 1$ block removed. For this we can build a pair of opposite edges as $n \times 1 \times 1$ blocks and fill the gaps with two $(n-2) \times 1 \times 1$ blocks.

Now suppose we can build S_n from $m \times 1 \times 1$ blocks. Denote each of the $N = n^3 - (n-2)^3$ unit cubes comprising S_n by the coordinates of the corner farthest from the origin. Partition the unit cubes into m color classes, where the cube in position (a, b, c) has color $k \equiv (a + b + c) \bmod m$. Let R be the rotation of 3-space taking (x, y, z) to (z, x, y). Clearly R permutes the cubes in S_n and preserves their colors. The only cubes of S_n fixed by R are those at $(1,1,1)$ and (n, n, n). Since R has period 3, we deduce $N \equiv 2 \bmod 3$.

Let N_i be the number of cubes having color i. Since any $m \times 1 \times 1$ block has one cube of each color, a construction of S_n from $m \times 1 \times 1$ blocks requires $N_1 = N_2 = \cdots = N_m$, and hence $N = mM$, where $M = N_i$. Unless m is 2 and n is even, we can choose a color k distinct from the colors at $(1,1,1)$ and (n, n, n)

(when $m = 2$ and n is odd, these cubes both have color 1, and we put $k = 2$). Now the rotation R fixes no cube of color k, so $M = N_k$ is divisible by 3. This contradicts the fact that M divides N and N is not divisible by 3, leaving only the case $m = 2$ and n even.

Editorial comment. Robert G. High derived several extensions and generalizations. For example, he proved that a d-dimensional hollow hypercube of side n can be tiled by $m \times 1 \times \cdots \times 1$ blocks if and only if $m = 2$ and n is even, or d is even and m divides $n - 1$.

Solved also by M. Gerstell, R. G. High, M. E. Kuczma (Poland), O. P. Lossers (The Netherlands), A. Nijenhuis, C. Soland (Switzerland), R. Stong, Anchorage Math Solutions Group, National Security Agency Problems Group, and the proposer.

The Antipedal Triangle in Perspective

E3407 [1990, 848]. *Proposed by Clark Kimberling, University of Evansville, Evansville, IN.*

Suppose ABC is a given triangle. Prove the existence of a triangle that is in perspective with every antipedal triangle of ABC. (If P is a point not collinear with any two of the points A, B, C, the lines through A, B, C perpendicular to PA, PB, PC respectively form a triangle called the *antipedal triangle* of P with respect to ABC. Two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are said to be *in perspective* if the lines A_1A_2, B_1B_2, C_1C_2 are concurrent.)

Solution by I. G. Macdonald, Queen Mary College, London, England. Let S be the circle through A, B, C, and let L, M, N be the points on S such that AL, BM, CN are diameters. Then LMN is the required triangle.

Let P be any point in the plane, and let l, m, n be the lines through A, B, C perpendicular to PA, PB, PC, respectively. Let l, m, n meet the circle S again at D, E, F, respectively. The angle ADL is a right angle, so DL is perpendicular to l and hence parallel to PA, and the parallel lines AP, DL are equidistant from the center O of S. Hence DL passes through the point Q collinear with O and P such that $PO = OQ$. Similarly, the lines EM, FN pass through Q, and therefore $QD \cdot QL = QE \cdot QM = QF \cdot QN$, a quantity we call r^2. (Note that r will be real if and only if Q (or P) lies outside S.)

It follows that L, M, N are the poles of l, m, n, respectively, with respect to the real or imaginary circle with center Q and radius r. The conclusion that the triangles LMN and lmn (the antipedal triangle of P) are in perspective follows from the following fact: two triangles in a plane are in perspective if and only if there is a conic with respect to which they are reciprocal, meaning that the sides of each triangle are the polars of the vertices of the other (see, for example, H. F. Baker, *Introduction to Plane Geometry*, Cambridge University Press (1943), 191).

Editorial comment. The Anchorage Math Solutions Group and Jordi Dou noted that the given triangle itself is such a triangle, for trivial reasons.

Solved also by J. Anglesio (France), J. Dou (Spain), Anchorage Math Solutions Group, and the proposer. One incorrect solution was received.

A Semigroup Related to Farey Sequences

E3413 [1990, 917]. *Proposed by Robert McNaughton, Rensselaer Polytechnic Institute, Troy, NY.*

Suppose n_1, n_2 are positive integers and m_1, m_2 are integers such that $m_1/n_1 < m_2/n_2$. Let V be the semigroup under componèntwise addition formed by all pairs (p, q), where p and q are integers, $q > 0$, and $m_1/n_1 \le p/q \le m_2/n_2$. Call a set B a generating set for V if $B \subseteq V$ and every element of V is equal to a finite sum of elements of B, repetitions allowed.

(a) Prove that V has a generating set B in which $q \le \max(n_1, n_2)$ for every $(p, q) \in B$.

(b) Prove that V has a unique minimal generating set, i.e., one contained in every other generating set.

Solution by Allan Pedersen, Soborg, Denmark. We first note that $B \subseteq V$ is generating if it generates all $(p, q) \in V$ with p/q in reduced terms; representing sums for non-reduced (p, q) are obtained by replication.

Also, suppose p/q is a reduced fraction with $q > 1$. Let a/b and c/d be respectively, the largest and smallest reduced fractions having positive denominators smaller than q such that $a/b < p/q < c/d$. Then $p = a + c$ and $q = b + d$. This follows from the "mediant property" of Farey sequences of proper reduced fractions. (Cf. G. H. Hardy & E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, 1979, §§ 3.1–3.8, or I. Niven, H. S. Zuckerman, and H. L. Montgomery, *An Introduction to the Theory of Numbers*, Wiley, 1991, §6.1.)

To prove (a), let $B = \{(r, s) \in V : s \le \max(n_1, n_2)\}$. By induction on q, we prove that every $(p, q) \in V$ is a finite sum of elements of B. Already B includes all such pairs with $q \le \max(n_1, n_2)$. For the inductive step, form a/b and c/d from p/q as described above. Since $m_1/n_1 \le a/b < p/q < c/d \le m_2/n_2$, we have (a, b), $(c, d) \in V$. By the observation in the preceding paragraph $(p, q) = (a, b) + (c, d)$. By the induction hypothesis, (a, b) and (c, d) are finite sums of elements of B, and hence so is (p, q).

To prove (b), let B' be the set of those $(p, q) \in V$ that are not sums of two or more elements of V. Clearly B' is contained in any generating set, and B' contains the elements $(p, q) \in V$ for which q is minimal. Furthermore, if $(p, q) \in V - B'$, then $(p, q) = (a, b) + (c, d)$ with $b,\ d < q$. Hence it follows by induction that B' generates V.

Solved also by D. Callan, R. High, K. S. Kedlaya (student), R. Stong, and the proposer.

The Sequence $\{(\sin n)^n\}$ Is Dense in $(-1, 1)$.

6645 [1990, 930]. *Proposed by Robert Kreczner, University of Wisconsin-Stevens Point.*

(a) Show that the sequence $\{(\sin n)^n\}_{n=1}^{\infty}$ is dense in $(0, 1)$.

(b)* Show that the same sequence is dense in $(-1, 0)$.

Solution by Richard Stong, Department of Mathematics, University of California at Los Angeles. We show that the sequence is dense in $(-1, 1)$.

Lemma. *There are arbitrarily large pairs of odd numbers* (p, q) *such that*

$$\left|\frac{p}{q} - \frac{\pi}{2}\right| < \frac{2}{q^2}.$$

Proof: Let A_n/B_n be the nth convergent of the continued fraction for $\pi/2$. Then $A_{n+1}B_n - A_nB_{n+1} = \pm 1$ and

$$\left|\frac{A_n}{B_n} - \frac{\pi}{2}\right| < \frac{1}{B_n^2}.$$

Clearly consecutive A_n's (or consecutive B_n's) cannot both be even. Hence we are done unless from some point on the A_n's and B_n's alternate even and odd values "out of phase with each other". But then take $p = A_n + A_{n+1}$ and $q = B_n + B_{n+1}$, which are both odd. Now $\pi/2$ lies between A_{n+2}/B_{n+2} and A_{n+1}/B_{n+1} and, hence, lies between p/q and A_{n+1}/B_{n+1}. Thus,

$$\left|\frac{p}{q} - \frac{\pi}{2}\right| < \left|\frac{p}{q} - \frac{A_{n+1}}{B_{n+1}}\right| = \frac{1}{qB_{n+1}} < \frac{2}{q^2}.$$

Now consider the sequence $(\sin rp)^{rp}$ for r odd. Write $p = (1/2)q\pi \pm \varepsilon$, where

$$0 \le \varepsilon < \frac{2}{q} < \frac{4}{p}.$$

Then $rp = (1/2)rq\pi \pm r\varepsilon$, so that

$$(\sin rp)^{rp} = \begin{cases} (\cos r\varepsilon)^{rp} & \text{if } rq \equiv 1 \bmod 4 \\ -(\cos r\varepsilon)^{rp} & \text{if } rq \equiv 3 \bmod 4. \end{cases}$$

Suppose x is a positive integer such that $x\varepsilon < \pi/3$. The sequence $(\cos r\varepsilon)^{rp}$ with $r = 1, 2, \ldots, x$ is monotonically decreasing from $(\cos \varepsilon)^p \ge (\cos(4/p))^p$, which is near 1 for p large, to $(\cos x\varepsilon)^{xp}$ which is near zero for p large, provided $x^3\varepsilon^2 p$ is large. (The latter fact follows from the fact that $\cos t \ge e^{-t^2/2}$ for $0 \le t \le \pi/3$.) A precise choice of x will be postponed until the end of the calculation. To show that $(\sin n)^n$ is dense in $(-1, 1)$ it is enough to show that the maximum difference between successive terms in this sequence goes to zero as p goes to infinity.

There is a positive constant c_1 such that for all $y \in [0, \pi/3]$ we have

$$\frac{-y^2}{2} - c_1 y^4 \le \ln \cos y \le \frac{-y^2}{2}.$$

Thus, for $1 \le r \le x$ we have

$$\exp\left\{-\frac{r^3\varepsilon^2 p}{2} - c_1 r^5\varepsilon^4 p\right\} \le (\cos r\varepsilon)^{rp} \le \exp\left\{-\frac{r^3\varepsilon^2 p}{2}\right\}.$$

If $x^5\varepsilon^4 p \le 1$, then there is a constant C with

$$\left|(\cos r\varepsilon)^{rp} - \exp\left\{-\frac{r^3\varepsilon^3 p}{2}\right\}\right| \le Cr^5\varepsilon^4 p \exp\left\{-\frac{r^3\varepsilon^2 p}{2}\right\}.$$

The right-hand side is easy to maximize as a function of r; the maximum occurs for $r^3 = c_2\varepsilon^{-2}p^{-1}$ for some constant c_2. Hence,

$$\left|(\cos r\varepsilon)^{rp} - \exp\left\{-\frac{r^3\varepsilon^2 p}{2}\right\}\right| \le C'\varepsilon^{2/3}p^{-2/3} \le C''p^{-4/3}.$$

Thus it is enough to show that the maximum difference between two consecutive terms in the sequence $f(r) = \exp\{-r^3\varepsilon^2 p/2\}$, $1 \le r \le x$ goes to zero for p large. Since

$$f'(r) = -\tfrac{3}{2}r^2\varepsilon^2 p \exp\{-r^3\varepsilon^2 p/2\} \le C'\varepsilon^{2/3}p^{1/3} \le C''p^{-1/3}$$

(the maximum of $f'(r)$ occurs for $r^3 = c_3\varepsilon^{-2}p^{-1}$ for some constant c_3), this is clear.

The above demonstration hinges upon being able to make an appropriate choice of x. The three conditions imposed on x are $x^3\varepsilon^2 p$ large, $x^5\varepsilon^4 p \le 1$, and $x\varepsilon < \pi/3$. These conditions are, however, easy to achieve. For example, take $x = \lfloor \varepsilon^{-3/4}p^{-1/4} \rfloor$; then all three conditions hold for p large.

Solved also by Shaw Chen (student). Part (a) was solved by Matthew Cook (student) and Harold G. Diamond.

Collaborating editors: *Paul T. Bateman, Bruce C. Berndt, Duane M. Broline, Barry W. Brunson, Frank S. Cater, Gulbank D. Chakerian, Michael A. Filaseta, Ira M. Gessel, Richard A. Gibbs, Douglas A. Hensley, John R. Isbell, Murray Klamkin, Daniel J. Kleitman, Frederick W. Luttmann, Marvin Marcus, Frank B. Miles, Richard Pfiefer, Stephen L. Portnoy, J. O. Shallit, John Henry Steelman, Kenneth B. Stolarsky, Daniel Ullman, and Edward T. H. Wang*

> The great mathematician, like the great poet or naturalist or great administrator, is born. My contention shall be that where the mathematic endowment is found, there will usually be found associated with it, as essential implications in it, other endowments in generous measure, and that the appeal of the science is to the whole mind, direct no doubt to the central powers of thought, but indirectly through sympathy to all, rousing, enlarging, developing, emancipating all, so that the faculties of will, of intellect and feeling learn to respond, each in its appropriate order and degree, like the parts of an orchestra to the "urge and ardor" of its leader and lord.
>
> —*C. J. Keyser*

UNSOLVED PROBLEMS

Edited by **Richard Guy**

In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial or related results. Typescripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Alberta. Canada T2N 1N4.*

A Pseudorandom Sequence—How Random Is It?

Andrzej Ehrenfeucht and Jan Mycielski

Let $\varepsilon_1, \varepsilon_2, \dots$ be a sequence of 0's and 1's. Suppose that we know $\varepsilon_1, \dots, \varepsilon_n$ and are asked to predict ε_{n+1}. A very simple way, which we will call the method M, is the following. Find the longest final segment $\varepsilon_j, \varepsilon_{j+1}, \dots, \varepsilon_n$ which occurs earlier in $\varepsilon_1, \dots, \varepsilon_n$. So $n - j$ is maximal such that $(\varepsilon_j, \varepsilon_{j+1}, \dots, \varepsilon_n) = (\varepsilon_{j-i}, \varepsilon_{j+i+1}, \dots, \varepsilon_{n-i})$ for some $i > 0$. Then find the smallest i (the most recent occurrence) for which this is so and let ε_{n-i+1} be your guess for ε_{n+1}. (Note that if $(\varepsilon_1, \dots, \varepsilon_n) = (\varepsilon, \varepsilon, \dots, \varepsilon, 1 - \varepsilon)$, then $(\varepsilon_j, \dots, \varepsilon_n)$ is empty and $i = 1$. Otherwise $(\varepsilon_j, \dots, \varepsilon_n)$ has length ≥ 1). The method M may seem to be very naive, but more or less refined variants of this method are used by all learning organisms. Perhaps every sensible method of prediction based on experience is equivalent to some kind of coding or description of the past by means of a sequence of 0's and 1's and the method M. Notice that if the sequence $\varepsilon_1, \varepsilon_2, \dots$ is eventually periodic, the predictions by M are eventually faultless.

In this note we do not consider any coding and use M only to produce a certain pseudorandom sequence $\rho_1, \rho_2, \dots$ We put $\rho_1 = 0$ and assume that whenever M predicts ρ_{n+1} to be ε, then in fact $\rho_{n+1} = 1 - \varepsilon$. Thus $\rho_1, \rho_2, \dots$ is characterized by the assumptions that $\rho_1 = 0$ and that M is always wrong. We could say that, from the point of view of M, the sequence $\rho_1, \rho_2, \dots$ is the most unpredictable one. It is easy to find by hand the first 40 values of this sequence:

$$(\rho_1, \rho_2, \dots) = (0,1,0,0,1,1,0,1,0,1,1,1,0,0,0,1,0,0,0,0,1,1,\\ 1,1,0,1,1,0,0,1,0,1,0,0,1,0,0,1,1,1,\dots).$$

Theorem. *Every finite sequence of* 0*'s and* 1*'s occurs as a segment in* $\rho_1, \rho_2, \dots$.

Proof: Assume that this theorem fails. Then there exists a finite sequence which does not occur infinitely many times as a segment of $\rho_1, \rho_2, \dots$. Let $\varepsilon_1, \dots, \varepsilon_k$ be

any such sequence which is the shortest. Then let S be the set of all left extensions of $\varepsilon_1, \ldots, \varepsilon_k$, that is sequences of the form $\eta_1, \ldots, \eta_r, \varepsilon_1, \ldots, \varepsilon_k$, which occur in $\rho_1, \rho_2, \ldots$. So, of course, S is finite. Since $\varepsilon_1, \ldots, \varepsilon_{k-1}$ occurs infinitely many times in $\rho_1, \rho_2, \ldots$, there exists a sequence of the form $\eta_1, \ldots, \eta_s, \varepsilon_1, \ldots, \varepsilon_{k-1}$ which occurs infinitely many times in $\rho_1, \rho_2, \ldots$ and is longer than any sequence in S. Of course, $\eta_1, \ldots, \eta_s, \varepsilon_1, \ldots, \varepsilon_k$ does not occur at all in $\rho_1, \rho_2, \ldots$. Let $\rho_j, \rho_{j+1}, \ldots, \rho_{j+s+k-2}$ and $\rho_{j-i}, \rho_{j-i+1}, \ldots, \rho_{j+i+s+k-2}$ be the first two occurrences of $\eta_1, \ldots, \eta_s, \varepsilon_1, \ldots, \varepsilon_{k-1}$ in $\rho_1, \rho_2, \ldots$. Since the method M never predicts correctly any ρ_n, it does not predict correctly $\rho_{j+s+k-1}$. Hence $\rho_{j+s+k-1} \neq \rho_{j-i+s+k-1}$. Therefore, either $\rho_j, \ldots, \rho_{j+s+k-1}$ or $\rho_{j-i}, \ldots, \rho_{j-i+s+k-1}$ equals $\eta_1, \ldots, \eta_s, \varepsilon_1, \ldots, \varepsilon_k$, which is a contradiction. So the theorem is proved.

Remark. The above theorem remains true if we modify the definition of $\rho_1, \rho_2, \ldots$ initiating it with any finite sequence of 0's and 1's.

Now our problem is how random is the sequence $\rho_1, \rho_2, \ldots$? And the same question can be raised about the modifications mentioned in the remark. Of course, from an algorithmic point of view, they are not random at all since there exist programs for producing them. But, from a statistical point of view, they could be quite random. For example, do they satisfy

$$\frac{\rho_1 + \cdots + \rho_n}{n} \to \frac{1}{2}?$$

The first 1300 values of the sequence, calculated by Walter Taylor.

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Further comments by I. J. Good. A Mycielski sequence can be expected to be flatter than "flat-random" because it is constructed to avoid repeated subsequences to some extent. An appropriate test for this purpose, over *finite* stretches, would be the *serial test*, the correct use of which is explained by Good (1953) and exemplified for the binary expansion of $\sqrt{2}$ by Good and Gover (1967). Since Walter Taylor has already written a program for generating M-sequences it would be easy for him to apply the serial test, and he will presumably thereby corroborate my expectation. Note, however, that the further one goes in the sequence the more one is avoiding longer repeats so the Mycielski sequence is not homogeneous. Meanwhile, I counted by hand the numbers of 1s in each of the 37 rows of length 35 in the printout and obtained a Pearson chi-squared value of only 15.7 with 36 degrees of freedom, corresponding to a P-value of 0.9987 (assuming the asymptotic chi-squared distribution). This supports my conjecture over the first 1295 bits.

A Mycielski sequence could also be called a Gambler's Fallacy sequence. Another class of Gambler's Fallacy sequences can be defined recursively in the following manner: at each stage of the construction choose a digit that will provide a new polybit of length k (a k-bit) where, at that stage, k is small as possible. When this rule does not determine whether a 0 or a 1 should be the next bit, decide by tossing a coin (or by a deterministic rule is preferred). Here is an example: $01\overset{*}{0}0\overset{*}{1}1\overset{*}{1}01\overset{*}{0}1100\overset{*}{0}010\ldots$ where the asterisks indicate the bits that had to be chosen at random. Presumably such a sequence is even more flatter-than-random than a Mycielski sequence.

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LETTERS

Minimal Surfaces

This letter concerns a new and beautiful relationship between the mathematics of minimal surfaces and one of the most fundamental components of biology and is respectfully submitted for publication in *The American Mathematical Monthly*.

It is well known that the minimal surface of the DNA molecule is a helicoid. This helicoid is generated by deforming a catenoid resulting in a helicoid with edges coincident with the double helix typical of a DNA molecule. Alternately, the helicoid can be generated by rotating and translating a line segment L along a perpendicular axis through its midpoint [**1**].

We recently repeated Plateau's minimal surface soap film experiment [**2**] to reproduce the DNA helicoid using Courant's wire model techniques [**3**]. Surprisingly, we found that a simple double helix structure model alone is insufficient by itself to generate a helicoid minimal surface. Instead, a simple catenoid ribbon of soap film runs down the double helix model between the pair of wires. When we added small wires to the model to represent the base pairs of DNA a soap film helicoid was formed immediately and naturally with no further effort on our part —in contrast to Courant's advice "to pierce and to destroy... surfaces" to get the desired surface ([**3**] p. 168). Indeed, on a model that did not have base pair wires on half of it, the catenoid film formed as before; but when it came to the section with the base pair wires, it dramatically twisted to assume the helicoid structure. What was also surprising was when we tried to wash the film off the model in running water, the catenoid film disappeared immediately, but the helicoid film was so robust that it remained intact after repeated washings.

We feel this represents a new discovery previously unreported in the literature of one of those beautiful correspondences that exist between mathematics and nature: *the generator of the minimal surface helicoid is analogous to the Watson–Crick base pairs of DNA*.

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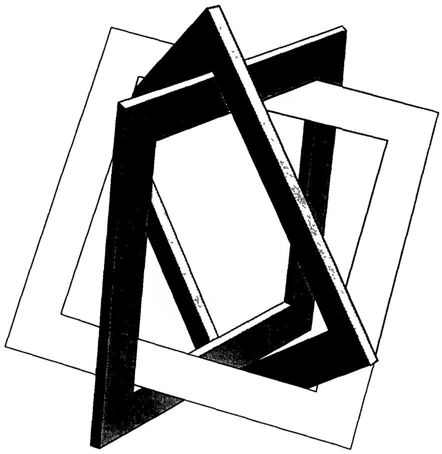
Borromean Circles

The article [**1**] shows that Borromean Circles are impossible. It is interesting therefore to know that Borromean Squares are certainly possible. Here is an illustration of such an object, as made by one of us, John Robinson.



This sculpture is called Creation, since it symbolizes that the whole is greater than the sum of its parts.
Creation by John Robinson. Illustration by Rhiannon Matthias.

Three 5′ high editions of Creation, in plain wood, have been donated by Edition Limitée to the Mathematics Departments at Bangor; the Universidad Autonoma de Barcelona; and the Universidad Zaragoza. The last two were associated with exhibitions of John Robinson sculptures (as in [2]). A 12′ high edition of Creation in redwood has been erected at Aspen, Colorado, as part of the sculpture collection of Robert Heffner III.

You can easily make your own (smaller) version from card. John Robinson has also made maquettes based on triangles, on lozenges, and on ellipses.

An exhibition of the sculptures at UCLA is under discussion.

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REVIEWS

Edited by Darrell Haile

Measure, Topology, and Fractal Geometry. By Gerald A. Edgar. Springer-Verlag, New York, 1990, xiii + 230 pp.

Alec Norton

Not long ago I helped judge the mathematics section of a statewide science fair for high school students. To qualify, a project first had to have won at the regional level, so the quality of the entries was enjoyably high. Moreover, I imagined that the projects represented what teachers and judges deemed successful mathematics, and for me this was an interesting aspect of the fair. Among the projects were one on chaos and one on fractals and nature. These attracted my attention immediately; yet both were disappointing in a way that got me thinking about the publicity these subjects have received lately, and about the question of what students and the public should understand mathematical achievement to be.

Unlike many of the other entries, neither of these projects contained any mathematical accomplishment that I recognized, except for the (legitimate but off-stage) writing of the programs that draw fractals. Once the pictures were drawn, the students did little more than classify them according to appearance and make an estimate of their "fractal dimension," remark on their similarity to natural shapes, and draw some dubious philosophical conclusions. As a result, these essentially expository projects did not compete well with those in which a problem was developed, made precise, and at least partly solved by mathematical reasoning.

Considering much of what the students and their teachers have been exposed to lately, one can't blame them too much. The recent popular attention and enthusiasm given to fractals (and chaos), on balance a great benefit to mathematics, has also evidently encouraged a confusion of phenomenology with mathematical accomplishment. It has been too easy for the casual observer to view fractal geometry as (i) little more than a zoology of shapes akin to clouds and coastlines, or (ii) representative of contemporary mathematics. Neither impression is accurate; together, they create a trivialized view of mathematics that omits the central role of precise logical reasoning toward the solution of a problem.

Mathematicians want *proof* to be recognized in its rightful place as the ultimate goal of any purely mathematical investigation. The formation of concepts and discovery of conceptual similarities (lately often inspired by machine computation) play a crucial role in the birth of mathematical discoveries—and can be the decisive step—but proofs provide the *explanations* that are the basis for mathematical understanding. If conjecture and proof are the two pillars of mathematical accomplishment, then the second pillar deserves as much recognition as the first.

Toward this end, fractal geometry needs to be better recognized as neither trivial nor typical. The first of these points is easier to address. In fact, the book under review is proof that fractal geometry contains interesting and nontrivial

mathematics. The books [3] and [4] are additional evidence of this. (For discussion of this point from various points of view, see the essays [**1, 2, 5, 6, 8, 9, 10**].)

As for the second point, it's worth remembering that the primary impact of fractal geometry is still not in pure mathematics but, rather, in physics and applied mathematics. For example, the language of fractals has often been taken up enthusiastically in connection with scaling properties for phenomena like diffusion-limited aggregation, percolation, and turbulence [**10**]. This helps to explain why fractal geometry should not be thought of as an enterprise of pure mathematics in the same sense as, say, algebraic topology or hyperbolic geometry. But how should it be thought of? Despite the list of recent essays cited above, this question still deserves a bit more attention.

The phrase "fractal geometry" was invented by Benoit Mandelbrot in 1975 to refer primarily to a *vocabulary* and *point of view*—one that brought many "fractal" objects in nature and mathematics, previously considered disparate pathologies, under one conceptual umbrella. The mathematical *explanations* were mostly intended to come later.

Traditionally, mathematical fields have acquired names and an independent status only after some systematic explanations have been developed. This makes fractal geometry—at least in its original form—exceptional in *kind*, and it should be contrasted with mathematical disciplines that are already composed of thematically related concepts interconnected by proofs. It deserves neither the credit nor the criticism that might be due a mature and fully formed subject.

The question of the definition of fractal illuminates the tentative status of the field. Some discussion of this may be helpful to the reader unfamiliar with the ideas involved. We first need to state some classical definitions. For convenience we assume in this essay that all sets are subsets of some Euclidean space.

The *topological dimension* $\dim_T(A)$ of a set A is an integer defined inductively as follows. Let $\dim_T(A) = 0$ if points of A have arbitrarily small neighborhoods with boundary disjoint from A (that is, if A is totally disconnected). Let $dim_T(A) \leqslant n$ if each point of A has arbitrarily small neighborhoods with boundary meeting A in a set of topological dimension $\leqslant n - 1$. Finally, we set $dim_T(A) = n$ if n is the smallest nonnegative integer such that $dim_T(A) \leqslant n$.

The topological dimension is a homeomorphism invariant. In the case of submanifolds, it assigns the intuitively correct value: curves have dimension one, surfaces two, etc. No set in R^n has dimension more than n.

A different notion of dimension is the *Hausdorff dimension* $\dim_H(A)$. This is not a topological quantity, but depends on the geometry of the set A, and can take non-integer values.

In certain cases, one can gain a crude intuition about Hausdorff dimension as follows. If we try to cover A with small balls of radius r, we'll need about $1/r$ balls, as r tends to zero, if A is a smooth curve (dimension 1); about $1/r^2$ balls if A is a smooth surface (dimension 2), etc. The dimension is indicated by the exponent. For the von Koch snowflake curve, the number of r-balls needed, as r tends to zero, is about $1/r^s$, where $s = \log 4/\log 3$. This number is the Hausdorff dimension of that set. (Actually this is the intuition behind the notion of *box dimension*, which often agrees with Hausdorff dimension in simple cases.)

We give a precise definition as follows.

For real $s \geq 0$, define the s-dimensional Hausdorff (outer) measure of A by $H^s(A) = \lim\{\inf \Sigma |U_i|^s\}$, where $| \cdot |$ denotes diameter, $\{U_i\}$ is a countable cover of A by sets of diameter at most epsilon, the infimum is taken over all such covers of

A for a fixed epsilon, and the limit is taken as epsilon tends to zero. (The limit always exists, but may be infinite.)

For example, H^1 is arclength measure for curves. In R^n, H^n is equivalent to Lebesgue measure. Caratheodory introduced these measures for integer values of s, and Hausdorff pointed out that they also made sense for noninteger s.

Now define

$$\dim_H(A) = \inf\{s\colon H^s(A) = 0\} = \sup\{s\colon H^s(A) = \text{infinity}\}.$$

Roughly speaking, $\dim_H(A) = s$ picks out the correct exponent for measuring the set A (although $H^s(A)$ itself may still be zero or infinity).

The Hausdorff dimension of a smooth submanifold agrees with its topological dimension; in general the two notions may disagree, but for any set A, $\dim_T(A) \leqslant \dim_H(A)$. For example, the standard middle thirds Cantor set C satisfies

$$\dim_T(C) = 0, \qquad \dim_H(C) = \log 2/\log 3.$$

In [7], Mandelbrot provisionally defined a fractal to be a set whose Hausdorff dimension strictly exceeds its topological dimension. This has been the most widely adopted, precise definition of the term, and it includes most of the standard examples; e.g., the middle thirds Cantor set mentioned above.

But, as Mandelbrot and others note, this definition is not very faithful to the motivating idea that a fractal is a shape that has "irregular structure" repeated at arbitrarily small scales. On the one hand, the definition includes sets with arbitrarily bad scaling properties, with little relation between one scale and another. On the other, it excludes sets—such as the graph of the Cantor function (the "devil's staircase") or a Cantor set with Hausdorff dimension zero—that ought to be considered fractals because of their recursively broken regular structure.

Since mathematics requires precise definitions, one should expect that gradually the intuitive idea of a fractal will become subordinate to some useful mathematical definition that captures the most important properties at the expense of the original intuition in special cases. (This has long been resolved with more mature concepts such as "the real line," or "connectedness." Even though there are, say, examples of connected sets that become totally disconnected upon removal of a single point, we now tend to blame our intuition of "connectedness" rather than the definition.)

Meanwhile there are a few competing alternatives. Edgar's book includes a discussion of a proposal by S. J. Taylor to define a fractal as a set whose Hausdorff and "packing" dimensions agree and exceed its topological dimension. This would exclude many sets with wild scaling properties too irregular to be a fractal by original intent.

One could *include* more sets that ought to be called fractals with the following definition: a set A is a fractal if it is not the countable union of subsets of finite H^s measure, where $s = \dim_T(A)$. This gives a Cantor set with $\dim_H = 0$ its rightful fractal credentials (note that H^0 is counting measure), includes all previously included sets, but unfortunately still bars the devil's staircase since it is a rectifiable curve (hence, of finite Hausdorff 1-measure).

Both of these definitions privilege the concepts of dimension and measure. Another approach is to try to use a version of "self-similarity," or its generalization "self-affinity." But this definition has its own shortcomings, for while the devil's staircase is in a certain precise sense "self-affine," so is a straight-line segment. Worse, the boundary of the Mandelbrot set, lately one of the most famous fractals, is in no sense self-affine since no neighborhood of any point is even homeomorphic to any other.

In mathematical practice, the fuzzy status of the word fractal is no impediment since one simply adopts a specific definition if necessary and proceeds. That fuzziness causes more trouble, however, when one is trying to explain what mathematics can be called fractal geometry. If we grant that the term refers primarily to a language or a viewpoint, it nevertheless has another unavoidable and growing meaning as a field of study. According to [**10**], it is the study of a geometry intermediate between Euclid and "geometric chaos," that is, the study of irregular shapes—not *too* irregular, but rather those with an orderly, scaling kind of irregularity.

Since the term was coined, its meaning has been influenced by association with other expositions. At present there is a spectrum of possibilities: at the narrow extreme, fractal geometry is the study of self-similar sets, their recursive constructions, and computation of their "fractal dimensions." At the broad extreme, it is the study of the metric properties of arbitrary point sets in Euclidean space, pioneered by Besicovitch. Probably most practitioners see it somewhere in between. However, a view of fractal geometry closer to the latter than the former view is reinforced by Mandelbrot's inclusive definition and especially by the influential text of K. J. Falconer, *The Geometry of Fractal Sets* [**3**], which was inspired by Besicovitch's unfinished monograph, *The Geometry of Sets of Points*.

However conceived, fractal geometry so far gives its best showing in a supporting role. And if the meaning of a term stems from its accepted use, then it should be fair to consider any mathematics as partaking of fractal geometry that makes use of the ideas of self-similarity or approximate self-similarity or its generalizations, scaling properties, or any of the metric tools of Besicovitch and followers (for example, Hausdorff measures) to study irregular or unrectifiable sets. If put this broadly, a good deal of interesting and important contemporary mathematics makes some use of fractal geometry—particularly in dynamical systems, probability, and parts of geometric analysis.

The student who has the good fortune to read Edgar's text will be exposed to most of these ideas, and more. The author develops metric topology to a solid level, using sequence spaces as a prime example. This leads to a very complete treatment of topological dimension. Self-similarity is discussed in depth, along with a generalization due to D. Mauldin and S. C. Williams called "graph self-similarity." Enough measure theory is developed to illuminate Hausdorff dimension and related concepts to a very satisfactory degree. The recursive constructions are further illuminated by the inclusion of programs in Logo that the student with access to that language can study to draw her own computer pictures.

By sacrificing breadth of coverage for systematic development, Edgar provides the best available course text about fractal geometry at its level (postcalculus undergraduate). Moreover, and no less important, a student reading this book can learn a lot of topology and analysis along the way. One advantage of teaching a junior-level analysis course with this book would be the attractive and concrete motivation for all the topics studied: the analysis of fractal shapes. This fact, combined with the adventurous treatment of the subject, should make it fun for students without any compromise in rigor. Of particular value are the many exercises, probably due in part to the book's prior incarnation as notes for a course in Arnold Ross's famous summer program at Ohio State for talented high school students.

The book is ideally suited for students who have had some introduction to elementary analysis and metric topology in the upper division, but have no background in measure theory or any further exposure to higher analysis. There-

fore, it supplies a good systematic textbook for use by students without the advanced background required by [3]. Most notably, this includes science and engineering students who, in increasing numbers, are returning to mathematics departments eagerly, and still often vainly, in search of courses on "fractal geometry."

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The Man Who Knew Infinity: A Life of the Genius Ramanujan. By Robert Kanigel. C. Scribner's, New York; Collier Macmillan Canada, Toronto; Maxwell Macmillan International, New York, 1991, ix + 438 pp.

Raghavan Narasimhan

The story of Srinivasa Ramanujan as it is usually told is a romantic one, a kind of rags-to-riches tale in which, from humble beginnings, he rose to recognition as an outstanding mathematician (as well as a Fellow of Trinity College, Cambridge, and a Fellow of The Royal Society). It is well known to people interested in mathematics or in mathematicians and need not be repeated here.

In some ways, Ramanujan was fortunate. People who knew him always seemed to have liked him, to have recognized him as exceptionally gifted, and to have been willing to do as much as they could to help him. It is hard to see how Ramanujan could have survived or continued working on mathematics without the help of many around him.

He was also fortunate in writing to Hardy. He wrote to two other English mathematicians before turning to Hardy. They are identified as H. F. Baker and E. W. Hobson by Mr. Kanigel in the book under review. Assuming that this is correct (and there is no reason to doubt the evidence he cites), it is not surprising that they did not react. Neither of them was even remotely capable of analyzing or judging the kind of work Ramanujan sent them. I think it safe to say that there

were few mathematicians in the world at the time besides Hardy and Littlewood to whom the pages sent by Ramanujan would not have been meaningless. Moreover Hardy took the time, and, with Littlewood's help, made the considerable effort needed to analyze Ramanujan's letter. This was Ramanujan's one real piece of luck.

All this is described in great detail in Mr. Kanigel's book. He has travelled to India and attempted to get as close to original sources as possible. He treats the book, rightly in my opinion, almost as a dual biography of Ramanujan and Hardy. He also considers that some appreciation of Ramanujan's mathematics is necessary to an understanding of his life. Let me say right away that the mathematical passages are awkward and contribute little to the book.

I think that Mr. Kanigel's treatment of Hardy is more successful than that of Ramanujan. Hardy's world is one familiar to the author, and despite his obvious sympathy, his understanding of the customs of south Indian Brahmins is incomplete. Let me give an example. In describing Ramanujan's mother, Mr. Kanigel refers to the photograph reproduced in the book and speaks of the raw intensity conveyed by the picture, and how she looks ready to spring because only the balls of her feet touch the floor (p. 19). In fact, I think that this is a rather conventional photograph of a south Indian Brahmin lady of a certain age. My family has a photograph of, for example, my grandmother at a comparable age which is practically identical with the one in the book, down to the scowl on the face and the position of the feet caused by the height of the chair in the photographer's studio.

This example is, of course, of no great importance. On the other hand, given the great deal of general attention that has been paid to Ramanujan's religious views and the importance that Mr. Kanigel himself attaches to them, his discussion of these views is a different matter.

Let me first recall Hardy's statement that he remembered well Ramanujan telling him (much to his surprise) that all religions seemed to him (Ramanujan) more or less equally true. Hardy went on to argue that this could only mean that Ramanujan was an agnostic. This is hardly surprising since Hardy's acquaintances and friends were almost exclusively Western intellectuals and he was a very strong atheist.

Mr. Kanigel, on the other hand, describes Ramanujan's rigid adherence to ritual and to extreme forms of Brahminical views on food and its preparation, and concludes that he must have been deeply religious. He claims that for Ramanujan, the split between mysticism and his mathematics was not sharp, and that he did not reveal to Hardy the "richness and extent of his spiritual life" for fear of alienating Hardy.

I think that both these views are wrong. I myself was brought up in an atmosphere of exactly the kind described in Mr. Kanigel's book. The existence of a Supreme Being, the attainment of Godhead by man and so on (which are described by Seshu Aiyar and Ramachandra Rao in the biographical sketch of Ramanujan in his Collected Works) were presented to us, from the earliest age, as matters of fact. But, at the same time, I, and all my acquaintances of like age, were taught a tolerance of other beliefs and other means of achieving Heaven. I think that this is an essential part of true Hinduism. We were told to practice certain rituals daily in order to reach spiritual goals, but were also taught to respect the different practices of others, even when we were told not to share a meal with them. It seems to me that Ramanujan's religious practices were not so different from those of most Brahmins of the day, and that his statement to Hardy was a

simple statement of fact. People in the West would probably not have been surprised by his behaviour if they had only met Ramanujan in India. Unlike many Indians abroad, he did not change his behaviour in England. I think that this would have been the case irrespective of his beliefs because of the promises he had made to his mother. I know others from Madras who carried out similar promises literally.

We come now to the question of Ramanujan as mathematician. Here Mr. Kanigel simply repeats the opinions of various mathematicians.

I said earlier that Ramanujan was fortunate in some ways. In his development as a mathematician, he was singularly unfortunate. He was born in an India dominated by the British in intellectual matters at a time when pure mathematics in Britain was at a low ebb. He was, moreover, cut off from most of even this work, and he wasted a lot of his time rediscovering results which had been long known. He was then chagrined when he found that these results were not new. Further, I do not think that being "discovered" by Hardy (Hardy's word) was the best thing that could have happened to him.

To explain the reasons for this opinion, I must first say something about my views about Hardy. Hardy was a true innovator and leader in real analysis, particularly in the work done with Littlewood. However, he did not understand the geometric aspects of complex analysis (see for example his cumbersome treatment of Abel's work in his tract *Integration of functions of a single variable*). What is more important in the context of Ramanujan, he had no feeling for the truly arithmetic aspects of number theory; certainly his work on the subject is purely analytic. He seems to have had an exaggerated respect for virtuosity; how else can one explain the following incident, cited by C. P. Snow in his Foreward to Hardy's *A Mathematician's Apology*? Hardy elevated Archimedes, Newton and Gauss from the Hobbs to the Bradman class [famous figures in the game of cricket] when he decided that Bradman was in a class of his own, as if the soaring imagination of these mathematicians could be measured in terms of virtuosity with a willow bat. It was Ramanujan's abundant virtuosity that he admired and encouraged.

It is also very possible that Hardy brought some pressure to bear on Ramanujan to work on matters Hardy thought interesting. The evidence cited by Mr. Kanigel in connection with their relationship when Ramanujan was in a sanatorium (pp. 254–255) is consistent with this view. Littlewood [who was, I think, broader than Hardy in his mathematical views, and just as deep] was away because of World War I, as were others, like Mordell. Ramanujan would have profited greatly from these people. Their absence was a very bad piece of luck indeed.

Ramanujan had a powerful mathematical imagination. I think he also had a deep feeling for the arithmetic aspects of number theory. This is especially apparent in the conjectures he made about the function $\tau(n)$. In his book *Ramanujan. Twelve lectures on subjects suggested by his life and work*, Hardy says of $\tau(n)$: "We may seem to be straying into one of the backwaters of mathematics, but the genesis of $\tau(n)$ as a coefficient in so fundamental a function compels us to treat it with respect". Ramanujan understood the importance of functions like $\tau(n)$ much better. The three statements he made have turned out to be central in some very profound mathematics. Ramanujan's first statement amounts to the existence of an Euler product for $\Sigma\tau(n)n^{-s}$; the theory of Hecke operators came out of an attempt to understand Dirichlet series arising from modular forms, as in the case of the τ-series, which have Euler products. Swinnerton-Dyer showed how the congruence properties of $\tau(n)$ conjectured by Ramanujan could be proved and explained by ideas of Serre and results of Serre and Deligne on Galois representa-

tions. These same ideas and results provided the link reducing the conjecture of Ramanujan on the size of $\tau(n)$ to the so-called Weil conjectures proved by Deligne on the basis of ideas of Grothendieck.

It is conceivable, even probable, that had Ramanujan been in the company of someone like Hecke, he would have pursued arithmetic questions further and developed all his powers more fully. But, given the time and place of his birth, it would have taken a miracle to make this possible.

Mr. Kanigel clearly has great sympathy for the conditions surrounding Ramanujan and great admiration for his achievements and gifts. He has cited his sources for a lot of information, so that one can decide which part of this information one wishes to treat with caution. This is important, since Ramanujan has become a famous and romantic figure. Memories of decades past not based on contemporary written records are likely to be coloured by this fact. However, it seems to me that Mr. Kanigel accepts too many stories uncritically. I should also add that I find some passages misleading, especially when he ascribes motives to people long gone from the scene. Some of this is due to lack of mathematical understanding. Typical is his discussion of Baker's reasons for not supporting Ramanujan when the latter wrote to him (p. 170). Baker was a geometer, and any mathematician can imagine the reaction of an algebraic geometer to a letter such as Ramanujan's.

Mr. Kanigel has certainly done a lot of research in trying to identify and locate both written sources on Ramanujan and the oral tradition which has grown around him. This should be useful to anyone attempting to study Ramanujan's life.

What many mathematicians, including myself, would like to see is a really competent mathematical biography of Ramanujan.

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TELEGRAPHIC REVIEWS

Edited by
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General, S(13-17), L. *Mathemtical Olympiad in China.* Chinese Mathematical Olympiad Committee, 1990, 314 pp. [ISBN: 7-5355-1152-X] China has taken part in the International Mathematical Olympiad since 1985, and has consistently ranked among the top countries (first place, 1989, 1990; second place, 1988, 1991). This volume contains seventeen articles featuring problems from various competitions (including the Chinese Olympiad and the Putnam Exam), together with instructive comments and related results and examples written by coaches and educators throughout the country. LCL

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Recreational Mathematics, S, L**.** *Polyominoes: A Guide to Puzzles and Problems in Tiling.* George E. Martin. MAA, 1991, ix + 184 pp, $21 (P). [ISBN: 0-88385-501-1] A systematic exploration with lots of examples and problems of the tiling properties of "polyominoes," a word coined by Solomon Golomb (this *Monthly*, 61 (1954) 675–682) to describe shapes formed from "rook-wise" connected squares. Excellent source for student papers, math club presentations, even for undergraduate research projects since dozens of the (seemingly simple) problems are unsolved. LAS

Precalculus, T(13). *Algebra and Trigonometry, Second Edition.* Thomas W. Hungerford, Richard Mercer. Saunders College, 1991, xviii + 804 pp, $35. [ISBN: 0-03-046928-7] This most comprehensive pre-calculus text will only disappoint those hoping that "lean and lively" might trickle down to pre-calculus. Everything one could want, including unusually thorough analytic geometry, linear systems and determinants, and discrete mathematics (sequences, the binomial theorem, induction, probability) are here. The text does not say much about graphing calculators, but a supplement is available. (*First Edition*, TR, November 1982.) AWR

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Complex Analysis, T(18), P. *Linear Differential Equations in the Complex Domain: Problems of Analytic Continuation.* Yasutaka Sibuya. Transl. of Math. Mono., V. 82. AMS, 1990, xiv + 267 pp, $78. [ISBN: 0-8218-4535-7] Original version appeared in 1976. Author translated current edition from Japanese, adding five appendices and 100 new references. Covers basic structures of analytic continuation, existence theorems of Grauert and Birkhoff, and characterizations of singular points. BL

Partial Differential Equations, P. *What Is Integrability?* Ed: V.E. Zakharov. Ser. in Nonlinear Dynamics. Springer-Verlag, 1991, xiv + 321 pp, $69. [ISBN: 0-387-51964-5] The integrability of systems of partial differential equations plays an important role in applied mathematics. The initial article of this book asks why certain systems are both widely applicable and integrable—that is, exactly solvable. Subsequent articles deal with their appearance in classical physics, the work of Poincaré in showing such systems to be rare exceptions, and the renewed interest stimulated in 1967 by the introduction of the inverse scattering transform. AWR

Analysis, T(17), P, L. *Fuzzy Set Theory—and Its Applications, Second, Revised Edition.* H.-J. Zimmermann. Kluwer Academic, 1991, xx + 399 pp, $69.96. [ISBN: 0-7923-9075-X] Originally the first introductory text to fuzzy set theory, this revised edition contains rewritten sections on possibility theory, fuzzy logic and approximate reasoning, expert systems and fuzzy control, decision making and fuzzy set models in operations research. Also, exercises have been added to nearly every chapter. Text begins with basic definitions and examples, extensions to algebraic and set-theoretic operations, fuzzy measures, fuzzy relations, fuzzy graphs, and fuzzy analysis. Includes numerous examples. MK

Algebraic Geometry, P. *Équations Différentielles à Coefficients Polynomiaux.* B. Malgrange. Progress in Math., V. 96. Birkhäuser, 1991, 232 pp, $49.50. [ISBN: 0-8176-3556-4] An algebraic geometric discussion of holonomic differential systems in one variable, and how these systems are affected by the Fourier-Laplace transform. JO

Geometry, S*, L*. *Journey into Geometries.* Marta Sved. MAA, 1991, xv + 182 pp, $21 (P). [ISBN: 0-88385-500-3] A serious introduction to non-Euclidean (hyperbolic) geometry through a whimsical Socratic conversation among three figures: Lewis Carroll (née Charles Dodgson), his fictional Alice, and a contemporary mathematical inquisitor Dr. What If. Can be read (as a sophisticated version of *Through the Looking Glass*) or studied (via exercises and problems at the end of each section, with full answers at the end). A wonderful narrative for math clubs. LAS

Geometry, S, C. *FractalQuilt.* Nicholas Strauss (Strauss Inc., 612 Shropshire Dr., West Chester, PA 19382). Macintosh Software. $30. A "shareware" graphics program that encodes into matrix form user-specified coloring of a small checkerboard, then computes Kronecker products of this matrix and displays the result as a fractal-

like quilt whose image can be saved and reused with standard Macintosh tools. LAS

Optimization, S(18), P, L. *Theory of Global Random Search.* Anatoly A. Zhigljavsky. Math. & Its Applic., V. 65. Kluwer Academic, 1991, xviii + 341 pp, $144. [ISBN: 0-7923-1122-1] Often global random search offers the only known way of solving complicated global optimization problems. These methods are attractive since they have simple structures and can be easily coded, they are insensitive to irregularity of objective function behavior and feasible region structure and growth of dimensionality. Yet, convergence rates can be slow and efficiency can be increased by means of increased complexity and decreased randomness. Topics include overview of global optimization, global random search algorithms, results on convergence, Markovian algorithms optimization in function spaces, and discrete optimization. Note price. MK

Systems Theory, P. *Information Dynamics.* Eds: Harald Atmanspacher, Herbert Scheingraber. NATO ASI Ser. B, V. 256. Plenum Pr, 1991, xi + 364 pp, $95. [ISBN: 0-306-43912-3] Wide-ranging series of papers presented at the 1990 NATO ASI conference. Underlying theme is that deterministic chaos and other types of behavior characterize systems which are deterministic (i.e., governed by mathematical equations) but not determinable (in the sense that arbitrarily accurate predictions are not possible). The limited predictability is regarded as the generation of information by the system. Papers on uncertainty, complexity, causality, quantum systems, computation, and information. RM

Probability, P, L. *Limit Theorems in Probability and Statistics.* Eds: I. Berkes, E. Csáki, P. Révész. North-Holland (US Distr: Elsevier Science), 1990, 561 pp, $200. [ISBN: 0-444-98758-4] Proceedings from the Third Hungarian Colloquium on limit theorems in probability and statistics of the Bolyai János Mathematical Society in Pécs, Hungary, July 3-7, 1989. Topics include limit theorems for partial sums of random variables, both dependent and independent cases, extreme values, empirical processes, local times, time series, etc. Note price. MK

Stochastic Processes, P. *Probability in Banach Spaces: Isoperimetry and Processes.* Michel Ledoux, Michel Talagrand. Ergebnisse der Math. und ihrer Grenzgebiete, Band 23. Springer-Verlag, 1991, xii + 480 pp, $129. [ISBN: 0-387-52013-9] An attempt to summarize the explosion of developments in the past twenty years. Focuses on two related topics: isoperimetric inequalities/methods, and the regularity of random processes. Highly technical. Contains a huge bibliography. Note price. TAV

Stochastic Processes, P. *Random Processes with Independent Increments.* A.V. Skorohod. Math. & Its Applic., V. 47. Kluwer Academic, 1991, xi + 279 pp, $118. [ISBN: 0-7923-0340-7] In spite of the fact that processes with independent increments are some of the most basic and elementary in the theory of stochastic processes, this treatment is hardly elementary. A revision of an earlier (1964) work in Russian. Valuable to the specialist to see these processes in a general (theoretical) setting. TAV

Elementary Statistics, S, P, L*.** *Perspectives on Contemporary Statistics.* Eds: David C. Hoaglin, David S. Moore. MAA Notes No. 21. MAA, 1992, xiii + 175 pp, $20 (P). [ISBN: 0-88385-075-3] Nine expositions of topics central to the teaching of statistics to beginners—data analysis, samples and surveys, design of experiments, probability, statistical inference, diagnostics, robust procedures—focused on current (often computer-based) practice, and illustrating the priorities and pitfalls of teaching. The goal is to reduce the time lag between changes in practice and changes in instruction at the beginning level. LAS

Elementary Statistics, T(13), L. *Statistics: Concepts and Controversies, Third Edition.* David S. Moore. WH Freeman, 1991, xvii + 439 pp, (P) [ISBN: 0-7167-2199-6]; *Instructor's Guide for Statistics, Concepts and Controversies, Third Edition,* 175 pp, (P). [ISBN: 0-7167-2247-X] A well-written, introductory text with pedagogical approach to statistics as a liberal art for non-mathematical students. Organization: producing data, organizing and analyzing data, drawing conclusions from data, and graphical explanations. Contains numerous examples and discussion exercises taken from journals, newspapers, and magazine articles. (*First Edition*, TR, August-September 1979; *Second Edition*, TR, April 1986.) MK

Statistical Methods, P. *Contextual Analysis.* Gudmund R. Iversen. Quantit. Applic. in Soc. Sci., V. 81. Sage Pub,

1991, 84 pp, $8.50 (P). [ISBN: 0-8039-4272-9] Lucid, not-too-technical exposition of contextual-effects, or hierarchical models. Aimed mainly at researchers in sociology and demography, though the methods are applicable to a variety of problems. Topics include contextual analyses with absolute and relative effects, random regression coefficients, parameter estimation, and more. MK

Computational Statistics, S*(13-18), C, P*. *StatView II: The Solution for Data Analysis and Presentation Graphics.* Daniel S. Feldman, Jr., *et al.* Macintosh Software. Abacus Concepts (1984 Bonita Ave., Berkeley, CA 94704-1038; 415-540-1949), 1986, vi + 278 pp, $495 (P). [ISBN: 0-944800-00-9] A flexible award-winning package combining elementary data analysis (descriptive statistics, comparative statistics, varied graphical presentations, regression, t-tests, contingency tables, ANOVA, factor analysis, nonparametric tests) with drawing tools (scattergrams, percentile plots, boxplots, confidence bands, outlier signals, legends, etc.) for formal presentations. Uses spreadsheet format for data entry and transformation. Can import data from other Macintosh programs, and can save images in PICT form for further graphical editing. Supports full color. Will run on all but the oldest Macs: requires 1Mb RAM; hard drive "strongly preferred." Thorough user guide includes discussion of algorithms and computational details. LAS

Computational Statistics, S(13-18), P, L. *SAS Applications Programming: A Gentle Introduction.* Frank C. DiIorio. Duxbury Ser. in Stat. & Decision Sci. PWS-Kent, 1991, xiv + 684 pp, $21.50 (P) net. [ISBN: 0-534-92390-9] A thorough introduction to the SAS system for data management, statistical analysis and reporting, with discussion of graphics, econometrics, and operations research. Gives a good background for using SAS without having to plough through the overwhelming SAS documentation. Philosophy is breadth, not depth. Audience is SAS users with a basic level of computer literacy. Includes lots of code and applications to real-world examples. MK

Statistics, S(18), P, L. *Lecture Notes in Statistics-67: Tools for Statistical Inference.* Martin A. Tanner. Springer-Verlag, 1991, vi + 110 pp, $20 (P). [ISBN: 0-387-97525-X] Excellent, though terse, monograph on Bayesian or likelihood-based analyses utilizing observed data and data augmentation methods. Topics include maximum likelihood, posterior density analysis, delta method, numerical integration, Laplace expansion, Monte Carlo methods, EM algorithm, Louis' method, predictive distributions via data augmentation, general imputation methods, chained data augmentation, the Gibbs sampler, the griddy Gibbs sampler. MK

Statistics, S(17), P. *The Taming of Chance.* Ian Hacking. Cambridge Univ Pr, 1990, xiii + 264 pp. [ISBN: 0-521-38014-6] A philosophical and historical discussion developing the connections between two theses: "the most decisive conceptual event of twentieth-century physics has been the discovery that the world is *not* deterministic," and "the enumeration of people and their habits" became pervasive and well-known as "society became statistical." Chapters include discussion of eighteenth-century public amateurs and secret bureaucrats, Condorcet, Select Committee of 1825, medical statistics as evidence for efficacy of rates of cure, Quetelet, and much more. MK

Statistics, S(18), P. *Survivorship Analysis for Clinical Studies.* Eugene K. Harris, Adelin Albert. Stat.: Textbooks & Mono., V. 114. Marcel Dekker, 1991, xiii + 200 pp, $75. [ISBN: 0-8247-8400-6] An excellent monograph concerning modern statistical methods for survival analysis in clinical trials. Contains the requisite mathematics though it is *not* a "theorem-proof" text. Full of exposition, motivation and examples, but no exercises. Includes estimation of survival probabilities, life tables, Kaplan-Meier estimation, confidence bands for survival rates and curves, Hall-Wellner band, Efron's bootstrap bands, hazard models, and survival analysis with time-dependent covariates. MK

Statistics, T(18). *Theory of Point Estimation.* E.L. Lehmann. Stat. & Prob. Ser. Wadsworth, 1991, xii + 506 pp, $49.95. [ISBN: 0-534-15978-8] Standard graduate-student fare. Classical statistical theory concerning point estimation in Euclidean sample spaces. Covers small-sample optimality problems with respect to unbiasedness, equivariance and minimax criteria, as well as large-sample theory for maximum likelihood estimators, Bayes estimators, asymptotic efficiency, and local asymptotic optimality. Numerous exer-

cises, a handful of examples, and little motivation. (1983 text, TR, April 1984.) MK

Statistics, T(18). *Testing Statistical Hypotheses, Second Edition.* E.L. Lehmann. Stat. & Prob. Ser. Wadsworth, 1991, xx + 603 pp, $49.95. [ISBN: 0-534-15984-2] New edition of classic, graduate-level text on testing theory. Contains much of the old fare, Neyman-Pearson theory, nonparametric tests, unbiasedness, invariance. Contains a new chapter on conditional inference, mixtures of experiments, ancillary and relevant subsets. Little to no discussion of Bayesian philosophy or sequential procedures. (*Second Edition*, TR, June-July 1987.) MK

Statistics, T(17-18), L. *Time Series: Theory and Methods, Second Edition.* Peter J. Brockwell, Richard A. Davis. Ser. in Stat. Springer-Verlag, 1991, xvi + 577 pp, $49.50. [ISBN: 0-387-97429-6] Highly mathematical treatment of time series methods. Makes extensive use of Hilbert space methods and recursive prediction techniques based on innovations, use of the exact Gaussian likelihood and AIC for inference, and a thorough treatment of asymptotic behavior of maximum likelihood estimators of coefficients of univariate ARMA models. This edition includes chapter on state-space models, is accompanied by diskette with ITSM (see below) for IBM-PC, and contains a multitude of exercises, mostly mathematical, but a number which use data and the ITSM package. (*First Edition*, TR, August-September 1987.) MK

Statistics, S(16-17), C, L. *ITSM: An Interactive Time Series Modelling Package for the PC.* Peter J. Brockwell, Richard A. Davis. Springer-Verlag, 1991, ix + 104 pp, $49.95 (P). [ISBN: 0-387-97482-2] A collection of programs for the IBM-PC written to accompany *Time Series: Theory and Methods.* Requires PC-compatible computer with at least 540K and a graphics card; a mathematical co-processor is recommended but not essential. Allows simple Box-Jenkins methods, diagnostics, transformations, spectral analyses, smoothing, transfer function analysis, multivariate autoregression; includes a screen editor. MK

Programming, S(16-17), P, L. *Methods and Programs for Mathematical Functions.* Stephen Lloyd Baluk Moshier. Ser. in Math. & Its Applic. Ellis Horwood (US Distr: Prentice Hall), 1989, vii + 415 pp, $35.95 (P). [ISBN: 0-470-21609-3] Aimed at programmers and engineers computing special functions *not* readily available in computer language, run-time libraries. Begins with discussion of floating point arithmetic, error analysis, and rational arithmetic. Continues with approximation methods (Taylor series, Padé, continued fractions and Newton-Raphson iterations), software notes (design, testing, utilities), elementary functions, probability distributions, Bessel functions, Airy functions, hypergeometric functions, Struve functions, elliptic functions, zeta functions. Contains few examples; discussion is purely technical. Includes source code for over 100 programs from an implementation of IEEE arithmetic to efficient calculation of special functions. MK

Programming, T(12-13: 1), L. *Practical C Programming.* Steve Oualline. O'Reilley & Assoc, 1991, xxii + 396 pp, $24.95 (P). [ISBN: 0-937175-65-X] Well-titled, this book is an introduction to C which contains much sage advice on program design, programming style, and the programming process. Examples are illustrated with excellent diagrams. RK

Languages, P. *Logic Programming and Non-monotonic Reasoning.* Eds: Anil Nerode, Wiktor Marek, V.S. Subrahmanian. MIT Pr, 1991, ix + 289 pp, $32.50 (P). [ISBN: 0-262-64027-9] Research papers on the relationship between logic programming semantics and non-monotonic reasoning presented at a workshop at the 1990 North American Conference on Logic Programming in Austin, Texas. RK

Languages, S(14-18), C, P, L. *C++ for Scientists and Engineers.* James T. Smith. McGraw-Hill, 1991, xii + 322 pp, $29.95 (P). [ISBN: 0-07-059180-6] Describes design, construction, and use of a numerical analysis software toolkit written in C++, Version 2.0 making essential use of the object-oriented features. Object-oriented programming allows abstractions at a level which helps to make the numerical application programs look like the mathematics they represent. Describes in detail implementation of real and complex arithmetic, elementary functions, vector and matrix algebra, polynomial algebras, solutions of transcendental and polynomial equations, solutions of linear systems of equations, eigenvalue problems, and solutions of nonlinear systems of equations. MK

Algorithms, P. *Intersection and Decom-*

position Algorithms for Planar Arrangements. Pankaj K. Agarwal. Cambridge Univ Pr, 1991, xvii + 277 pp, $39.50. [ISBN: 0-521-40446-0] Algorithmic and combinatorial study of some computational geometry problems on arrangements of lines, segments, and curves in the plane. Topics include Davenport-Schinzel sequences, random sampling and deterministic partitioning, spanning trees and stabbing number, and applications, e.g., to motion planning and implicit point location. JPH

Computer Systems, S(16-17), C, P, L. *Guide to OSF/1: A Technical Synopsis.* Open Software. O'Reilly & Assoc, 1991, ix + 280 pp, $21.95 (P). [ISBN: 0-937175-78-1] Provides technical overview concerning what *is* OSF/1, how is it being described by the people who develop it, and what promises and commitments for its future are being made by those people. Includes brief discussion of open systems, Mach kernel, architecture (e.g., threads, tasks, and processes), messages and ports, virtual memory, dynamic device configuration, file systems, and security. Also discusses the programming environment, the loader, internationalization, and distributed computing environments. MK

Computer Graphics, P. *Oriented Projective Geometry: A Framework for Geometric Computations.* Jorge Stolfi. Academic Pr, 1991, vii + 237 pp, $39.95. [ISBN: 0-12-672025-8] "Programmers who use homogeneous coordinates for geometric computations are implicitly—and often unknowingly—working in the so-called projective space." So saying, the author presents a geometric model that preserves the advantages of doing graphics computations in projective space (simpler formulas, few special cases, etc.) while eliminating some of its disadvantages (nonorientability, ambiguous notions of direction, etc.). Aimed at programmers, this book is short on theory and long on examples. JO

Artificial Intelligence, P. *Uncertainty and Vagueness in Knowledge Based Systems: Numerical Methods.* R. Kruse, E. Schwecke, J. Heinsohn. Springer-Verlag, 1991, xi + 491 pp, $69. [ISBN: 0-387-54165-9] Mathematical models of uncertainty (doubt about the actual state of affairs) and vagueness (ambiguity or imprecision) are developed using measure-theoretic methods. Three approaches use probability theory for probabilistic reasoning, L-sets for fuzzy reasoning, and weighted sets for evidentiary reasoning. Interpretation of vague data is an extension of interval analysis. RK

Artificial Intelligence, T(18), S, P, L. *Representing and Reasoning with Probabilistic Knowledge: A Logical Approach to Probabilities.* Fahiem Bacchus. MIT Pr, 1990, 233 pp, $29.95. [ISBN: 0-262-02317-2] Addresses the question "How can probabilities be applied in artificial intelligence?" Develops logics for probability in an attempt to 1) solve problem of epistemological adequacy by developing logics capable of expressing a wide range of qualitative probability assertions; 2) develop first-order logics for probabilities providing a smooth integration with first-order logic; and 3) develop two distinct logics, each suitable for representing and reasoning with a distinct interpretation of probability (statistical interpretation—relative frequency, and degrees of belief). Audience is researchers and graduate students of artificial intelligence and logic. Basics of probability theory and logic (proof theory) are covered. MK

Computer Science, P. *Advances in Computers, Volume 32.* Ed: Marshall C. Yovits. Academic Pr, 1991, x + 331 pp, $69.95. [ISBN: 0-12-012132-8] Survey articles on "Computer-Aided Logic Synthesis for VLSI Chips;" "Sensor-Driven Intelligent Robotics;" "Multidatabase Systems: An Advanced Concept in Handling Distributed Data;" "Models of the Mind and Machine: Information Flow and Control between Humans and Computers;" and "Computerized Voting." RK

Computer Science, T(16-17: 2). *The Design and Analysis of Spatial Data Structures.* Hanan Samet. Addison-Wesley, 1990, xvii + 493 pp. [ISBN: 0-201-50255-0] A second course in data structures emphasizing representation of spatial data. Focuses on divide-and-conquer methods. Applications include computer graphics, computational geometry, database management systems, and image processing. MK

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The American Mathematical Monthly

Volume 99, Number 5 / MAY 1992

Joseph Fourier (p. 427)

NOTICE TO AUTHORS

The *Monthly* publishes articles, notes, and other features about mathematics and the profession. The readership of the *Monthly* is intended to include everybody who is mathematically inclined, including of course professional mathematicians and students of mathematics at all collegiate levels. While no single article or feature is likely to appeal to everyone, material should interest and be accessible to a large number of readers. This is the most important criterion for acceptance.

Articles may be expositions of old results or presentations of new ones. They may concern all of mathematics or one small area, a broad development or a single application, historical reminiscences or one important event. While some articles may contain the author's new research, the novelty of material and generality of the results is far less important than the clarity of exposition and general interest. Discussing one illuminating case of a well known result is far better than providing all the details of an obscure but new proposition. Articles in the *Monthly* are supposed to inform and to entertain; they are meant to be read rather than archived.

Notes are short and possibly informal articles. A note may concern a clever new proof of an old theorem, a novel way to present tired material, or a lively discussion of a philosophical (but still mathematical) issue. Also any topic is suitable, so long as it is related to mathematics. Because a note is short, the first few sentences are the most important part: They should explain the purpose and invite the reader in. Photographs or diagrams often will attract the reader's attention.

All articles and notes should be sent to the editor:

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ARTICLES

FEATURES

COMMENTS

In 1993 the MONTHLY will be 100 years old. We want to celebrate. We will mark the occasion, of course, in the MONTHLY itself, publishing a few special articles and adding some features based on past. But a major part of our celebration will be a centennial volume containing a collection of articles and tidbits from the MONTHLY.

A collection of articles sounds pretty dull. These articles are supposed to serve a purpose, however. By selecting interesting articles, notes, problems, and announcements from the past 99 volumes of the MONTHLY, we hope to give a feeling for mathematics and its culture over the past century. We want the articles to be interesting (mathematically), but we also want them to be representative. The articles and the tidbits should remind us of past fads and fashions, both the respectable ones and those best forgotten. They should recall famous people and famous theorems, but they should also recall the quieter mathematics that makes up our everyday lives. They should be selected over a broad period time and a broad range of topics.

Mixed with the articles and historical material, we want to include photographs—lots of them. Again, the photographs are supposed to represent mathematics over the past 100 years, providing a glimpse of both famous and ordinary mathematicians and their institutions.

How do we select 200 pages of material from 9000 pages of the MONTHLY? How do we find those forgotten photographs of people and places? That's where you can help. We want to use the readers of the MONTHLY as a resource. We want your advice, and we want your old photographs.

Your advice. If you have ever looked at old issues of the MONTHLY, you have likely found some mathematical gems. You may also have found some curious articles that reminded you of a fashion in mathematics long since forgotten. Old issues of the MONTHLY make wonderful browsing.

What are the five best articles the MONTHLY has published?
What are the five articles in the MONTHLY (spread over time) that best represent the mathematics of the period in which they were written?
What are the five best problems (reviews) ever published?

We would like to have your suggestions. We also want to have ideas for interesting filler and curious tidbits from the old MONTHLY. We need advice.

Your photographs. Photographs make our history and culture a bit more real; almost everyone would rather *see* Emil Borel standing in a stately pose than to read a description of him. We need photographs, and we hope the readers of the MONTHLY will supply them. The photgraphs can be of famous mathematicians or of not-so-famous ones; they can be of groups or of buildings or of places. The only *crucial* information we need is the name of the person or place; the date (approximate) and the story behind the photo would help. Upon request, we will make copies of the photos and return the originals to you within 6 weeks. While we may not use all the photographs in the centennial volume, the entire collection will serve as an archive for the MONTHLY (and the Association) in the future.

Please send your ideas and your photographs to: *MONTHLY Centennial, Dept. of Mathematics, Indiana University, Bloomington, IN 47405.* Thanks for your help.

—John Ewing

Two Notes on Notation

Donald E. Knuth

Mathematical notation evolves like all languages do. As new experiments are made, we sometimes witness the survival of the fittest, sometimes the survival of the most familiar. A healthy conservatism keeps things from changing too rapidly; a healthy radicalism keeps things in tune with new theoretical emphases. Our mathematical language continues to improve, just as "the *d*-ism of Leibniz overtook the dotage of Newton" in past centuries [**4**, Chapter 4].

In 1970 I began teaching a class at Stanford University entitled Concrete Mathematics. The students and I studied how to manipulate formulas in continuous and discrete mathematics, and the problems we investigated were often inspired by new developments in computer science. As the years went by we began to see that a few changes in notational traditions would greatly facilitate our work. The notes from that class have recently been published in a book [**15**], and as I wrote the final drafts of that book I learned to my surprise that two of the notations we had been using were considerably more useful than I had previously realized. The ideas "clicked" so well, in fact, that I've decided to write this article, blatantly attempting to promote these notations among the mathematicians who have no use for [**15**]. I hope that within five years everybody will be able to use these notations in published papers without needing to explain what they mean.

The notations I'm talking about are (1) Iverson's convention for characteristic functions; and (2) the "right" notation for Stirling numbers, at last.

1. IVERSON'S CONVENTION. The first notational development I want to discuss was introduced by Kenneth E. Iverson in the early 60s, on page 11 of the pioneering book [**21**] that led to his well known *APL*.

> "If α and β are arbitrary entities and $\mathscr{R}$ is any relation defined on them, the *relational statement* $(a\mathscr{R}b)$ is a logical variable which is true (equal to 1) if and only if α stands in the relation $\mathscr{R}$ to β. For example, if x is any real number, then the function
>
> $$(x > 0) - (x < 0)$$
>
> (commonly called the *sign function* or $\operatorname{sgn} x$) assumes the values 1, 0, or -1 according as x is strictly positive, 0, or strictly negative."

When I read that, long ago, I found it mildly interesting but not especially significant. I began using his convention informally but infrequently, in class discussions and in private notes. I allowed it to slip, undefined, into an obscure corner of one of my books (see page 117 of [**16**]). But when I prepared the final manuscript of [**15**], I began to notice that Iverson's idea led to substantial improvements in exposition and in technique.

Before I can explain why the notation now works so well for me, I need to say a few words about the manipulation of sums and summands. I realized long ago that

"boundary conditions" on indices of summation are often a handicap and a waste of time. Instead of writing

$$(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k, \tag{1.1}$$

it is much better to write

$$(1+z)^n = \sum_{k} \binom{n}{k} z^k; \tag{1.2}$$

the sum now extends over all integers k, but only finitely many terms are nonzero. The second formula (1.2) is instantly converted to other forms:

$$(1+z)^n = \sum_{k} \binom{n}{k} z^k = \sum_{k} \binom{n}{k+1} z^{k+1} = \sum_{k} \binom{n}{\lfloor n/2 \rfloor - k} z^{\lfloor n/2 \rfloor - k}; \tag{1.3}$$

by contrast, we must work harder when dealing with (1.1), because we have to think about the limits:

$$(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k = \sum_{k=-1}^{n-1} \binom{n}{k+1} z^{k+1} = \sum_{k=-\lceil n/2 \rceil}^{\lfloor n/2 \rfloor} \binom{n}{\lfloor n/2 \rfloor - k} z^{\lfloor n/2 \rfloor - k}. \tag{1.4}$$

Furthermore, (1.2) and (1.3) make sense also when n is not a positive integer.

Even when limits are necessary, it is best to keep them as simple as possible. For example, it's almost always a mistake to write

$$\sum_{k=2}^{n-1} k(k-1)(n-k) \quad \text{instead of} \quad \sum_{k=0}^{n} k(k-1)(n-k); \tag{1.5}$$

the additional zero terms are more helpful than harmful (and the former sum is problematical when $n = 0$, 1, or 2).

Finally it dawned on me that Iverson's convention allows us to write *any* sum as an infinite sum without limits: If $P(k)$ is any property of the integer k, we have

$$\sum_{P(k)} f(k) = \sum_{k} f(k)[P(k)]. \tag{1.6}$$

For example, the sums in (1.5) become

$$\sum_{k} k(k-1)(n-k)[0 \le k \le n] = \sum_{k} k(k-1)(n-k)[k \ge 0][k \le n]. \tag{1.7}$$

(At the time I made this observation, I had forgotten that Iverson originally defined his convention only for single relational operators enclosed in parentheses; I began to put *arbitrary* logical statements in square brackets, and to assume that this would produce the value 0 or 1.) In this particular case nothing much has been gained when passing from (1.5) to (1.7), although we might be able to make use of identities like

$$k[k \ge 0] = k[k \ge 1]. \tag{1.8}$$

But in general, the ability to manipulate "on the line" instead of "below the line" turns out to be a great advantage.

For example, in my first book [25] I had found it necessary to include the rule

$$\sum_{k \in A} f(k) + \sum_{k \in B} f(k) = \sum_{k \in A \cup B} f(k) + \sum_{k \in A \cap B} f(k) \tag{1.9}$$

as a separate axiom for Σ manipulation. But this axiom is unnecessary in **[15]**, because it can be derived easily from other basic laws: The left-hand side is

$$\begin{aligned}\sum_{k\in A} f(k) + \sum_{k\in B} f(k) &= \sum_k f(k)[k\in A] + \sum_k f(k)[k\in B]\\ &= \sum_k f(k)([k\in A]+[k\in B])\end{aligned}$$

and the right-hand side is the same, because we have

$$[k\in A]+[k\in B] = [k\in A\cup B]+[k\in A\cap B]. \tag{1.10}$$

The interchange of summation order in multiple sums also comes out simpler now. I used to have trouble understanding and/or explaining why

$$\sum_{j=1}^{n}\sum_{k=1}^{j} f(j,k) = \sum_{k=1}^{n}\sum_{j=k}^{n} f(j,k); \tag{1.11}$$

but now it's easy for me to see that the left-hand sum is

$$\begin{aligned}\sum_{j,k} f(j,k)[1\le j\le n][1\le k\le j] &= \sum_{j,k} f(j,k)[1\le k\le j\le n]\\ &= \sum_{j,k} f(j,k)[1\le k\le n][k\le j\le n],\end{aligned}$$

and this is the right-hand sum.

Here's another example: We have

$$[k \text{ even}] = \sum_m [k=2m] \quad\text{and}\quad [k \text{ odd}] = \sum_m [k=2m+1]; \tag{1.12}$$

therefore,

$$\begin{aligned}\sum_k f(k) &= \sum_k f(k)([k\text{ even}]+[k\text{ odd}])\\ &= \sum_{k,m} f(k)[k=2m] + \sum_{k,m} f(k)[k=2m+1]\\ &= \sum_m f(2m) + \sum_m f(2m+1).\end{aligned} \tag{1.13}$$

The result in (1.13) is hardly surprising; but I like to have mechanical operations like this available so that I can do manipulations reliably, without thinking. Then I'm less apt to make mistakes.

Let lg stand for logarithms to base 2. Then we have

$$\begin{aligned}\sum_{k\ge1}\binom{n}{\lfloor \lg k\rfloor} &= \sum_{k\ge1}\sum_m \binom{n}{m}[m=\lfloor \lg k\rfloor]\\ &= \sum_{k,m}\binom{n}{m}[m\le \lg k< m+1][k\ge1]\\ &= \sum_{m,k}\binom{n}{m}[2^m\le k<2^{m+1}][k\ge1]\\ &= \sum_m \binom{n}{m}(2^{m+1}-2^m)[m\ge0]\\ &= \sum_m \binom{n}{m}2^m = 3^n.\end{aligned} \tag{1.14}$$

If we are doing infinite products we can use Iversonian brackets as exponents:

$$\prod_{P(k)} f(k) = \prod_{k} f(k)^{[P(k)]}. \qquad (1.15)$$

For example, the largest squarefree divisor of n is

$$\prod_{p} p^{[p \text{ prime}][p \text{ divides } n]}.$$

Everybody is familiar with one special case of an Iverson-like convention, the "Kronecker delta" symbol

$$\delta_{ik} = \begin{cases} 1, & i = k; \\ 0, & i \neq k. \end{cases} \qquad (1.16)$$

Leopold Kronecker introduced this notation in his work on bilinear forms [**30**, page 276] and in his lectures on determinants (see [**31**, page 316]); it soon became widespread. Many of his followers wrote δ_j^k, which is a bit more ambiguous because it conflicts with ordinary exponentiation. I now prefer to write $[j = k]$ instead of δ_{jk}, because Iverson's convention is much more general. Although '$[j = k]$' involves five written characters instead of the three in 'δ_{jk}', we lose nothing in common cases when '$[j = k + 1]$' takes the place of '$\delta_{j(k+1)}$'.

Another familiar example of a 0-1 function, this time from continuous mathematics, is Oliver Heaviside's unit step function $[x \geq 0]$. (See [**44**] and [**37**] for expositions of Heaviside's methods.) It is clear that Iverson's convention will be as useful with integration as it is with summation, perhaps even more so. I have not yet explored this in detail, because [**15**] deals mostly with sums.

It's interesting to look back into the history of mathematics and see how there was a craving for such notations before they existed. For example, an Italian count named Guglielmo Libri published several papers in the 1830s concerning properties of the function 0^{0^x}. He noted [**32**] that 0^x is either 0 (if $x > 0$) or 1 (if $x = 0$) or ∞ (if $x < 0$), hence

$$0^{0^x} = [x > 0]. \qquad (1.17)$$

But of course he didn't have Iverson's convention to work with; he was pleased to discover a way to denote the discontinuous function $[x > 0]$ without leaving the realm of operations acceptable in his day. He believed that "la fonction $0^{0^{x-n}}$ est d'un grand usage dans l'analyse mathématique." And he noted in [**33**] that his formulas "ne renferment aucune notation nouvelle... Les formules qu'on obtient de cette manière sont très simples, et rentrent dans l'algèbre ordinaire."

Libri wrote, for example,

$$(1 - 0^{0^{-x}})(1 - 0^{0^{x-a}})$$

for the function $[0 \leq x \leq a]$, and he gave the integral formula

$$\frac{2}{\pi}\int_0^\infty \frac{dq \cos qx}{1 + q^2} = e^x \cdot 0^{0^{-x}} + e^{-x}(1 - 0^{0^{-x}}) = \frac{e^x}{0^{-x} + 1} + \frac{e^{-x}}{0^x + 1}.$$

(Of course, we would now write the value of that integral as $e^{-|x|}$, but a simple notation for absolute value wasn't introduced until many years later. I believe that the first appearance of '$|z|$' for absolute value in Crelle's journal—the journal containing Libri's papers [**32**] and [**33**]—occurred on page 227 of [**56**] in 1881. Karl Weierstrass was the inventor of this notation, which was applied at first only to complex numbers; Weierstrass seems to have published it first in 1876 [**55**].)

Libri applied his 0^{0^x} function to number theory by exhibiting a complicated way to describe the fact that x is a divisor of m. In essence, he gave the following recursive formulation: Let $P_0(x) = 1$ and for $k > 0$ let

$$P_k(x) = 0^{0^{x-k}}P_0(x) - 0^{0^{x-k+1}}P_1(x) - \cdots - 0^{0^{x-1}}P_{k-1}(x).$$

Then the quantity

$$\frac{1 - m \cdot 0^{0^{x-m}}P_0(x) - (m-1)0^{0^{x-m+1}}P_1(x) - \cdots - 2 \cdot 0^{0^{x-2}}P_{m-2}(x) - 0^{0^{x-1}}P_{m-1}(x)}{x}$$

turns out to equal 1 if x divides m, otherwise it is 0. (One way to prove this, Iverson-wise, is to replace $0^{0^{x-k}}$ in Libri's formulas by $[x > k]$, and to show first by induction that $P_k(x) = [x \text{ divides } k] - [x \text{ divides } k-1]$ for all $k > 0$. Then if $a_k(x) = k[x > k]$, we have

$$\begin{aligned}\sum_{k=0}^{m-1} a_{m-k}(x)P_k(x) &= \sum_{k=0}^{m-1} a_{m-k}(x)([x \text{ divides } k] - [x \text{ divides } k-1])\\ &= \sum_{k=0}^{m-1} [x \text{ divides } k](a_{m-k}(x) - a_{m-k-1}(x)).\end{aligned}$$

If the positive integer x is not a divisor of m, the terms of this new sum are zero except when $m - k = m \bmod x$, when we have $a_{m-k}(x) - a_{m-k-1}(x) = 1$. On the other hand if x is a divisor of m, the only nonvanishing term occurs for $m - k = x$, when we have $a_{m-k}(x) - a_{m-k-1}(x) = 0 - (x - 1)$. Hence the sum is $1 - x\,[x \text{ divides } m]$. Libri obtained his complicated formula by a less direct method, applying Newton's identities to compute the sum of the mth powers of the roots of the equation $t^{x-1} + t^{x-2} + \cdots + 1 = 0$.)

Evidently Libri's main purpose was to show that unlikely functions can be expressed in algebraic terms, somewhat as we might wish to show that some complicated functions can be computed by a Turing Machine. "Give me the function 0^{0^x}, and I'll give you an expression for $[x \text{ divides } m]$." But our goal with Iverson's notation is, by contrast, to find a simple and natural way to express quantities that help us solve problems. If we need a function that is 1 if and only if x divides m, we can now write $[x \text{ divides } m]$.

Some of Libri's papers are still well remembered, but **[32]** and **[33]** are not. I found no mention of them in *Science Citation Index*, after searching through all years of that index available in our library (1955 to date). However, the paper **[33]** did produce several ripples in mathematical waters when it originally appeared, because it stirred up a controversy about whether 0^0 is defined. Most mathematicians agreed that $0^0 = 1$, but Cauchy [**5**, page 70] had listed 0^0 together with other expressions like $0/0$ and $\infty - \infty$ in a table of undefined forms. Libri's justification for the equation $0^0 = 1$ was far from convincing, and a commentator who signed his name simply "S" rose to the attack **[45]**. August Möbius **[36]** defended Libri, by presenting his former professor's reason for believing that $0^0 = 1$ (basically a proof that $\lim_{x \to 0+} x^x = 1$). Möbius also went further and presented a supposed proof that $\lim_{x \to 0+} f(x)^{g(x)} = 1$ whenever $\lim_{x \to 0+} f(x) = \lim_{x \to 0+} g(x) = 0$. Of course "S" then asked **[3]** whether Möbius knew about functions such as $f(x) = e^{-1/x}$ and $g(x) = x$. (And paper **[36]** was quietly omitted from the historical record when the collected works of Möbius were ultimately published.) The debate stopped there, apparently with the conclusion that 0^0 should be undefined.

But no, no, ten thousand times no! Anybody who wants the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{1.18}$$

to hold for at least one nonnegative integer n *must* believe that $0^0 = 1$, for we can plug in $x = 0$ and $y = 1$ to get 1 on the left and 0^0 on the right.

The number of mappings from the empty set to the empty set is 0^0. It *has* to be 1.

On the other hand, Cauchy had good reason to consider 0^0 as an undefined *limiting form*, in the sense that the limiting value of $f(x)^{g(x)}$ is not known *a priori* when $f(x)$ and $g(x)$ approach 0 independently. In this much stronger sense, the value of 0^0 is less defined than, say, the value of $0 + 0$. Both Cauchy and Libri were right, but Libri and his defenders did not understand why truth was on their side.

Well, it's instructive to study mathematical history and to observe how tastes change as progress is made. But let's come closer to the present, to see how Iverson's convention might be useful nowadays. Today's mathematical literature is, in fact, filled with instances where analogs of Iversonian brackets are being used—but the concept must be expressed in a roundabout way, because his convention is not yet established. Here are two examples that I happened to notice just before writing this paper:

(1) Hardy and Wright, in the course of proving the Staudt-Clausen theorem about the denominators of Bernoulli numbers [**20**, §7.9], consider the sum

$$\sum_{p-1 \text{ divides } k} \frac{1}{p}$$

where p runs through primes. They define $\varepsilon_k(p)$ to be 1 if $p - 1$ divides k, otherwise $\varepsilon_k(p) = 0$; then the sum becomes

$$\sum_p \frac{\varepsilon_k(p)}{p}.$$

They proceed to show that $\sum_{m=1}^{p-1} m^k \equiv -\varepsilon_k(p) \pmod p$ whenever p is prime, and the theorem follows with a bit more manipulation.

(2) Mark Kac, introducing the relation of ergodic theory to continued fractions [**24**, §5.4], says: "Let now $P_0 \in \Omega$ and $g(P)$ the characteristic function of the measurable set A; i.e.,

$$g(P) = \begin{cases} 1, & p \in A, \\ 0, & p \overline{\in} A. \end{cases}$$

It is now clear that $t(\tau, P_0, A)$ is given by the formula

$$t(\tau, P_0, A) = \int_0^{\tau} g(T_t(P_0))\, dt,$$

and... ".

I hope it is now clear why my students and I would find it quite natural to say directly that

$$t(\tau, P_0, A) = \int_0^{\tau} [T_t(P_0) \in A]\, dt.$$

Also, in the context of Hardy and Wright, we would evaluate $(\sum_{m=1}^{p-1} m^k) \bmod p$ and discover that it is $(p-1)[p-1 \text{ divides } k]$.

If you are a typical hard-working, conscientious mathematician, interested in clear exposition and sound reasoning—and I like to include myself as a member of that set—then your experiences with Iverson's convention may well go through several stages, just as mine did. First, I learned about the idea, and it certainly seemed straightforward enough. Second, I decided to use it informally while solving problems. At this stage it seemed too easy to write just '$[k \geq 0]$'; my natural tendency was to write something like '$\delta(k \geq 0)$', giving an implicit bow to Kronecker, or '$\tau(k \geq 0)$' where τ stands for truth. Adriano Garsia, similarly, decided to write '$\chi(k \geq 0)$', knowing that χ often denotes a characteristic function; he has used χ notation effectively in dozens of papers, beginning with **[10]**, and quite a few other mathematicians have begun to follow his lead. (Garsia was one of my professors in graduate school, and I recently showed him the first draft of this note. He replied, "My definition from the very start was

$$\chi(\mathscr{A}) = \begin{cases} 1 & \text{if } \mathscr{A} \text{ is true} \\ 0 & \text{if } \mathscr{A} \text{ is false} \end{cases}$$

where $\mathscr{A}$ is any statement whatever. But just like you, I got it by generalizing from Iverson's APL. ... I don't have to tell you the magic that the use of the χ notation can do.")

If you go through the stages I did, however, you'll soon tire of writing δ, τ, or χ, when you recognize that the notation is quite unambiguous without an additional symbol. Then you will have arrived at the philosophical position adopted by Iverson when he wrote **[21]**. And I had also reached that stage when I completed the first edition of **[15]**; I adopted Iverson's original suggestion to enclose logical statements in ordinary parentheses, not square brackets.

Unfortunately, not all was well with that first edition. Students found cases where I had parenthesized a complicated logical statement for clarity, for example when I wrote something of the form 'α and (β or γ)'; they pointed out that the simple act of putting parentheses around 'β or γ' automatically caused it to be evaluated as either 0 or 1, according to a strict interpretation of Iverson's rule as I had extended it.

Worse yet, as I began to read the first edition of **[15]** with fresh eyes, I found that the formulas involved too many parentheses. It was hard for me to perceive the structure of complex expressions that involved Iversonian statements; the statements had been clear to me when I wrote them down, but they looked confusing when I came back to them several months later. A computer could readily parse each expression, but good notation must be engineered for human beings.

Therefore in the second and subsequent printings of **[15]**, my co-authors and I now use square brackets instead of parentheses, whenever we wish to transform logical statements into the values 0 or 1. This resolves both problems, and we now believe that the notation has proved itself well enough to be thrust upon the world. Square brackets are used also for other purposes, but not in a conflicting way, and not so often that the multiple uses become confusing.

One small glitch remains: We want to be able to write things like

$$\sum_p [p \text{ prime}][p \leq x]/p \tag{1.19}$$

to denote the sum of all reciprocals of primes $\leq x$. But this summand unfortunately reduces to $0/0$ when $p = 0$. In general, when an Iverson-bracketed statement is false, we want it to evaluate into a "very strong 0," namely a zero so strong

that it annihilates anything it is multiplied by—even if that other factor is undefined.

Similarly, in formulas like (1.2) it is convenient to regard $\binom{n}{k}$ as strongly zero when k is negative, so that, for example, $\binom{n}{-10}z^{-10} = 0$ when $z = 0$.

The strong-zero convention is enough to handle 99% of the difficult situations, but we may also be using $1 - [P(k)]$ to stand for the quantity [not $P(k)$]; then we want $[P(k)]$ to give a "strong 1." And paradoxes can still arise, whenever irresistible forces meet immovable objects. (What happens if a strong zero appears in the denominator? And so on.)

In spite of these potential problems in extreme cases, Iverson's convention works beautifully in the vast majority of applications. It is, in fact, far less dangerous than most of the other notations of mathematics, whose dark corners we have learned to avoid long ago. The safe use of Iverson's simple and convenient idea is quite easy to learn.

2. STIRLING NUMBERS. The second plea I wish to make for perspicuous notation concerns the famous coefficients introduced by James Stirling at the beginning of his *Methodus Differentialis* in 1730 [**52**]. The lack of a widely accepted way to refer to these numbers has become almost scandalous. For example, Goldberg, Newman, and Haynsworth begin their chapter on Combinatorial Analysis in the NBS Handbook [**1**] by remarking that notations for Stirling numbers "have never been standardized... We feel that a capital S is natural for Stirling numbers of the first kind; it is infrequently used for other notation in this context. But once it is used we have difficulty finding a suitable symbol for Stirling numbers of the second kind. These numbers are sufficiently important to warrant a special and easily recognizable symbol, and yet that symbol must be easy to write. We have settled on a script capital $\mathscr{S}$ without any certainty that we have settled this question permanently."

The present predicament came about because Stirling numbers are indeed important enough to have arisen in a wide variety of applications, yet they are not quite important enough to have deserved a prominent place in the most influential textbooks of mathematics. Therefore they have been rediscovered many times, and each author has chosen a notation that was optimized for one particular application.

The great utility of Stirling numbers has become clearer and clearer with time, and mathematicians have now reached a stage where we can intelligently choose a notation that will serve us well in the whole range of applications.

I came into the picture rather late, having never heard of Stirling numbers until after receiving my Ph.D. in mathematics. But I soon encountered them as I was beginning to analyze the performance of algorithms and to write the manuscript for my books *The Art of Computer Programming*. I quickly realized the truth of Imanuel Marx's comment that "these numbers have similarities with the binomial coefficients $\binom{n}{k}$; indeed, formulas similar to those known for the binomial coefficients are easily established" [**35**]. In order to emphasize those similarities and to facilitate pattern recognition when manipulating formulas, Marx recommended using bracket symbols $\left[{n \atop k}\right]$ for Stirling numbers of the first kind and brace symbols $\left\{{n \atop k}\right\}$ for Stirling numbers of the second kind. A similar proposal was being made at about the same time in Italy by Antonio Salmeri [**46**].

I was strongly motivated by Charles Jordan's book, *Calculus of Finite Differences* **[23]**, which introduced me to the important analogies between sums of factorial powers and integrals of ordinary powers. But I kept getting mixed up when I tried to use Stirling numbers as he defined them, because half of his "first kind" numbers were negative and the other half were positive. I had similar problems with Marx's suggestions in **[35]**; he made all Stirling numbers of the first kind positive, but then he attached a minus sign to half the numbers of the *second* kind. I decided that I'd never be able to keep my head above water unless I worked with Stirling numbers that were entirely signless.

And I soon learned that the signless Stirling numbers have important combinatorial significance. So I decided to try a definition that combined the best qualities of the other notations I'd seen; I defined the quantities $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$ as follows:

$$\left[{n \atop k}\right] = \text{the number of permutations of } n \text{ objects having } k \text{ cycles;}$$

$$\left\{{n \atop k}\right\} = \text{the number of partitions of } n \text{ objects into } k \text{ nonempty subsets.}$$

For example, $\left[{4 \atop 2}\right] = 11$, because there are eleven different ways to arrange four elements into two cycles:

[1, 2, 3] [4]	[1, 2, 4] [3]	[1, 3, 4] [2]	[2, 3, 4] [1]
[1, 3, 2] [4]	[1, 4, 2] [3]	[1, 4, 3] [2]	[2, 4, 3] [1]
[1, 2] [3, 4]	[1, 3] [2, 4]	[1, 4] [2, 3].	

And $\left\{{4 \atop 2}\right\} = 7$, because the partitions of $\{1, 2, 3, 4\}$ into two subsets are

{1, 2, 3}{4}	{1, 2, 4}{3}	{1, 3, 4}{2}	{2, 3, 4}{1}
{1, 2}{3, 4}	{1, 3}{2, 4}	{1, 4}{2, 3}.	

Notice that this notation is mnemonic: The meaning of $\left\{{n \atop k}\right\}$ is easily remembered, because braces { } are commonly used to denote sets and subsets. We could also adopt the convention of writing cycles in brackets, as in my examples above, where $[1, 2, 3] = [2, 3, 1] = [3, 1, 2]$ is a typical three-cycle; that would make the notation $\left[{n \atop k}\right]$ equally mnemonic. But I don't insist on this.

I have never decided how to pronounce '$\left[{n \atop k}\right]$' and '$\left\{{n \atop k}\right\}$' when I'm reading formulas aloud in class. Many people have begun to verbalize '$\binom{n}{k}$' as "n choose k"; hence I've been saying "n cycle k" for $\left[{n \atop k}\right]$ and "n subset k" for $\left\{{n \atop k}\right\}$. But I have also caught myself calling them "n bracket k" and "n brace k."

One of the advantages of these notational conventions is that binomial coefficients and Stirling numbers can be defined by very simple recurrence relations having a nice pattern:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}; \tag{2.1}$$

$$\left[{n+1 \atop k}\right] = n\left[{n \atop k}\right] + \left[{n \atop k-1}\right]; \tag{2.2}$$

$$\left\{{n+1 \atop k}\right\} = k\left\{{n \atop k}\right\} + \left\{{n \atop k-1}\right\}. \tag{2.3}$$

Moreover—and this is extremely important—these identities hold for all integers n and k, whether positive, negative, or zero. Therefore we can apply them in the

midst of any formula (for example, to "absorb" an n or a k that appears in the context $n\left[{n \atop k}\right]$ or $k\left\{{n \atop k}\right\}$), without worrying about exceptional circumstances of any kind.

I introduced these notations in the first edition of my first book **[25]**, and by now my students and I have accumulated some 25 years of experience with them; the conventions have served us well. However, such brackets and braces have still not become widely enough adopted that they could be considered "standard." For example, Stanley's magnificent book on *Enumerative Combinatorics* **[51]** uses $c(n,k)$ for $\left[{n \atop k}\right]$ and $S(n,k)$ for $\left\{{n \atop k}\right\}$. His notation conveys combinatorial significance, but it fails to suggest the analogies to binomial coefficients that prove helpful in manipulations. Such analogies were evidently not important enough in his mind to warrant an extravagant two-line notation—although he does use $\left(\!\binom{n}{k}\!\right)$ to denote $\binom{n+k-1}{k} = (-1)^k\binom{-n}{k}$, the number of combinations with repetitions permitted. (In a sense, Stanley's $\left(\!\binom{n}{k}\!\right)$ is a signless version of the numbers $\binom{-n}{k}$.)

When I wrote *Concrete Mathematics* in 1988, I explored Stirling numbers more carefully than I had ever done before, and I learned two things that really clinch the argument for $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$ as the best possible Stirling number notations. Ron Graham sent me a preview copy of a memorandum by B. F. Logan **[34]**, which presented a number of interesting connections between Stirling numbers and other mathematical quantities. One of the first things that caught my attention was Logan's Table 1, a two-dimensional array that contained the numbers $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$ simultaneously—implying that there really is only one "kind" of Stirling number. Indeed, when I translated Logan's results into my own favorite notation, I was astonished to find that his arrangement of numbers was equivalent to a beautiful and easily remembered law of duality,

$$\left\{{n \atop k}\right\} = \left[{-k \atop -n}\right]. \tag{2.4}$$

Once I had this clue, it was easy to check that the recurrence relations (2.2) and (2.3) are equivalent to each other. And the boundary conditions

$$\left[{0 \atop k}\right] = \left\{{0 \atop k}\right\} = [k=0] \quad \text{and} \quad \left[{n \atop 0}\right] = \left\{{n \atop 0}\right\} = [n=0] \tag{2.5}$$

yield unique solutions to (2.2) and (2.3) for all integers k and n, when we run the recurrences forward and backward; the "negative" region for Stirling numbers of one kind turns out to contain precisely the numbers of the other kind. For example, the following subset of Logan's table gives the values of $\left[{n \atop k}\right]$ when $|n|$ and $|k|$ are at most 4:

	$k=-4$	$k=-3$	$k=-2$	$k=-1$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$n=-4$	1	0	0	0	0	0	0	0	0
$n=-3$	6	1	0	0	0	0	0	0	0
$n=-2$	7	3	1	0	0	0	0	0	0
$n=-1$	1	1	1	1	0	0	0	0	0
$n=0$	0	0	0	0	1	0	0	0	0
$n=1$	0	0	0	0	0	1	0	0	0
$n=2$	0	0	0	0	0	1	1	0	0
$n=3$	0	0	0	0	0	2	3	1	0
$n=4$	0	0	0	0	0	6	11	6	1

The reflection of this matrix about a 45° diagonal gives the value of $\left\{{n \atop k}\right\} = \left[{-k \atop -n}\right]$.

Naturally I wondered how I could have been working with Stirling numbers for so many years without having been aware of such a basic fact. Surely it must have been known before? After several hours of searching in the library, I learned that identity (2.4) had indeed been known, but largely forgotten by succeeding generations of mathematicians, primarily because previous notations for Stirling numbers made it impossible to state the identity in such a memorable form. These investigations also turned up several things about the history of Stirling numbers that I had not previously realized.

During the nineteenth century, Stirling's connection with these numbers had been almost entirely forgotten. The numbers themselves were studied, in the role of "sums of products of combinations of the numbers $\{1, 2, \ldots, n\}$ taken k at a time." Let $C_k(n)$ and $\Gamma_k(n)$ denote those sums, when the combinations are respectively without or with repetitions; thus, for example,

$$\begin{aligned} C_4(4) &= 1 \cdot 2 \cdot 3 + 1 \cdot 2 \cdot 4 + 1 \cdot 3 \cdot 4 \cdot + 2 \cdot 3 \cdot 4 = 50; \\ \Gamma_3(3) &= 1 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 2 + 1 \cdot 1 \cdot 3 + 1 \cdot 2 \cdot 2 + 1 \cdot 2 \cdot 3 \\ &\qquad + 1 \cdot 3 \cdot 3 + 2 \cdot 2 \cdot 2 + 2 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 3 + 3 \cdot 3 \cdot 3 = 90. \end{aligned}$$

It turns out that

$$C_k(n) = \left[{n+1 \atop n+1-k} \right] \quad \text{and} \quad \Gamma_k(n) = \left\{ {n+k \atop n} \right\}. \tag{2.6}$$

Christian Kramp [**28**] proved near the end of the eighteenth century that

$$C_k(n) = \sum \binom{n+1}{k+l} \frac{(k+l)!}{j_1!\,2^{j_1}\,j_2!\,3^{j_2} j_3!\,4^{j_3} \cdots}, \tag{2.7}$$

$$\Gamma_k(n) = \sum \binom{n+k}{k+l} \frac{(k+l)!}{j_1!\,2!^{j_1}\,j_2!\,3!^{j_2} j_3!\,4!^{j_3} \cdots}, \tag{2.8}$$

where the sums are over all sequences of nonnegative integers $\langle j_1, j_2, j_3, \ldots \rangle$ such that we have $j_1 + 2j_2 + 3j_3 + \cdots = k$ (i.e., over all partitions of k), and where $l = j_1 + j_2 + j_3 + \cdots$. For example,

$$C_2(n) = \binom{n+1}{4}\frac{1}{8} + \binom{n+1}{3}\frac{1}{3}; \quad \Gamma_2(n) = \binom{n+2}{4}\frac{1}{8} + \binom{n+2}{3}\frac{1}{6}.$$

Notice that $C_k(n)$ and $\Gamma_k(n)$ are polynomials in n, of degree $2k$. The duality law (2.4) and the notational transformations of (2.6) are equivalent to the amazing polynomial identity

$$C_k(n-1) = \Gamma_k(-n); \tag{2.9}$$

but hardly anybody was aware of this surprising fact, otherwise we would almost certainly find it mentioned explicitly in the comprehensive surveys compiled in the 1890s [**19**, **38**].

On the other hand, a rereading of Stirling's original treatment [**52**] makes it clear that Stirling himself would not have found the duality law (2.4) at all surprising. From the very beginning, he thought of the numbers as two triangles hooked together in tandem. Indeed, his entire motivation for studying them was the general identity

$$z^n = \sum_k \left\{ {n \atop k} \right\} z^{\underline{k}}, \tag{2.10}$$

which expresses ordinary powers in terms of falling factorial powers. When n is

positive, the nonzero terms in this sum occur for positive values of $k \le n$; but when n is negative, the nonzero terms occur for negative $k \le n$. Stirling presented his tables by displaying $\left\{ {n \atop k} \right\}$ with k as the row index and $\left[{n \atop k} \right]$ with k as the column index; thus, he visualized a tandem arrangement exactly as in the matrix of numbers above, with each column containing a sequence of coefficients for (2.10).

I need to digress a bit about factorial powers. If n is a positive integer and z is a complex number, I like to write

$$z^{\underline{n}} = z(z-1)\dots(z-n+1), \tag{2.11}$$

which I call "z to the n falling," and

$$z^{\overline{n}} = z(z+1)\dots(z+n-1), \tag{2.12}$$

which is "z to the n rising." More generally, if α is any complex number, factorial powers are defined by

$$z^{\underline{\alpha}} = z!/(z-\alpha)! \quad \text{and} \quad z^{\overline{\alpha}} = \Gamma(z+\alpha)/\Gamma(z), \tag{2.13}$$

unless these formulas reduce to ∞/∞ (when limiting values are used). My use of underlined and overlined exponents is still controversial, but I cannot resist mentioning a curious fact: Many people (e.g., specialists in hypergeometric series) have become accustomed to the notation $(z)_n$ for rising factorial powers, while many other people (e.g., statisticians) use the same notation for *falling* powers. The curious fact is that this notation is called "Pochhammer's symbol," but Pochhammer himself **[43]** used $(z)_n$ to stand for the binomial coefficient $\binom{z}{n}$. I prefer the underline/overline notation because it is unambiguous and mnemonic, especially when I'm doing work that involves factorial powers of both kinds. (Moreover, I know that $z^{\underline{n}}$ and $z^{\overline{n}}$ are easy to typeset, using macros available in the file `gkpmac.tex` in the standard UNIX distribution of $\TeX$.)

In the special case $n = 3$, Stirling's formula (2.10) gives

$$z^3 = \left\{ {3 \atop 3} \right\} z^{\underline{3}} + \left\{ {3 \atop 2} \right\} z^{\underline{2}} + \left\{ {3 \atop 1} \right\} z^{\underline{1}} = z(z-1)(z-2) + 3z(z-1) + z.$$

And in the special case $n = -1$, it reduces to the infinite sum

$$\begin{aligned} \frac{1}{z} &= \sum_k \left\{ {-1 \atop k} \right\} z^{\underline{k}} \\ &= \sum_k \left[{k \atop 1} \right] z^{\underline{-k}} \\ &= \frac{0!}{z+1} + \frac{1!}{(z+1)(z+2)} + \frac{2!}{(z+1)(z+2)(z+3)} + \cdots, \end{aligned} \tag{2.14}$$

because

$$\left[{n \atop 1} \right] = (n-1)!\,[n > 0]. \tag{2.15}$$

(Stirling did not discuss convergence; he was, after all, writing in 1730. We have the partial sum

$$\frac{1}{z} = \sum_{k=1}^{n} \frac{(k-1)!}{(z+1)\dots(z+k)} + \frac{n!}{z(z+1)\dots(z+n)};$$

this is a special case of the general identity

$$\frac{1}{z} = \sum_{k=1}^{n} \frac{z_1 \dots z_{k-1}}{(z+z_1)\dots(z+z_k)} + \frac{z_1 \dots z_n}{z(z+z_1)\dots(z+z_n)} \tag{2.16}$$

discovered by François Nicole [**39**] a few years before Stirling's treatise appeared. Therefore the infinite series (2.14) converges if and only if $\mathrm{Re}(z) > 0$. By induction on n, the same condition is necessary and sufficient for (2.10) when n is any negative integer. See [**41**, §30] for further discussion of (2.10).)

We noted above that the numbers $\left[{m \atop k}\right]$ can be regarded as sums of products of combinations. The first identity in (2.6) is equivalent to the formula

$$z^{\overline{n}} = \sum_k \left[{n \atop k}\right] z^k, \tag{2.17}$$

when n is a nonnegative integer, if we expand the product $z^{\overline{n}}$ and sum the coefficients of each power of z. Similarly, we have

$$z^{\underline{n}} = \sum_k \left[{n \atop k}\right] (-1)^{n-k} z^k. \tag{2.18}$$

These equations are valid also when n is a negative integer; in that case both infinite series converge for $|z| > |n|$. Notice that (2.10) and (2.18) tell us how to convert back and forth between ordinary powers and factorial powers.

Let's turn now to the nineteenth century. Kramp [**29**] decided to explore a slightly generalized type of factorial power, for which he used the notations

$$a^{n|r} = a(a+r)\dots(a+(n-1)r) \tag{2.19}$$

$$a^{-n|r} = 1/(a-r)(a-2r)\dots(a-nr) \tag{2.20}$$

when n is a positive integer. Then he considered the expansion

$$a^{n|r} = a^n + n\mathfrak{f}1.a^{n-1}r + n\mathfrak{f}2.a^{n-2}r^2 + \cdots, \tag{2.21}$$

where the coefficients $n\mathfrak{f}m$ are independent of a and r [**29**, §§539–540]; thus $n\mathfrak{f}m$ was his notation for $\left[{n \atop n-m}\right]$. He obtained [**29**, §557] a series of formulas equivalent to

$$m\left[{n \atop n-m}\right] = \sum_{k=0}^{m-1} \binom{n-k}{m+1-k} \left[{n \atop n-k}\right], \tag{2.22}$$

thereby giving a new proof that $\left[{n \atop n-m}\right]$ is a polynomial in n of degree $2m$. This proof, independent of his earlier formulas (2.7) and (2.8), works for both positive and negative values of n.

Kramp implicitly understood the duality principle (2.4), in the sense that he regarded the coefficients $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$ as the positive and negative portions of a doubly infinite array of numbers. In fact, he assumed that equation (2.21) would hold for arbitrary real values of n. He differentiated $a^{x|r}$ with respect to x and gave formal derivations of several interesting series. However, his expansion (2.21) is equivalent to

$$z^{\overline{n}} = \sum_k \left[{n \atop n-k}\right] z^{n-k} \tag{2.23}$$

(a slight variation of (2.17)), and this series is not always convergent for noninteger

n. We can show, for example, that

$$\left\| \begin{bmatrix} 1/2 \\ 1/2 - k \end{bmatrix} \right\| > k!/7^k \quad \text{for infinitely many } k; \tag{2.24}$$

hence (2.23) diverges for all z when $n = 1/2$. Kramp lived before the days when convergence of infinite series was understood. (See [**29**, §574], where he says that the divergent series $\sum_{k>0} B_k y^k/k$ is "très convergente pour peu que y soit une petite fraction"!)

Several other nineteenth-century authors developed the theory of factorial powers, notably Andreas von Ettingshausen [**6**], Ludwig Schläfli [**41**, **48**], and Oskar Schlömilch [**49**], who used the respective notations

$$\overset{n}{\mathrm{F}}_m, \ \overset{n}{\mathrm{A}}_m, \quad \text{and} \quad \overset{n}{\mathrm{C}}_m$$

for the coefficients $\begin{bmatrix} n \\ n-m \end{bmatrix}$. All of these authors considered both positive and negative integers n. Thus, for example, Ettingshausen's notation for a Stirling number such as $\left\{ \begin{matrix} n+m \\ n \end{matrix} \right\} = \begin{bmatrix} -n \\ -n-m \end{bmatrix}$ was

$$\overset{-n}{\mathrm{F}}_m$$

(see [**6**, §151]).

Incidentally, these works of Kramp and Ettingshausen proved to be important in the history of mathematical notations. Kramp's book introduced the notation $n!$ for factorials [**29**, pages V and 219], and Ettingshausen's book introduced the notation $\binom{n}{k}$ for binomial coefficients [**6**, page 30]. Ettingshausen wrote his book shortly after Fourier [**8**] had invented Σ-notation for sums; Ettingshausen tried a German variation, writing $\mathfrak{S}^k_{a,b}$ for what has evolved into $\sum_{k=a}^b$. He also wrote $(a, r)^n$ for Kramp's $a^{n|r}$; thus, for example, Ettingshausen [**6**, §153 and §156] gave the equations

$$(a, d)^n = \overset{w}{\underset{0}{\mathfrak{S}}} \overset{n}{\mathrm{F}}_w a^{n-w} d^w \quad \text{and} \quad a^n = \overset{r}{\underset{0}{\mathfrak{S}}} (-1)^r \overset{-n+r}{\mathrm{F}_r} (a, d)^{n-r} d^r$$

as equivalents of Kramp's (2.21) and Stirling's (2.10). He presented Kramp's (2.22) in the form

$$v \overset{n}{\mathrm{F}}_v = \overset{w}{\underset{0,\, v-1}{\mathfrak{S}}} \binom{n-w}{v+1-w} \overset{n}{\mathrm{F}}_w,$$

and remarked [**6**, §154] that this holds for both negative and positive n. Ettingshausen had related the F coefficients to sums of products of combinations with and without repetition; thus he implicitly confirmed (2.9).

The first person to attach Stirling's name to the numbers we now call Stirling numbers was Niels Nielsen in 1904 [**40**]; he said that this new nomenclature had been suggested to him by T. N. Thiele. (The numbers may have been studied before Stirling's time; for example, I once found the values of $\begin{bmatrix} n \\ k \end{bmatrix}$ for $1 \le n \le 7$ in some unpublished manuscripts of Thomas Harriot, dating from about 1600, in the British Museum [**26**, page 241]. But Stirling almost surely deserves the credit for being first to deduce nontrivial facts about $\begin{bmatrix} n \\ k \end{bmatrix}$ and $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$.)

Nielsen wrote C_n^k for $\begin{bmatrix} n \\ n-k \end{bmatrix}$, which he called a "Stirling number of rank n"; and he wrote $\mathfrak{C}_n^k$ for $\left\{ \begin{matrix} n+k-1 \\ n-1 \end{matrix} \right\}$, which he called a "Stirling number of rank $-n$." (He should really have defined its rank to be $1 - n$). In equation (41) of his paper,

Nielsen obtained a rigorous proof of the duality law (2.4); but he had to state it in a peculiar way, because he had defined C_n^k and $\mathfrak{C}_n^k$ only for nonnegative n and k. Thus, he could not write $C_n^k = \mathfrak{C}_{1-n}^k$; he had to say instead that $f_k(n) = g_k(1-n)$, where $f_k(n)$ and $g_k(n)$ were the polynomials defined by C_n^k and $\mathfrak{C}_n^k$. Tweedie [**54**] expressed (2.4) with similar circumlocutions.

When Jordan took up Stirling numbers [**22**], he wrote S_n^k for $(-1)^{n-k}\left[{n \atop k}\right]$ and $\mathfrak{S}_n^k$ for $\left\{{n \atop k}\right\}$. He does not seem to have known the duality law (2.4), probably because he had learned about Stirling numbers from Nielsen's book [**41**], which omitted some of the details in Nielsen's paper [**40**]. And as far as I know, the duality law largely disappeared from mathematicians' collective consciousness during most of the twentieth century; it seems to have been mentioned explicitly only in a few scattered places: (1) Hansraj Gupta, "working in a small township away from what was then the only University in the Panjab" [**18**, page 5], rediscovered Stirling numbers and Stirling duality by himself, in the early 1930s. This became part of his Ph.D. dissertation [**17**], and he included it in a book on number theory prepared many years later [**18**, Chapter 5]. (2) H. W. Gould [**12**] was probably the first twentieth-century mathematician to observe that we can use the polynomials $\left[{n \atop n-k}\right]$ and $\left\{{n \atop n-k}\right\}$ to extend the domain of Stirling numbers to negative values of n. Gould's way of writing (2.4) was $S_1(-n-1, k) = S_2(n, k)$; and shortly thereafter [**13**], he mentioned the equivalent formula

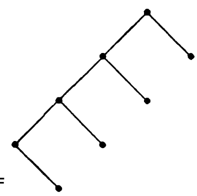
$$S_{-k}^{-n} = (-1)^{n-k}\mathfrak{S}_n^k,$$

in Jordan's notation. (3) R. V. Parker [**42**], like Gupta, displayed both of Stirling's triangles in tandem, presenting them in a single table as Logan later did. (4) In 1976, Ira Gessel and Richard Stanley investigated some of the deeper structure underlying the Stirling polynomials $f_k(n) = \left\{{n+k \atop n}\right\}$ and $g_k(n) = \left[{n \atop n-k}\right]$. They noted in particular [**11**, equation (3)] that $f_k(-n) = g_k(n)$. This fact is equivalent to the duality law (2.4).

Stanley had discovered a beautiful theorem in his Ph.D. thesis a few years earlier [**50**, Proposition 13.2(i)], now called the reciprocity theorem for order polynomials: If P is any finite partially ordered set, let $\Omega(P, n)$ be the number of order-preserving mappings from P into the totally ordered set $\{1, 2, \ldots, n\}$; and let $\overline{\Omega}(P, n)$ be the number of such mappings that are strictly order-preserving. Thus, if $x \prec y$ in P, the mappings f enumerated by $\Omega(P, n)$ must satisfy $f(x) \le f(y)$, and the mappings g enumerated by $\overline{\Omega}(P, n)$ must satisfy $g(x) < g(y)$. Stanley's theorem states that, in general, we have $f(-n) = (-1)^p g(n)$, where p is the number of elements of P. For example, if P consists of p isolated points with no order constraints whatever, we have $\Omega(P, n) = \overline{\Omega}(P, n) = n^p$. And if the points of P are themselves totally ordered, then $\Omega(P, n)$ is $\binom{n+p-1}{p}$, the number of combinations of n things p at a time with repetitions permitted, and $\overline{\Omega}(P, n)$ is $\binom{n}{p}$, the combinations without repetition. In both cases we have $\Omega(P, -n) = (-1)^p\overline{\Omega}(P, n)$.

I showed Stanley the first draft of this note and asked him whether the Stirling duality law (2.4) could be derived as a special case of his general reciprocity law. Sure enough, he replied that Gessel had noticed a simple way to do exactly that, shortly after the paper [**11**] was written. Let P_k be the partial order on $2k$ points

typified by

$P_4 =$  ;

then

$$\Omega(P_k, n) = \sum_{1 \le x_1, \ldots, x_k, y_1, \ldots, y_k \le n} [x_1 \le \cdots \le x_k][x_1 \ge y_1] \ldots [x_k \ge y_k]$$
$$= \sum_{1 \le x_1, \ldots, x_k \le n} [x_1 \le \cdots \le x_k] x_1 \ldots x_k,$$

and

$$\overline{\Omega}(P_k, n) = \sum_{1 \le x_1, \ldots, x_k, y_1, \ldots, y_k \le n} [x_1 < \cdots < x_k][x_1 > y_1] \ldots [x_k > y_k]$$
$$= \sum_{2 \le x_1, \ldots, x_k \le n} [x_1 < \cdots < x_k](x_1 - 1) \ldots (x_k - 1)$$
$$= \sum_{1 \le x_1, \ldots, x_k \le n-1} [x_1 < \cdots < x_k] x_1 \ldots x_k.$$

Thus the sums are, respectively, $\Gamma_k(n)$ and $C_k(n-1)$; by (2.6) we have $\Omega(P_k, n) = \left\{ {n+k \atop n} \right\}$ and $\overline{\Omega}(P_k, n) = \left[{n \atop n-k} \right]$, hence (2.4) is indeed an instance of Stanley's theorem.

Now we are ready to discuss the second reason why I became convinced that $\left[{n \atop k} \right]$ is the right symbolism for these coefficients after I had translated Logan's memo [**34**] into that notation: We know that $\left[{n \atop n-k} \right]$ is a polynomial in n, when k is an integer; hence, as Kramp knew, we can sensibly define the quantity $\left[{\alpha \atop \alpha-k} \right]$ for arbitrary complex α and integer k, using that same polynomial. Then—and here comes the punch line—Logan noticed that the fundamental equations (2.17) and (2.18) generalize to *asymptotic formulas*, valid for arbitrary exponents α: If $z \to \infty$ and if m is any nonnegative integer, we have

$$z^{\overline{\alpha}} = \sum_{k=0}^{m} \left[{\alpha \atop \alpha - k} \right] z^{\alpha - k} + O(z^{\alpha - m - 1}); \tag{2.25}$$

$$z^{\underline{\alpha}} = \sum_{k=0}^{m} \left[{\alpha \atop \alpha - k} \right] (-1)^k z^{\alpha - k} + O(z^{\alpha - m - 1}). \tag{2.26}$$

(See [**15**, exercise 9.44]; equation (2.25) is a correct way to formulate Kramp's divergent series (2.23). These equations are special cases of a still more general result proved by Tricomi and Erdélyi [**53**, **9**].) The easily remembered expansions in (2.25) and (2.26) were quite a revelation to me. I had often spent time laboriously calculating approximations to ratios such as $z^{\overline{1/2}} = \Gamma(z + 1/2)/\Gamma(z)$, the hard way: I took logarithms, then used Stirling's approximation, and then took exponentials. But equations (2.25) and (2.26) produce the answer directly.

Moreover Stirling's original identity (2.10) can be generalized in a similar way: If α is any complex number, we have

$$z^{\alpha} = \sum_k \left\{ {\alpha \atop \alpha - k} \right\} z^{\underline{\alpha - k}}, \qquad \mathrm{Re}(z) > 0. \tag{2.27}$$

When I wrote the first draft of this note, I knew only that the series (2.27) was convergent, and that it was asymptotically correct as $z \to \infty$; so I conjectured that equality might hold. Soon afterward, B. F. Logan found the following proof (although he naturally stated it in his own notation): Suppose first that $\mathrm{Re}(\alpha) < 1$. Then we have the well known identity

$$z^{\alpha - 1} = \frac{1}{\Gamma(1 - \alpha)} \int_0^{\infty} e^{-zt} t^{-\alpha}\, dt, \qquad \mathrm{Re}(z) > 0, \tag{2.28}$$

and we can substitute $e^{-t} = 1 - u$ to get

$$z^{\alpha - 1} = \frac{1}{\Gamma(1 - \alpha)} \int_0^1 (1 - u)^{z - 1} u^{-\alpha} \left(\frac{1}{u} \ln \frac{1}{1 - u} \right)^{-\alpha} du.$$

Now it turns out that the powers of $(1/u)\ln 1/(1 - u)$ generate the Stirling numbers $\left\{ {\alpha \atop \alpha - k} \right\} = \left[{k - \alpha \atop -\alpha} \right]$, in the sense that

$$\left(\frac{1}{u} \ln \frac{1}{1 - u} \right)^{-\alpha} = \sum_k \left\{ {\alpha \atop \alpha - k} \right\} \frac{u^k}{(k - \alpha) \ldots (1 - \alpha)}, \tag{2.29}$$

a series that converges for $|u| < 1$ (see [**15**, equations (6.45), (6.53), (7.50)]). Therefore

$$\begin{aligned} z^{\alpha} &= \sum_k \left\{ {\alpha \atop \alpha - k} \right\} \frac{z}{\Gamma(k + 1 - \alpha)} \int_0^1 (1 - u)^{z - 1} u^{k - \alpha}\, du \\ &= \sum_k \left\{ {\alpha \atop \alpha - k} \right\} \frac{\Gamma(z + 1)}{\Gamma(z + 1 + k - \alpha)} = \sum_k \left\{ {\alpha \atop \alpha - k} \right\} \frac{z!}{(z + k - \alpha)!}, \end{aligned}$$

and (2.27) is verified when $\mathrm{Re}(\alpha) < 1$. To complete the proof, we need only show that (2.27) holds for $\alpha + 1$ if it holds for α; but this is easy, because

$$\begin{aligned} z^{\alpha + 1} &= \sum_k \left\{ {\alpha \atop \alpha - k} \right\} z \cdot z^{\underline{\alpha - k}} \\ &= \sum_k \left\{ {\alpha \atop \alpha - k} \right\} \left(z^{\underline{\alpha + 1 - k}} + (\alpha - k) z^{\underline{\alpha - k}} \right) \\ &= \sum_k \left\{ {\alpha \atop \alpha - k} \right\} z^{\underline{\alpha + 1 - k}} + \sum_k \left\{ {\alpha \atop \alpha + 1 - k} \right\} (\alpha + 1 - k) z^{\underline{\alpha + 1 - k}} \\ &= \sum_k \left\{ {\alpha + 1 \atop \alpha + 1 - k} \right\} z^{\underline{\alpha + 1 - k}} \end{aligned}$$

by the basic recurrence equation (2.3).

Notice that in all of the general identities (2.25)–(2.27), as in the original formulas (2.10), (2.17), and (2.18) that inspired them, the lower index within the braces or brackets is the same as the exponent of z. This makes the relations easy to remember, by analogy with the binomial theorem

$$(1 + z)^{\alpha} = \sum_k \binom{\alpha}{k} z^k, \quad \text{when } |z| < 1. \tag{2.30}$$

Some readers will have been thinking, "This all looks fairly plausible, but unfortunately Knuth is overlooking a key point that ruins the whole proposal: We *can't* use the notation $\left[{n \atop k}\right]$ for Stirling numbers, because it has already been used for more than a century as the standard notation for Gauss's generalized binomial coefficients."

Well, there is a down side to every good idea, but this objection is not really severe. For one thing, the standard notation for Gaussian binomial coefficients involves a hidden parameter q, and it's not unusual for modern researchers to make transformations that change q. Therefore Gauss's notation is incomplete, and Andrews (for example) has used the notation $\left[{n \atop k}\right]_{q^2}$ for the Gaussian coefficient with q^2 as the hidden parameter [**2**, page 49]. Such examples suggest that it is appropriate to denote Gaussian binomials as $\binom{n}{k}_q$, especially since they reduce to ordinary binomials when $q = 1$. This notation also generalizes nicely to such things as Fibonomial coefficients $\binom{n}{k}_{\mathscr{F}}$; see [**27**]. We can then reserve the notation $\left[{n \atop k}\right]_q$ for a q-generalization of $\left[{n \atop k}\right]$. (This reverse strategy was unfortunately adopted in [**14**].) Secondly, I do not believe that any existing mathematical works, including books like [**2**] which use Gaussian coefficients extensively, would become seriously cluttered if the Gaussian $\left[{n \atop k}\right]$ were changed everywhere to $\binom{n}{k}_q$. Even so, such changes are not necessary; there is obviously no harm in beginning a mathematical paper or a book chapter or an entire book with a statement to the effect that "$\left[{n \atop k}\right]$ will denote a Gaussian binomial coefficient with parameter q in what follows." All notation can be redefined for special purposes. Therefore Stirling number enthusiasts are not encroaching on Gaussian territory when they write $\left[{n \atop k}\right]$, if they also mumble something about Stirling in order to set the context.

One further point is worth noting in conclusion: As soon as the notations $\left[{n \atop k}\right]$ and/or $\left\{{n \atop k}\right\}$ are adopted, there will no longer be a need to speak about Stirling numbers "of the first and second kind," except as a concession to history. Nielsen wrote a superb book [**41**], but he did the world a disservice by originating the *Erster Art* and *Zweier Art* terminology, because that terminology has no mnemonic value and is historically inaccurate. Stirling introduced the numbers $\left\{{n \atop k}\right\}$ first and brought in $\left[{n \atop k}\right]$ second. Indeed, practical applications have always tended to involve the numbers $\left\{{n \atop k}\right\}$ much more often than their $\left[{n \atop k}\right]$ counterparts. It seems far better to speak of $\left\{{n \atop k}\right\}$ as a Stirling subset number, and to call $\left[{n \atop k}\right]$ a Stirling cycle number. Then the names are tied to intuitive, student-friendly concepts, not to arbitrary and offputting concepts of the kth kind.

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Representing Primes by Binary Quadratic Forms

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The study of integral binary quadratic forms

$$f(x,y) = ax^2 + bxy + cy^2 \ (a, b, c \text{ integers})$$

has its origins in the work of Fermat, Euler, Lagrange, and Legendre (see for example [**5**, Chapter 1]). An integer n is said to be represented by f if there exist integers x and y such that $n = f(x, y)$. An important problem in the theory of binary quadratic forms is to determine the set of positive primes represented by $f(x, y)$. For this problem we restrict ourselves to those f which are (i) primitive, that is $\text{GCD}(a, b, c) = 1$; (ii) irreducible, that is the discriminant $D = b^2 - 4ac$ is not a square; and (iii) positive-definite if $D < 0$. This avoids those f which represent at most one prime or for which the representation problem can be solved by factoring f. If f satisfies (i), (ii), (iii) it will be called a form for short. Dirichlet in 1840 (see [**7**, Vol. I, pp. 497–502]) was the first to show that a form $ax^2 + bxy + cy^2$ represents infinitely many primes for a certain class of discriminants and Weber [**10**] in 1882 was the first to give a proof valid for any discriminant.

In the seventeenth century Fermat characterized the set of primes represented by the form $x^2 + y^2$. He showed that this set consists of the prime 2 together with all primes $p \equiv 1 \pmod 4$. If we exclude the prime 2, which divides the discriminant -4 of the form $x^2 + y^2$, Fermat's theorem can be stated: for a prime $p \neq 2$ we have

$$p = x^2 + y^2 \ (x, y \text{ integers}) \quad \text{if and only if } p \equiv 1 \pmod 4.$$

Fermat also stated, and Euler proved, the following similar results: for a prime $p \neq 2$ we have

$$p = x^2 + 2y^2 \quad \text{if and only if } p \equiv 1, 3 \pmod 8,$$

and for a prime $p \neq 2, 3$

$$p = x^2 + 3y^2 \quad \text{if and only if } p \equiv 1 \pmod 3.$$

These and other similar results suggest a theorem of the following type: if $ax^2 + bxy + cy^2$ is a form of discriminant D then there exist positive integers $s, a_1, \ldots, a_s, m$ (depending on a, b and c) such that for an odd prime p not dividing D we have

$$p = ax^2 + bxy + cy^2 \quad \text{if and only if } p \equiv a_1, \ldots, a_s \pmod m. \tag{1}$$

However such a result does not hold for every form $ax^2 + bxy + cy^2$. This fact is often stated in number theory textbooks [**1**, p. 345], [**2**, p. 242], [**4**, p. 2], [**5**, p. 62], [**6**, p. 145] but when this claim is addressed [**2**, p. 242], [**4**, §1] reference is usually

made to class field theory. It seems desirable to give a more transparent justification of this assertion. We will do this by appealing to the following generalization of Weber's theorem to quadratic polynomials $ax^2 + bxy + cy^2 + dx + ey + f$ in two variables, where $a, b, \dots, f$ are integers: the polynomial $g(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$ represents infinitely many primes provided $\deg g = 2$, $\mathrm{GCD}(a, b, c, d, e, f) = 1$, $g(x, y)$ is irreducible in $Q[x, y]$, $g(x, y)$ represents arbitrarily large odd integers, and $g(x, y)$ is genuinely a function of two variables. This result follows from a theorem of Iwaniec [**9**], which can be proved without class field theory. The failure of a result of type (1) will be demonstrated for the particular form $x^2 + 14y^2$. Other forms for which (1) also fails can be treated in a similar manner. We prove

Theorem. *There do not exist positive integers* $s, a_1, \dots, a_s, m$ *with* $\mathrm{GCD}(a_i, m) = 1$ $(i = 1, \dots, s)$ *such that for primes* $p \neq 2, 7$

$$p = x^2 + 14y^2 \quad \text{if and only if } p \equiv a_1, \dots, a_s \pmod{m}. \tag{2}$$

We will also need the concept of a genus (plural genera) of form classes (see for example [**3**, Chapter 4], [**5**, Chapter 1]). The theory of genera was Gauss' major contribution to the study of binary quadratic forms. Two forms $ax^2 + bxy + cy^2$ and $a'x^2 + b'xy + c'y^2$ are said to be equivalent if there exist integers r, s, t, u with $ru - st = 1$ such that

$$ax^2 + bxy + cy^2 = a'(rx + sy)^2 + b'(rx + sy)(tx + uy) + c'(tx + uy)^2.$$

Equivalent forms have the same discriminant. It is a classical result that the set of equivalence classes (called form classes) for a given discriminant is finite. It is clear that forms in the same class represent the same integers and hence represent the same primes. Gauss partitioned the set of form classes for a given discriminant into genera in such a way that the primes represented by the forms in the form classes in each genus could be characterized by means of congruences. Two form classes with representatives $f_1(x, y)$ and $f_2(x, y)$ are in the same genus if and only if $f_1(x,y)$ and $f_2(x, y)$ are equivalent modulo m for all nonzero integers m, that is, there are integers r, s, t, u (depending on f_1, f_2 and m) with $\mathrm{GCD}(ru - st, m) = 1$ such that

$$f_1(x, y) \equiv f_2(rx + sy, tx + uy) \pmod{m}$$

for all x and y. For those discriminants possessing only one form class per genus Gauss could therefore say which forms represented which primes. Euler knew of discriminants with this property. It is known that there are only finitely many such discriminants with $D < 0$. An example of such a discriminant is $D = -24$. There are 2 form classes with representatives $x^2 + 6y^2$ and $2x^2 + 3y^2$. Each form class belongs to a different genus, and by Gauss' theory of genera we can deduce: if p is a prime $\neq 2, 3$ we have

$$p = x^2 + 6y^2 \quad \text{if and only if } p \equiv 1, 7 \pmod{24}$$

and

$$p = 2x^2 + 3y^2 \quad \text{if and only if } p \equiv 5, 11 \pmod{24}.$$

In this article we are concerned with the other situation where there are at least 2 form classes in the same genus. This occurs for example when $D = -56$. Here there are 4 form classes but only 2 genera. The classes of the forms $x^2 + 14y^2$ and $2x^2 + 7y^2$ belong to the same genus, and Gauss' theory of genera tells us only that

for primes $p \neq 2, 7$ we have

$$p = x^2 + 14y^2 \text{ or } 2x^2 + 7y^2 \quad \text{if and only if } p \equiv 1, 9, 15, 23, 25, 39 \pmod{56}$$

[**4**, p.2].

Proof of Theorem. If positive integers $s, a_1, \dots, a_s, m$ exist for which (2) holds, then m may be taken to be even, since for m odd the congruence $p \equiv a_i \pmod m$ is equivalent to $p \equiv b_i \pmod{2m}$, where $b_i = a_i$, if a_i is odd, $b_i = a_i + m$, if a_i is even, as p is odd.

We prove the theorem by showing that any arithmetic progression $A(a, m) = \{a + km\colon k = 0, 1, 2, \dots\}$, where $m \equiv 0 \pmod 2$ and $\mathrm{GCD}(a, m) = 1$, either contains no primes of the form $x^2 + 14y^2$ or it contains primes of both forms $x^2 + 14y^2$ and $2x^2 + 7y^2$.

Suppose that $A(a, m)$ contains a prime p of the form $x^2 + 14y^2$. As the two forms $x^2 + 14y^2$ and $2x^2 + 7y^2$ are in the same genus of discriminant -56, they are equivalent modulo every positive integer and thus in particular equivalent modulo m. Hence there exist integers r, s, t, u such that

$$p = x^2 + 14y^2 \equiv 2(rx + sy)^2 + 7(tx + uy)^2 \pmod m,$$

where $\mathrm{GCD}(ru - st, m) = 1$ [**5**, Theorem 3.21] [**8**, §12.5]. Let X and Y be integral variables and let $Q(X, Y)$ be the quadratic function

$$Q(X, Y) = 2m^2X^2 + 7m^2Y^2 + 4mAX + 14mBY + (2A^2 + 7B^2),$$

where

$$A = rx + sy, \quad B = tx + uy.$$

Clearly we have

$$\begin{aligned} Q(X, Y) &= 2(A + mX)^2 + 7(B + mY)^2 \\ &\equiv 2A^2 + 7B^2 \pmod m \\ &\equiv a \pmod m. \end{aligned}$$

It is easily checked that $Q(X, Y)$ is primitive, irreducible, represents arbitrarily large odd integers as m is even, and depends genuinely on the two variables X and Y. By Iwaniec's theorem [**9**] $Q(X, Y)$ represents infinitely many primes. Choosing X and Y so that $Q(X, Y) = q$ is prime, we see that $A(a, m)$ contains a prime of the form $2x^2 + 7y^2$. ∎

We have shown that every such arithmetic progression either contains no primes of the form $x^2 + 14y^2$ or it contains primes of both forms $x^2 + 14y^2$ and $2x^2 + 7y^2$. Thus congruences cannot be used to distinguish the representability of a prime by $x^2 + 14y^2$ from that by $2x^2 + 7y^2$.

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Another Proof of the Fundamental Theorem of Algebra

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The fundamental theorem can be stated in the following manner: every polynomial $P(z)$ of positive degree having complex coefficients is a surjective map from $\mathbf{C}$ to $\mathbf{C}$. The proof involves examining the boundary of the image of $\mathbf{C}$ under P. First, note that the image is closed. One way this can be seen is by extending P continuously to the Riemann sphere, and noting that the continuous image of a compact set is compact. Identifying $\mathbf{C}$ with $\mathbf{R}^2$, the Jacobian of $P(z)$ is non-singular precisely when $P'(z) \neq 0$. The inverse function theorem implies that $P(z)$ is a homeomorphism in a neighborhood of z whenever $P'(z) \neq 0$. Thus, if w is in the boundary of $P(\mathbf{C})$, then $w = P(z)$ and $P'(z) = 0$ for some z in $\mathbf{C}$. However, the number of zeroes of $P'(z)$ is at most the degree of P'. This shows that $P(\mathbf{C})$ has non-empty interior, and that its boundary consists of at most finitely many points. But the boundary of a proper subset of $\mathbf{R}^2$ with non-empty interior cannot consist of only a finite set of points.

Connections in Mathematical Analysis: the Case of Fourier Series

Enrique A. González-Velasco

INTRODUCTION. Napoleon Bonaparte's expedition to Egypt took place in the summer of 1798, the expeditionary forces arriving on July 1 and capturing Alexandria the following day. On the previous March 27 a young professor at the newly founded *École Polytechnique*, Jean-Joseph Fourier (1768–1830), was summoned by the Minister of the Interior in no uncertain terms [**16**, p. 64]:

> Citizen, the Executive Directory having in the present circumstances a particular need of your talents and of your zeal has just disposed of you for the sake of public service. You should prepare yourself and be ready to depart at the first order.

It was in this manner, perhaps not entirely reconcilable with the idea of *Liberté*, that Fourier joined the Commission of Arts and Sciences of Bonaparte's expedition. The military forces conquered Cairo on July 24, and by August 20 Bonaparte had decreed the foundation of the *Institut d'Egypte* in Cairo to promote the advancement of science in Egypt. Its first meeting, with Fourier appointed as its permanent secretary, was held on August 25.

After several military encounters the French surrendered to invading British forces on August 30, 1801, and were forced to depart from Egypt. Upon his return to France, Fourier resumed his post at the *École Polytechnique* but only briefly. In February of 1802 Bonaparte appointed him *Préfet* of the Department of Isère in the French Alps. It was here, in the city of Grenoble, that Fourier returned to his research endeavors, with which we shall presently occupy ourselves.

But Fourier's stay in Egypt had left a permanent mark on his health that was to influence the direction of his research. He contracted rheumatic pains during the siege of Alexandria and the sudden change of climate, from that of Egypt to that of the Alps, was distressing to him. The facts are that he lived in overheated rooms, that he covered himself with an excessive amount of clothing even in the heat of summer, and that his preoccupation with heat extended to the subject of heat propagation in solid bodies, heat loss by radiation and heat conservation. It was then on the subject of heat that he concentrated his main research efforts.

The results were first presented to the *Institut de France* on December 21, 1807 as a *Mémoire sur la propagation de la chaleur*. It was not entirely well received, and the committee that was to judge it and publish a report on it never did so (it appeared first in [**11**]). Instead, criticisms were made personally to Fourier in one of his visits to Paris in 1808 or 1809. They came mainly from Laplace and Lagrange and referred to two major points: Fourier's derivation of the equations of heat propagation and his use of some series of trigonometric functions known today as *Fourier series*. He replied to these objections and, as a means to settle the question,

suggested that a public competition be set up and a prize awarded by the *Institut* to the best work on the propagation of heat. Laplace—who had by then become supportive of Fourier's work—was probably instrumental in converting this suggestion into reality, and this was indeed the subject chosen for a prize essay for the year 1811. Another committee, including Lagrange and Laplace, was to judge on the only two entries, and on January 6, 1812, the prize was awarded to Fourier's *Théorie du mouvement de la chaleur dans les corps solides*. However, the committee's report expressed some reservations, specifically stating that [**11**, p. 452]

> the manner in which the Author arrives at his equations is not exempt from difficulties, and that his analysis, to integrate them, still leaves something to be desired in the realms of both generality and even rigor.

Fourier protested but to no avail, and his new work, like his previous memoir, was not published by the *Institut* at that time. He was to ultimately prevail, and in 1822 he gathered the larger part of his researches on heat in his monumental work *Théorie analytique de la chaleur* [**10**].

There is no doubt that today this book stands as one of the most daring, innovative, and influential works of the nineteenth century on mathematical physics. The methods that Fourier used to deal with heat problems were those of a true pioneer because he had to work with concepts that were not yet properly formulated. He worked with discontinuous functions when others dealt with continuous ones, used integral as an area when integral as an antiderivative was popular, and talked about the convergence of a series of functions before there was a definition of convergence. At the end of his 1811 prize essay, he even integrated 'functions' that have value ∞ at one point and are zero elsewhere. But such methods were to prove fruitful in other disciplines such as electromagnetism, acoustics and hydrodynamics. It was the success of Fourier's work in applications that made necessary a redefinition of the concept of function, the introduction of a definition of convergence, a reexamination of the concept of integral, and the ideas of uniform continuity and uniform convergence. It also provided motivation for the discovery of the theory of sets, was in the background of ideas leading to measure theory, and contained the germ of the theory of distributions. In the remaining sections we shall examine the steps that led from Fourier's work to the development of each of these pillars of classical analysis.

CONVERGENCE AND UNIFORM CONVERGENCE. One of the first problems studied by Fourier was that of a thin bar made of some conducting material, which, for convenience, we shall assume to be of length π and located along the x-axis with endpoints at $x = 0$ and $x = \pi$. If the temperature at a point x at time t is denoted by $u(x, t)$, Fourier deduced that it satisfies the equation

$$u_t = ku_{xx}, \tag{1}$$

where k is a positive constant. If its endpoints are maintained at zero temperature for $t \geq 0$ and if its initial temperature distribution is given by a known function f, we must solve (1) subject to the conditions $u(0, t) = u(\pi, t) = 0$ for $t \geq 0$ and $u(x, 0) = f(x)$ for $0 \leq x \leq \pi$. Fourier found that, for any positive integer n and any real constant c_n, the function $c_n e^{-n^2kt} \sin nx$ is a solution of (1) that vanishes at the endpoints. So is the sum of any number of such functions, but none of these sums need satisfy the initial condition because f may not be a sum of sine

functions. Fourier then proposed an infinite sum

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 kt} \sin nx, \tag{2}$$

and set out to find the constants c_n such that

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin nx = f(x). \tag{3}$$

This is easy if we assume that the last equality holds, if each term of (3) is multiplied by $\sin mx$, and if the resulting expression can be integrated term by term. Then

$$c_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx. \tag{4}$$

The series in (3) is a particular instance of a more general form that contains cosine terms in addition to sine terms, the usual *Fourier series*.

Now, the idea that an infinite sum of trigonometric functions can add up to an arbitrary function was rejected by the mathematical establishment. The main obstacle was precisely the concept of function popular at the time. Mathematicians were used to functions given by analytic expressions such as roots, logarithms and so on. How, they demanded, can $f(x) = e^x$ be the sum of an infinite series of sines on an interval $[-\pi, \pi]$? Why, this function is not even periodic while the sine functions are and, consequently, so is the sum of a series of sines. Surprisingly, they failed to realize that it could coincide with a periodic function over a bounded interval. Fourier gave numerous examples in which adding more and more terms of (3), where the c_n are computed from a given function f, results in a sum that is closer and closer to f. But an abundance of examples is not a proof that (3) converges. The problem that mathematicians faced in the early nineteenth century is that there was no definition of convergence. Surely, the concept did exist in some vague manner, but mathematics deals with quantities and comparisons between quantities, with equalities and inequalities. What was needed was a definition of convergence involving comparisons between the partial sums of a series and its proposed sum, such comparisons to be established by means of inequalities. One of the first definitions of convergence along these lines was given by Fourier himself in his prize essay of 1811, later incorporated into his book of 1822. He stated that to establish the convergence of a series [**10**, pp. 196–197]

> it is necessary that the values at which we arrive on increasing continually the number of terms, should approach more and more a fixed limit, and should differ from it only by a quantity which becomes less than any given magnitude: this limit is the value of the series.

The use of inequalities is already implicit in his *less than any given magnitude*. More precise and influential was the definition of convergence given by Augustin-Louis Cauchy (1789–1857). He was the first to understand the importance of rigor in analysis and the first to use inequalities in his definitions of limits and continuity. We shall never know whether or not Fourier's earlier definition helped him in shaping his own ideas. But once in possession of a rigorous definition of limit, Cauchy published the following in his 1821 textbook *Cours d'analyse de l'École Royale Polytechnique* [6, series 2; **3**, p. 114]:

> Let $s_n = u_0 + u_1 + u_2 + \cdots + u_{n-1}$ be the sum of the first n terms [of the series under consideration], n being any natural number. If, for always increasing values of n, the sum s_n approaches a certain limit s, the series will be called convergent and the limit in question will be called the sum of the series.

This is essentially the modern definition. More remarkably, Cauchy did not limit himself to stating it. On the next page he gave theorems containing tests for convergence: the Cauchy criterion and the root and ratio tests. A proof of the convergence of Fourier series was attempted by Poisson in 1820, by Cauchy in 1823 and, of course, by Fourier himself throughout his life. He never succeeded, but one of his sketches for a proof [**10**, pp. 438–440] would be of value to the man who finally did.

In 1822 a West Prussian teenager, Johann Peter Gustav Lejeune-Dirichlet (1805–1859), came to Paris to study mathematics. There he became acquainted with Fourier, who encouraged him to complete his sketch of the convergence proof. It would be some time, however, before Dirichlet could do so. In 1829, already a professor at Berlin, he published a paper entitled *Sur la convergence des séries trigonométriques qui servent a représenter une fonction arbitraire entre des limites données* [7, **1**, pp. 117–132]. After replacing a certain trigonometric identity in Fourier's sketch of proof with one of his own, he succeeded in giving sufficient conditions for convergence: if f is piecewise continuous and has a finite number of maxima and minima, then its Fourier series converges to the average of the right-hand and left-hand limits of f at each x.

Dirichlet's theorem is in flagrant contradiction with an earlier one by Cauchy. In his *Cours d'analyse* Cauchy had stated that the sum of a convergent series of continuous functions is continuous [6, series 2, **3**, p. 120]. Already in 1826 Abel had remarked that this theorem is wrong [1, **1**, pp. 224–225], and then, in 1829, Dirichlet's theorem made this abundantly clear. This is not mentioned to show a blemish in Cauchy's work, but because of its connection with an important discovery. Probably at Dirichlet's prompting, one of his students, Phillip Ludwig von Seidel (1821–1896), was led to investigate this matter in 1847. Here is his report: if $\sum_{n=1}^{\infty} u_n(x)$ is a convergent series of continuous functions with sum $f(x)$, I is an interval in the domain of these functions, and $\varepsilon > 0$ is given, let N be the smallest positive integer such that

$$\left| \sum_{n=N+1}^{\infty} u_n(x) \right| < \varepsilon$$

for all x in I. Then the given series is said to converge *arbitrarily slowly* on I if $N \to \infty$ as $\varepsilon \to 0$. Using this new concept, that was unavailable to Cauchy in 1821, Seidel was able to prove Cauchy's theorem provided that the convergence is not arbitrarily slow on any interval [**20**]. However, he did not pursue the matter, nor did he realize that he had put forth a powerful new kind of convergence.

As it happens, this idea of a different kind of convergence was not entirely new. Already in 1838 Christof Gudermann (1798–1852) had referred to a kind of convergence at the same rate—*im ganzen gleichen Grad*—that is the precursor of the modern concept of uniform convergence [**13**, pp. 251–252]. But its importance escaped him, as it would escape Seidel later on. This realization was left to Gudermann's student Karl Theodor Wilhelm Weierstrass (1815–1897), one of the giants of modern mathematics. Uninspired by the lectures at the University of Bonn, where he was a student, he went to Münster in 1839 to attend Gudermann's

lectures. Gudermann was to influence Weierstrass' research and it is quite likely that, while at Münster, they discussed the new concept of convergence. Weierstrass never finished his doctorate and became a *Gymnasium* teacher in 1841. During his tenure, until 1854, he produced an incredible amount of first-rate research in manuscript form that, regretably, remained unpublished. The fact that he referred to uniform convergence—*gleichmässige Convergenz*—in an 1841 manuscript [23, **1**, pp. 68–69] supports the idea that he may have learned about it from Gudermann. Weierstrass' many research achievements eventually earned him a position at the University of Berlin in 1856, where he frequently discussed uniform convergence. He defined it formally, for functions of several variables, in [23; **2**, pp. 201–233, Art. 1]. Adapted to the one variable case, his definition was:

> An infinite series $\sum_{\nu=0}^{\infty} u_\nu$ converges uniformly in a subset B of the region of convergence if given an arbitrarily small positive quantity δ a whole number m can be found such that the absolute value of the sum $\sum_{\nu=n}^{\infty} u_\nu$ is smaller than δ for each value of $n \geq m$, and for each value of the variable in B.

Still, the importance of Weierstrass' contribution stems from the fact that he realized the usefulness of uniform convergence and incorporated it in theorems on the integrability and differentiability of series of functions term by term.

G. Lejeune-Dirichlet

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THE CONCEPT OF FUNCTION. A lasting controversy over the concept of function started in 1747 when Jean Le Rond d'Alembert (1717–1783), of Paris, published his researches on the vibrating string [3]. If a piece of string, initially located along the x-axis and tied down at its endpoints at $x = 0$ and $x = a$, is displaced and then released, and if its vertical displacement at x at time t is denoted by $u(x, t)$, d'Alembert showed that it satisfies the equation

$$u_{tt} = c^2 u_{xx}, \tag{5}$$

where c is a constant. He also showed that if the initial displacement is given by a known function f, then the displacement of the string at any point x and at any time $t \geq 0$ is given by

$$u(x,t) = \tfrac{1}{2}\big[\tilde{f}(x + ct) + \tilde{f}(x - ct)\big],$$

where $\tilde{f}$ is the odd periodic extension of f to $\mathbb{R}$ of period $2a$. It is quite clear that f has to be twice differentiable for u to satisfy (5). However, this differentiability was rejected by Leonhard Euler (1707–1783) who, in a paper of 1748 written at Berlin, allowed a function with a discontinuous derivative as a better model for a plucked string than a twice differentiable function [8, series 2; **10**, pp. 63–77]. d'Alembert would not accept such functions [**2**], and this disagreement marked the beginning of a lively mathematical argument between the two men. The fact is that Euler's proposal represented something very new, since the concept of function at the time was that of an analytic expression or formula. In fact, this was the year of publication of Euler's enormously influential treatise *Introductio in analysin infinitorum* [8, series 1, **8** and **9**], the standard text on analysis for the next half century. At the very beginning, in the fourth paragraph, he defined a function of a variable quantity as

> any analytic expression made up in any manner whatever from that variable quantity and numbers and constants.

But then, that very same year, the vibrating string problem made him realize that this definition was too narrow to fit the needs of applied mathematics.

d'Alembert's solution completely describes the motion of the string, for it specifies the position of each of its points at each time. Mathematically that is all very well, but where is the musical description of the phenomenon? Where are the vibrations? This solution does not show a periodicity in t. It was Euler who stated that the motion of the string is periodic in time and made up of individual vibrations. In fact, in 1748 he wrote down the equation

$$u(x,t) = \sum c_n \sin \frac{n\pi}{a} x \cos \frac{n\pi}{c} t, \tag{6}$$

meant to be valid only if f is a sum of sines, but did not specify whether these sums are finite or infinite. Upon reading d'Alembert's and Euler's papers, Daniel Bernoulli (1700–1782), of Basel, decided to publish his own ideas on the subject, which he did in 1753 [**4**]. Perhaps there was an element of irritation in the fact that Euler now stated what he had known for some time. In a previous paper Bernoulli had already stated that the shape of the string at a given instant is the superposition of individual vibrations. Now, after having a bit of fun criticizing d'Alembert and Euler—he referred to the former as a great mathematician *in abstractis*—he asserted that this shape can be represented by an infinite series of sines. In

particular, for $t = 0$,

$$f(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi}{a} x. \tag{7}$$

If we accept this equation, we can combine it with (6) to arrive at the following expression for the solution of the vibrating string problem.

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi}{a} x \cos \frac{n\pi}{c} t.$$

Although Bernoulli never actually wrote this equation, it is called nowadays *Bernoulli's solution*, and it clearly shows that the motion of the string is periodic in time. Bernoulli based his equation (7) on physical considerations alone and provided no mathematical reasons whatsoever to back it up. Euler pounced on it immediately, the very same year, refusing to accept it [8, series 2; **10**, pp. 232–254]. For one thing, its right-hand side is a periodic function, which f need not be. Moreover, harping on his earlier idea that f need not be differentiable at all points, he rejected (7) because the sine functions on the right are differentiable. d'Alembert published a similar attack on Bernoulli's paper, but he did not surrender his position for, he said, he had infinitely many coefficients to choose to make the equality true. All this created a heated controversy that raged through the 1770's, without any of the participants giving an inch to the others' point of view. It was later revived through Fourier's researches on heat and eventually settled once and for all: the sum of an infinite series of sines can be a function that is not differentiable at all points.

With all this, Euler's wider concept of function emerged as the winner over the idea of function as a formula. In his *Institutiones calculi differentialis* of 1755, Euler himself gave the new definition as follows [8, series 1; **10**, p. 4]:

> If some quantities depend on other quantities so that they change when the latter are varied, then the former quantities are called functions of the latter.

This would not be the last word, however. For one thing, it is vague, lacking the precision demanded by the publication of Cauchy's *Cours d'analyse*. For another it was not totally accepted. What definitely won the day was Fourier's work, his use of discontinuous functions, and Dirichlet's proof of Fourier's assertion that a trigonometric series could converge to such a function. After this there was no turning back to the purely analytic concept of function. Fourier himself tried his hand at a new definition as follows [**10**, p. 432]:

> The function $f(x)$ denotes a function completely arbitrary, that is to say a succession of given values, subject or not to a common law, and answering to all the values of x between 0 and any magnitude X.

But, in spite of this *completely arbitrary* qualifier (what does it mean, anyway?), it is clear from an examination of his work that Fourier never had in mind a function with more than a finite number of discontinuities.

Neither did Dirichlet up to a point. But then he realized that a full generalization of his convergence theorem should allow integrable functions with infinitely many discontinuities [**7**, p. 131]. If this motivated him to search for a general definition of function, then he must have lost track of what he was after for the

fact is that, contrary to what many have asserted, he never stated such a definition. Later on, during the years 1847–1849, Dirichlet had the good fortune of counting a very gifted young man among his students at the University of Berlin. Georg Friedrich Bernhard Riemann (1826–1866) had transferred from the University of Göttingen to Berlin, and here Dirichlet was his favorite teacher and was instrumental in shaping some of Riemann's research interests. We do not know whether or not they discussed the concept of function before Riemann returned to Göttingen, where he received his doctorate in 1851. The fact is that in the opening paragraphs of his thesis we read [**18**, p. 3]:

> If we let z be a variable quantity that can gradually assume all possible real values, when to each of its values there corresponds a unique value of the undetermined quantity w, then we say that w is a function of $z \ldots$ this definition does not specify any fixed law between the individual values of the function, because, after it is defined on a particular interval, the way it can be extended outside remains entirely arbitrary.

Bernhard Riemann

Dirk J. Struik, *A Concise History of Mathematics*, 1948, Dover Publications, Inc., New York. Reprinted with permission.

Which is what Fourier had been saying all along: no *common law*, and it does not matter how the function is extended beyond $[-\pi, \pi]$. But with Riemann we have precision, we have this correspondence of a unique value of the function to each value of the variable. In short, the first entirely general and modern definition of function. With it ends, once and for all, an era of misconception. For it may once have been believed, when functions were just given by analytic expressions, that every continuous function has a derivative but not necessarily an integral. In fact, the opposite is true: not every continuous function has a derivative, while they all have integrals. But this is another topic.

INTEGRATION. The popular concept of integral in the eighteenth century was that of antiderivative. Leibniz had defined the integral much earlier as a sum, but his idea did not quite catch for some time. How could it, involving, as it does, the sum of infinitely many infinitely small quantities? Fourier changed that. He was used to handling functions not given by analytic expressions, but by curves and pieces of curves, and found antiderivatives to be impractical. Instead he remarked that, whether or not f is continuous, the integral defining the constant c_n in (4) can be viewed as the area under the graph of $f(x)\sin nx$ from 0 to π [**10**, p. 186]. It may have been responding to this interpretation of the integral as an area that Cauchy gave the following definition in his *Resumé des leçons donnés à l'École Royale Polytechnique sur le calcul infinitésimal* of 1823 [6, series 2, **4**, p. 125], which we reproduce in the current notation. If f is continuous on an interval $[a, b]$ and if $x_0, x_1, \ldots, x_n$ are points such that $a = x_0 < x_1 < \cdots < x_n = b$, then

$$\int_a^b f = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}), \tag{8}$$

provided that $x_i - x_{i-1} \to 0$ for each i as $n \to \infty$. Cauchy was then able to prove —not rigorously because he lacked the concept of uniform continuity—the existence of this limit. Notice also that if f is piecewise continuous it is still integrable because $[a, b]$ can be partitioned into a finite number of subintervals where f is continuous, and then the integrals over each of these subintervals can be added together. Incidentally, this notation for the definite integral, adopted by Cauchy, is due to Fourier [**10**, p. 463].

This definition suffices to prove Dirichlet's convergence theorem. In fact, Dirichlet had limited the discontinuities of his functions to a finite number to make them integrable. In order to generalize the theorem to functions with infinitely many discontinuities, he only needed to make sure that they could be integrated. That is, what he needed is what Cauchy's definition did not provide, namely, a condition for integrability. Dirichlet never achieved his goal of integrating functions with infinitely many discontinuities, but Riemann, who had acquired an interest in these topics from Dirichlet, would succeed. In 1854, wishing to qualify for a position at Göttingen as *Privatdozent*, he wrote a *Habilitationsschrift*, which at Dirichlet's suggestion was *Über die Darstellbarkeit einer Function durch eine trigonometrische Reiche*. Here he modified Cauchy's definition by replacing the factor $f(x_{i-1})$ in (8) by $f(t_i)$, where t_i is any point in the subinterval $[x_{i-1}, x_i]$, and by removing the continuity requirement on f. Instead, he turned things around and *defined* f to be integrable if the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}), \tag{9}$$

exists, provided that, for each $i, x_i - x_{i-1} \to 0$ as $n \to \infty$ [**18**, p. 239]. Next he stated a theorem giving conditions for the integral to exist [**18**, pp. 240–241], and to show the wide applicability of his definition, he gave an example of an integrable function with infinitely many discontinuities [**18**, p. 242].

Of course, not every function is integrable. For instance, at the end of his 1829 paper, Dirichlet pointed out that if c and d are constants and if $f(x) = c$ when x is rational and $f(x) = d$ when x is irrational, then the integrals that define the Fourier coefficients of f lose all significance [**7**, p. 132]. Indeed, the sum in (9) has value c if each t_i is rational and value d if each t_i is irrational, so that the limit does not exist. However, this is a rather weird function and the fact that it is not

integrable was regarded as unimportant. It seemed for quite some time that Riemann's definition of integral was the most general imaginable. Reality, in its usual fashion, would soon dispel this illusion.

THE THEORY OF SETS. The coefficients in (3) were obtained by assuming that the series converges and can be integrated term by term. Can it? A theorem of Weierstrass states that it can if the convergence is uniform. Then we ask: when does a Fourier series converge uniformly? We are not just posing a purely theoretical question because the needs of applications demand an answer. For instance, in order for (2) to be the solution of the problem posed earlier, it must be continuous for $t \geq 0$ and $0 \leq x \leq \pi$. This is true if (2) converges uniformly, as shown by Abel, unknowingly using the idea of uniform convergence [1; **1**, pp. 224–225]. But then, in particular, the convergence of (2) must be uniform for $t = 0$, that is, the Fourier series in (3) must be uniformly convergent. So, once again, when does a Fourier series converge uniformly? This is the question that Heinrich Eduard Heine (1821–1881), of the University of Halle, posed himself, and in 1870 he showed that if a function satisfies Dirichlet's conditions on $[-\pi, \pi]$, then its Fourier series converges uniformly on the set that results after removing arbitrarily small neighborhoods of the points where it is discontinuous [**15**].

Now, in his integration paper Riemann had also considered trigonometric series on $[-\pi, \pi]$ of the usual form

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{10}$$

but with arbitrary coefficients, not necessarily the Fourier coefficients of some function [**18**, p. 245]. In principle, there may be several choices of the coefficients for which (10) converges to the same function. But this is impossible if (10) converges uniformly, for then term by term integration shows that they must be the Fourier coefficients of its sum. It was at this point that Heine posed a second problem: how to weaken the hypothesis of uniform convergence and still be able to conclude that the coefficients are unique. He found that if (10) converges uniformly on the subset of $[-\pi, \pi]$ that remains after removing arbitrarily small neighborhoods of a finite number of points, then the coefficients are unique [**15**].

Notice that Heine, even though geographically removed from the Weierstrassian world at Berlin, used uniform convergence. He had been a student of Weierstrass and may have learned about it before leaving Berlin, or he may have heard about it from a new arrival from Berlin, Georg Ferdinand Louis Philippe Cantor (1845–1918), who had become a *Privatdozent* in 1869 at Halle. In any case, Heine encouraged Cantor to do some further work on the problem of uniqueness of the coefficients of (10). Cantor started with the idea of discarding uniform convergence entirely, and succeeded fairly soon, but had to assume that (10) converges at every point [**5**, pp. 80–83]. Then, in 1871, he was able to allow (10) to diverge a finite number of points and still conclude that its coefficients are unique [**5**, pp. 84–86]. But Cantor was ambitious and found these results short of what he wanted to do, namely to reach the same conclusion after allowing the convergence of (10) to fail at infinitely many points. But then, what kind of infinite set of points should this be? In 1872, Cantor found that, in order to construct such a set, he needed to develop first a theory of the real numbers. Having accom-

plished this, he defined the concept of limit point [5, p. 98]:

> Given a set of points P, if there are an infinite number of points of P in every neighborhood, no matter how small, of a point p, then p is said to be a limit point of the set P.

By a neighborhood of p Cantor meant an open interval containing p. Then he defined the *derived set* P' of P as the set of all limit points of P, the second derived set P'' of P as the derived set of P', and so on until, after k iterations, the k-th derived set $P^{(k)}$ of P is the derived set of $P^{(k-1)}$. Then he proved his most general uniqueness theorem in the following form: if (10) vanishes for all values of x in $[-\pi, \pi]$ except for those corresponding to a subset P such that $P^{(k)}$ is empty for some k, then all its coefficients are zero [5, p. 99].

Having found his motivation on questions about trigonometric series, Cantor had just laid the foundations on which he would then build his acclaimed and controversial theory of sets.

MEASURE-THEORETIC INTEGRATION. This is, then, the way it was: in 1870 Cantor gave the first steps toward the theory of sets by investigating the set of points where (10) may fail to vanish and still conclude that $a_n = b_n = 0$. This is, instead, the way it could have been: in 1870 Hermann Hankel (1839–1873) could have given the first steps toward the theory of sets by investigating the set of points where a function may be discontinuous and still integrable. A professor at Tübingen, Hankel had been a student of Riemann at Göttingen and was seeking a necessary and sufficient condition for integrability. In view of Riemann's example of a highly discontinuous integrable function, Hankel wanted to characterize integrability in terms of the set of points where a function is discontinuous, and started by defining the *jump* of f at a point x_0 to be the largest—*i.e.*, the supremum—of all numbers $\sigma > 0$ such that in any interval containing x_0 there is an x for which $|f(x) - f(x_0)| > \sigma$ [**14**, p. 87]. Then, if S_σ denotes the set of points where the jump of f is greater than σ, Hankel concluded that a bounded function is integrable if and only if for every $\sigma > 0$ the set S_σ can be enclosed in a finite collection of intervals of arbitrarily small total length, a fact that we express by saying that S_σ *has content zero*. On the other hand, if a set cannot be so enclosed it is said to have *positive content*. With this result Hankel initiated the set-theoretic approach to integration.

But instead of developing these ideas, Hankel next made a mistake and stated the wrong theorem. First he defined a set to be *scattered*—the modern term, due to Cantor, is *nowhere dense*—if between any two of its points there is an entire interval that contains no points of the set. And then, erroneously thinking that a set has content zero if and only if it is scattered, he stated that a bounded function is integrable if and only if for every $\sigma > 0$ the set S_σ is scattered. Henry John Stephen Smith (1826–1883), of Oxford, carefully read Hankel's paper, found the error and, in 1875, gave several methods to construct nowhere dense sets of positive content [**21**, p. 148]. It is easy to see that if S is one such set contained in an interval I and if $f \equiv 1$ on S and $f \equiv 0$ on $I - S$ then f is not integrable.

Then, in 1881 Vito Volterra (1860–1940), a student at Pisa, used a nowhere dense set of positive content to construct a function f on $[0, 1]$ such that f' exists and is bounded at every point, but is not integrable [**22**]. Therefore, while f' always has an integral in the sense of antiderivative, it may not have an integral in Riemann's sense. It can then be said that Riemann's definition is beginning to

show some rough edges. Furthermore, it was known, at least since 1875, that it is not always possible to interchange passage to the limit and integration in a sequence of integrable functions.

All this meant that the definition of integrability had to come up for review and, in view of Hankel's characterization of it in terms of sets of content zero, the new approach had to be set-theoretic. After some preliminary work by Marie-Ennemond Camille Jordan (1838–1922), this was accomplished by Henri-Léon Lebesgue (1875–1941) in his doctoral dissertation of 1902 at the Sorbonne, later expanded into a book [**17**]. Here he introduced a theory of the measure of sets and, based on it, a definition of integral that generalizes that of Riemann but is free of the defects pointed out above [**17**, pp. 110–121].

THE THEORY OF DISTRIBUTIONS. In his 1811 memoir Fourier considered heat propagation in an ideal bar of infinite length whose initial temperature is a known function f. A series solution was not possible in this case and he proposed, instead, an integral solution. To satisfy the initial condition, it must equal f for $t = 0$, leading—in modern notation—to the integral equation

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega, \tag{11}$$

that must be solved for the unknown function $\hat{f}$. The solution is

$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx. \tag{12}$$

Fourier's proof was unrigorous but interesting because it contains the germ of further discoveries, and we shall examine it next. If we substitute (12) into the right-hand side of (11), reverse the order of integration and simplify, we obtain

$$\int_{-\infty}^{\infty} f(s)\left(\frac{1}{\pi}\int_{0}^{\infty} \cos\omega(x-s)\, d\omega\right) ds = \int_{-\infty}^{\infty} f(s)\frac{1}{\pi} \lim_{p\to\infty} \frac{\sin p(x-s)}{x-s}\, ds. \tag{13}$$

Then Fourier stated that the right-hand side is equal to

$$\int_{-\infty}^{\infty} f(s)\frac{\sin p(x-s)}{\pi(x-s)}\, ds, \tag{14}$$

where, he said, $p = \infty$. Let's just say that if $p > 0$ is fixed and very large (14) is an approximation of the right-hand side of (11). For p very large, $\sin p(x-s)$ undergoes a complete oscillation on every interval $[x + k\pi/p, x + (k+2)\pi/p]$, where k is any integer, and $f(s)/(x-s)$ is approximately constant in each for $k \neq -1$. In the remaining interval $f(s) \approx f(x)$, and then

$$\int_{-\infty}^{\infty} f(s)\frac{\sin p(x-s)}{\pi(x-s)}\, ds \approx f(x)\int_{x-\pi/p}^{x+\pi/p} \frac{\sin p(x-s)}{\pi(x-s)}\, ds = f(x)\int_{-\pi/p}^{\pi/p} \frac{\sin pu}{\pi u}\, du.$$

But, as above, the integral of the quotient on the right over the rest of the real line is negligible, and then

$$\int_{-\infty}^{\infty} f(s)\frac{\sin p(x-s)}{\pi(x-s)}\, ds \approx f(x)\int_{-\infty}^{\infty} \frac{\sin pu}{\pi u}\, du = \frac{f(x)}{\pi}\int_{-\infty}^{\infty} \frac{\sin t}{t}\, dt = f(x). \tag{15}$$

Fourier, however, kept $p = \infty$ throughout his argument [**10**, pp. 426–429]. It seems

that he would have us believe that there is a function δ defined by

$$\delta(x) = \lim_{p\to\infty} \frac{\sin px}{\pi x}$$

and such that, as suggested by (15),

$$\int_{-\infty}^{\infty} f(s)\delta(x-s)\,ds = f(x). \tag{16}$$

(15) also suggests that the integral of δ over the whole real line is one, while the argument following (14) shows that its integral over any interval that excludes the origin is zero. In short, $\delta \equiv 0$ outside the origin and $\delta(0) = \infty$.

There is, of course, no such function. But we wish there were for the sake of applications. For instance, in *An essay on the application of mathematical analysis to the theories of electricity and magnetism* of 1828, George Green (1793–1841), of Cambridge, considered the problem of solving the equation

$$u_{xx} + u_{yy} + u_{zz} = f \tag{17}$$

in a bounded region of space that contains the origin. Here u is the electrostatic potential created by a charge distribution given by f. He showed that he could solve this problem if he could first solve it for the restricted case in which there is just one point charge—infinite charge density—at the origin and none elsewhere [**12**, pp. 32–33]. Now, let's say that there is a δ function on $\mathbb{R}^3$ with the properties listed above except that the integrals are three-dimensional. Since, in particular, $\delta \equiv 0$ outside the origin and $\delta(0) = \infty$, we can rephrase Green's claim as follows: a solution of (17) can be obtained from a solution of

$$u_{xx} + u_{yy} + u_{zz} = \delta. \tag{18}$$

Indeed, let u^δ be a solution of (18), denote the function of x defined by the left-hand side of (16) by $f * \delta$, and define $f * u^\delta$ in the same way but replacing δ with u^δ in the integrand. Then $u = f * u^\delta$ is a solution of (17) because, if differentiation under the integral sign is permitted,

$$u_{xx} + u_{yy} + u_{zz} = f * \left(u^\delta_{xx} + u^\delta_{yy} + u^\delta_{zz}\right) = f * \delta = f,$$

where the last equality is just (16).

The power of wishful thinking cannot be underestimated. During the period 1945–1948 Laurent Schwartz (1915–), working in isolation at Grenoble as Fourier had done before, developed a complete, rigorous, and applicable theory of this δ and similar 'functions', which he called *distributions*, culminating in the publication of his two-volume work *Théorie des distributions* [**19**].

EPILOGUE. Back in 1811, disappointed by the committee's reaction to his memoir, Fourier returned to Grenoble and, being far from Paris, lacked the power and the influence to have his prize essay published by the *Institut*. But new political events would soon change his fortune. A European Alliance against Napoleon forced his unconditional abdication on April 11, 1814, restoring the monarchy in the person of Louis XVIII. Fourier remained as *Préfet* of Isère under the new regime, a tribute to his diplomatic abilities, but early the following March he learned that Napoleon had returned from his exile at Elba. Fearing the consequences of his temporary allegiance to the Crown, he fled to Lyons, but by the time he arrived there the Emperor had forgiven his ungrateful behavior and appointed him *Préfet* of the Rhône. He was dismissed from this position on

May 17 and, having been granted a pension of 6,000 francs by Napoleon, Fourier finally returned to Paris. A new allied army defeated Napoleon on June 18, 1815, at the Battle of Waterloo, and he was forever banished to the island of St. Helena. Fourier's pension never materialized under the King's restored government, and he found himself penniless. However, with the influence of a friend and former student at the *École Polytechnique*, the Count of Chabrol de Volvic, he secured the position of Director of the Bureau of Statistics of the Department of the Seine, and this allowed him to remain in Paris permanently and to set down to business.

First, there was the publication of the prize essay, a matter in which he succeeded after a considerable amount of insistence. It finally appeared in 1824 and 1826 in volumes 4 and 5 of the *Mémoires de l'Académie Royale des Sciences de l'Institut de France* [9; **2**, pp. 1–94]. But before this, in May of 1816, two new members of the Academy of Sciences were to be elected. Fourier lobbied vigorously on his own behalf and, after several rounds of voting, was elected to the second position. The King, resentful of Fourier's activities during Napoleon's second period in power, refused to give his approval. But a regular vacancy was created again in 1817, and on the election of May 12 Fourier obtained forty seven of the fifty votes. The King was then compelled to grant his approval.

Fourier's scientific standing was no longer in doubt. In 1822 his *Théorie analytique de la chaleur* was printed in Paris, and on November 18 of the same year he became Permanent Secretary of the mathematics section of the Academy of Sciences. His last years were marked by honors and poor health. He was elected to the Royal Society of London and to the *Académie Française* in 1826. Then, the next year, he became president of the *Conseil de perfectionnement de l'École Polytechnique*. But already in 1826, in a letter to Auger, permanent secretary of the French Academy, he claimed to *see the other bank where one is healed of life* [**16**, p. 137]. In addition to his rheumatism, which never left him, he developed a shortness of breath that was particularly acute if not standing up. Resourceful to the very end, he invented a contraption in the form of a box with holes for his arms and head to protrude, and carried on in this fashion. The end came at about four o'clock in the afternoon of May 16, 1830 in the form of a heart attack, and shortly afterward he died.

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Tessellations

Chandler Fulton

It is easy to see that the only way to tile the plane with a single regular polygon is to use a triangle, a square, or a hexagon. If the restriction of regularity is removed, one can also use a pentagon. But no polygon of more than six sides will work. What is not so obvious is that no infinite variety of polygons each with more than six edges will tile the plane, provided that their areas are bounded below and that their diameters are bounded above. I. Niven proved this fact in 1978 using Euler's theorem, and he gave a bound for the area of the largest square that can be tiled by polygons satisfying these conditions [**3**]. (A simplification of his proof appeared in [**2**]; for history and related facts see [**1**].) Here we derive Niven's results by an elementary argument, measuring angles of the polygons. This also gives a sharper bound on the largest region that can be tessellated.

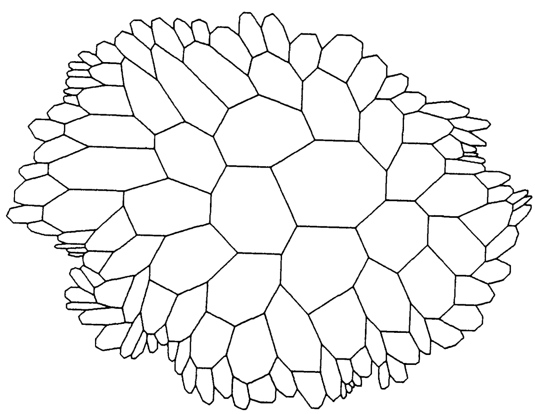

Diagram 1. One cannot tile the plane using convex polygons of more than six edges and limited diameters without letting their areas shrink to zero.

Theorem. *It is impossible to tessellate the plane by convex polygons having areas bounded below and diameters bounded above if each has more than six edges.*

Proof: Suppose that a tessellation covers a disk of diameter l. Let k be the number of polygons which contain one or more points of the disk, and k^* the number of these polygons which contain one or more points of the boundary of the disk. Let N be the sum of the number of vertices on each of the k polygons, and N^* the sum of the number of vertices on the k^* boundary polygons. We define a "junction" to be any point which is a vertex of one or more polygons, and an "improper" junction to be any junction which is on an edge of, but not at a vertex of, some polygon. (See Diagram 2.) Let M be the total number of improper junctions on the k polygons, and M^* the number of improper junctions on the k^* boundary polygons.

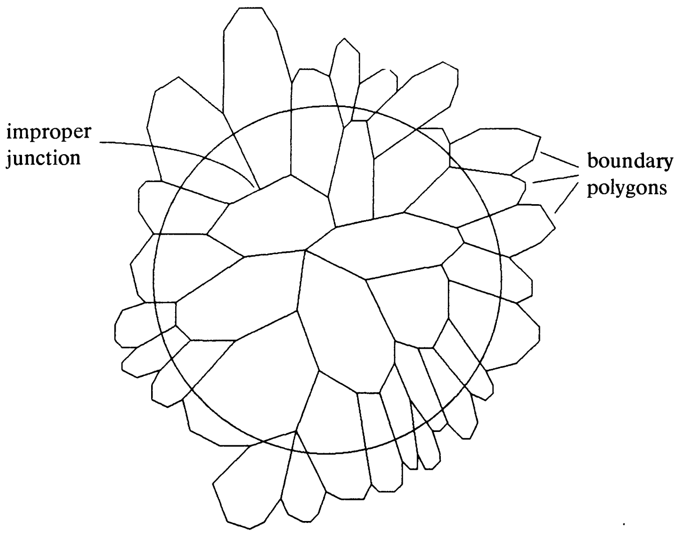

Diagram 2. The theorem holds even if some vertices are at the middle of edges.

The idea is to calculate the total interior angle two ways, first counting within each polygon to produce an exact figure, then at each junction to obtain an upper bound. The total interior angle is $(N - 2k)\pi$, but we want to know the total angle contained inside any of the k polygons at all the junctions, including improper ones. Thus we need to add π for each of the M improper junctions, since at these points only half the total angle has been counted so far by interior angles of polygons. The total angle at all junctions is then $(N + M - 2k)\pi$. On the other hand, we know that the total angle at each junction inside the circle is 2π, and that each such junction must be shared by at least three convex polygons. Thus we can, in effect, count each angle of each polygon as $2\pi/3$ to obtain an upper bound on the total angle measure. At each junction outside the circle, however, there are not necessarily as many as three polygons which contain a point of the disk, so each angle at each junction outside the circle must be counted as π to retain the true upper bound. Therefore, we must add $\pi/3$ for each angle exterior to the circle. Since there are fewer than $N^* + M^*$ such angles, we have

$$(N + M - 2k)\pi < \tfrac{1}{3}(N + M)2\pi + \varepsilon,$$

where

$$\varepsilon = \left(\tfrac{1}{3}\right)(N^* + M^*)\pi.$$

Putting these together, we get

$$(N - N^*)/k + (M - M^*)/k < 6. \tag{1}$$

Since M is always greater than or equal to M^*, we have

$$(N - N^*)/k < 6. \tag{2}$$

Setting e equal to the least number of edges of any of the k polygons, we know that the sum of the number of edges of the $k - k^*$ non-boundary polygons is at least $k - k^*$ times e, or

$$N - N^* \geq (k - k^*)e.$$

Then,

$$e - \frac{N - N^*}{k} \le e - \frac{k - k^*}{k} \cdot e = \frac{k^*}{k} \cdot e. \tag{3}$$

By the Lemma below, k^*/k is bounded above by c/l for a constant c which depends on the minimum area and maximum diameter of the polygons. Equations (2) and (3) therefore give

$$e < 6 + ce/l. \tag{4}$$

Now, if we allow l to get larger, so that $ce/l \le 1$, we get $e \le 6$. In other words, we can always choose a disk which is too large to be tiled by polygons of more than six edges. □

Lemma. *Let k, k^*, and l be defined as above. Let d be the largest diameter, and A the greatest area, of any of the k polygons; and let α be the minimum area of any of the k^* boundary polygons. Then*

$$k^*/k \le 4dA/\alpha l.$$

Proof: We have

$$\pi l^2/4 \le A \cdot k. \tag{5}$$

Also, the sum of the areas of the k^* boundary polygons is bounded above by $\pi[(l + d)^2 - (l - d)^2]/4 = \pi \cdot l \cdot d$, because each of these k^* polygons lies between two circles of diameters $l - d$ and $l + d$, where these two circles have the same center as the given circle of diameter l. Thus

$$\alpha \cdot k^* \le \pi \cdot l \cdot d. \tag{6}$$

Combining equations (5) and (6), we get

$$k^*/k \le 4dA/\alpha l. \quad \square$$

Note that in the preceding Lemma, d could be replaced by the largest diameter of any boundary polygon. But with d as in the Lemma, A is bounded above by $\pi d^2/4$, so

$$k^*/k \le \pi d^3/\alpha l. \tag{7}$$

With this result we can now establish the size of the largest region which can be tiled by convex polygons of more than six edges.

Corollary. *The largest disk which can be tessellated by convex polygons, each with at least seven sides, diameter at most d, and area at least α, has diameter less than $7\pi d^3/\alpha$.*

Proof: Substituting $c = \pi d^3/\alpha$ into equation (4), we get

$$l < \frac{\pi d^3}{\alpha} \cdot \frac{e}{e - 6}.$$

Since $e \ge 7$, we obtain the desired result: $l < 7\pi d^3/\alpha$. □

In particular, if the perimeters of the polygons are bounded above by β, since $d \le \beta/2$, the diameter of the largest disk is less than $7\pi\beta^3/8\alpha$. This improves Niven's bound of $4\beta + 32\beta^3/\alpha$ for the side of the largest square.

We say two polygons of a tessellation are adjacent if they have more than a single point on an edge in common. The following was also proved by Niven [3].

Proposition. *Any tessellation of the plane, by convex polygons having areas bounded below and diameters bounded above, has an infinite number of polygons which are adjacent to fewer than seven others.*

Proof: Suppose, in a given tessellation, there are only Q polygons which are adjacent to fewer than 7 others. Consider a disk of diameter l which contains these Q polygons. Let k, k^*, N, N^*, M, and M^* be defined as in the Proof of the Theorem. Let j be the least number of polygons adjacent to any of the $k - k^* - Q$ polygons which are adjacent to more than 6 others. Then,

$$(N - N^*) + (M - M^*) \geq (k - k^* - Q)j + 3Q,$$

since all but Q of the $k - k^*$ interior polygons each is adjacent to at least j others, while each of the other Q certainly is adjacent to at least 3. Thus,

$$j - (N - N^*)/k - (M - M^*)/k \leq (k^*/k)j + Q(j - 3)/k.$$

Since $k^*/k \leq c/l$, and $1/k \leq 4A/\pi l^2$ by equation (5), there exists an l such that

$$j - (N - N^*)/k - (M - M^*)/k < 1.$$

Thus, by equation (1), $j < 6 + 1 = 7$, a contradiction. □

Remarks. 1. Since the edges of a convex polygon with p vertices must meet the edges of at least p other convex polygons, the Proposition implies that there must be an infinite number of polygons with fewer than seven edges in a tessellation satisfying the given conditions of the Proposition.

2. A simple calculation shows that there must be at least $(l^2/7d^2) - (\pi l d/\alpha)$ polygons which are adjacent to fewer than seven others (and therefore at least the same number with fewer than seven edges) in any circle of diameter l of a given tessellation.

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Rewriteability in Finite Groups

J. L. Leavitt, G. J. Sherman and M. E. Walker

INTRODUCTION. What's the probability that two elements in a finite group commute? A formal answer,

$$Pr_2(G) = \frac{|\{(x, y) \in G^2 | xy = yx\}|}{|G|^2}, \tag{1}$$

begs our next question. How many ordered pairs of elements of a finite group commute?

Let's be specific. Consider the "commutativity matrix" for the symmetric group on three symbols.

S_3	id	$(1,2)$	$(1,3)$	$(2,3)$	$(1,2,3)$	$(1,3,2)$
id	1	1	1	1	1	1
$(1,2)$	1	1	0	0	0	0
$(1,3)$	1	0	1	0	0	0
$(2,3)$	1	0	0	1	0	0
$(1,2,3)$	1	0	0	0	1	1
$(1,3,2)$	1	0	0	0	1	1

The xth row of this matrix identifies the subgroup, $C(x)$, of elements which commute with x; i.e., the centralizer of x. Here's the way to parse the commutativity count for S_3.

$$18 = 6 + 2 + 2 + 2 + 3 + 3 = 1 \cdot 6 + 3 \cdot 2 + 2 \cdot 3 = 6 + 6 + 6 = 3 \cdot 6$$

The elementary group theory at work in this count is:

- conjugate elements have centralizers of the same order

$$y = g^{-1}xg \text{ implies } C(y) = g^{-1}C(x)g,$$

- the order of a conjugacy class is the index of the centralizer of any element in the class

$$|x^G| = |\{g^{-1}xg | g \in G\}| = [G : C(x)],$$

- Lagrange's theorem

$$|G| = [G : H] \cdot |H|.$$

An abstraction of this example, originally due to Erdös and Turán [4], answers our second question.

$$\begin{aligned}
|\{(x, y) \in G^2 | xy = yx\}| &= \sum_{x \in G} |C(x)| \\
&= \sum_{i=1}^{k} |x_i^G| \cdot |C(x_i)| \\
&= \sum_{i=1}^{k} [G : C(x_i)] \cdot |C(x_i)| \\
&= k \cdot |G| \qquad (2)
\end{aligned}$$

where $\{x_1, x_2, \ldots, x_k\}$ is a complete set of conjugacy class representatives of G.

Thus, an informative answer to our first question is

$$Pr_2(G) = \frac{k}{|G|}.$$

It comes as no surprise that G is abelian precisely when $Pr_2(G) = 1$. But what may surprise you is that if G is not abelian, then

$$Pr_2(G) = \frac{k}{|G|} \leq \frac{p_s^2 + p_s - 1}{p_s^3} \leq \frac{5}{8}. \qquad (3)$$

where p_s is the smallest prime divisor of the order of G. The essence of these bounds is that the index of the center of a nonabelian group is at least p_s^2; i.e., $|G:Z] \geq p_s^2$.

The $5/8$ bound, which is assumed by the dihedral and quaternion groups of order eight, has been around for a long time. Yet, it doesn't seem to be commonly known—so be sure to tell your students about it. We do not know with whom it originated. Some say Max Zorn. But, many years ago, during a conversation with one of the authors (Sherman), Zorn declined credit for the bound. To the best of our knowledge the bound first appeared in print in 1973 when Gustafson [**7**] showed that an analogous bound holds for compact nonabelian groups. Gallian's recent textbook ([**6**, pages 329, 330]) also includes a discussion of the bound. Both upper and lower bounds on $Pr_2(G)$ for various classes of groups have been obtained ([**1**], [**4**], [**5**], [**7**], [**10**], [**13**]). And, since commutativity can be defined in terms of conjugation, analogous results have been pursued for various group actions ([**11**], [**13**], [**15**]).

Commutativity is a special case of rewriteability. Let $S \subseteq S_n - \{\mathrm{id}\}$; i.e., S is a set of nontrivial permutations of $\{1, 2, \ldots n\}$. An n-tuple $(x_1, x_2, \ldots, x_n)$ of elements of G is S-rewriteable if $x_1 x_2 \cdots x_n = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$ for some $\sigma \in S$. We generalize (1) by setting

$$Pr_n(G; S) = \frac{|Rw_n(G; S)|}{|G|^n} \qquad (4)$$

where

$$Rw_n(G; S) = \{(x_1, x_2, \ldots x_n) \in G^n | (x_1, x_2, \ldots x_n) \text{ is S-rewriteable}\}. \qquad (5)$$

Those groups for which $Pr_n(G; S_n - \{\mathrm{id}\}) = 1$ will be referred to as n-rewriteable groups. The notion of rewriteability has its origins in automata theory and is currently of considerable interest in group theory [**2**].

In particular, Curzio, Longobardo and Maj [3] have provided elementary proofs that the following three statements are equivalent.

i) *G is 3-rewriteable; i.e., $xyz \in \{yxz, zyx, xzy, zxy, yzx\}$ for all $x, y, z \in G$.*
ii) *The order of the derived subgroup of G, $G' = \langle x^{-1}y^{-1}xy | x, y \in G \rangle$ is one or two.*
iii) *The order of the centralizer of each element of G is $|G|$ or $|G|/2$.*

The equivalence of ii) and iii) revolves around the relationship between commutators (elements of the form $x^{-1}y^{-1}xy$) and conjugates: $x^{-1}y^{-1}xy = g$ if, and only if $y^{-1}xy = xg$. The equivalence of i) with ii) or iii) is case-driven. For example, an application of the definition of 3-rewriteability to the product xyx^2 places x^2 in the center of the group. This means the centralizer of x is a "large" normal subgroup. In view of iii) and our discussion prior to (2), we may add the following statement to the list.

iv) *The order of each conjugacy class of G is one or two.*

Each of ii), iii) and iv) suggests a connection between 3-rewriteability and the probability of two elements commuting. In particular, the size of a group's derived subgroup is a classic measure of the degree of commutativity the group enjoys. If G' is small, then "most" commutators are trivial; i.e., it is "likely" that $xy = yx$.

Let's formalize this connection. Notice that the average order of a conjugacy class of a 3-rewriteable group is less than two; i.e., $|G|/k < 2$. Thus $Pr_2(G) = k/|G| > 1/2$ for 3-rewriteable groups. An appeal to character theory establishes the converse. G has k irreducible characters and $|G|/|G'|$ irreducible characters of degree one. Thus

$$|G| \geq (|G|/|G'|) \cdot 1^2 + (k - |G|/|G'|) \cdot 2^2$$

which implies

$$1 \geq -3/|G'| + 4k/|G|.$$

If $k/|G| > 1/2$, then $1 > -3/|G'| + 2$ from which it follows that $|G'| \leq 2$. We have the following theorem.

Theorem. *A finite group G is 3-rewriteable if, and only if, $Pr_2(G) > 1/2$.*

It's interesting to formulate this theorem in terms conjugacy classes

Each conjugacy class has order one or two if, and only if, the average conjugacy class order is less than two.

and in terms of conditional probability.

The probability of x and y commuting, given y, is at least $1/2$ for each y, if and only if $Pr_2(G) > 1/2$.

AN ELEMENTARY PROOF. An elementary proof that *if $Pr_2(G) > 1/2$, then G is 3-rewriteable* follows. Think of "3-rewriteable" as a generic label for your favorite from among statements i)–iv) above. We will assume that G is not 3-rewriteable and prove that $Pr_2(G) \leq 1/2$.

The proof and subsequent discussion hinge on relationships among the orders of three subsets of G:

$$X = \{x \in G \mid [G:C(x)] \geq 3\},$$
$$Y = \{x \in G \mid [G:C(x)] = 2\},$$
$$Z = \{x \in G \mid [G:C(x)] = 1\}; \text{ i.e., the center of } G.$$

The following three lemmas, which are of some interest in their own right, help organize the proof.

Lemma 1. *If x and y are elements of G for which $[G:C(x)] = 2$ and $C(y) \cap (G - C(x)) \neq \varnothing$, then $[G:C(xy)] \geq [G:C(y)]$.*

Proof: The conjugacy class of y in G, y^G, may be written $\{y^{g_1}, y^{g_2}, \ldots, y^{g_n}\}$ where $\{g_1, g_2, \ldots, g_n\}$ is a complete set of right coset representatives for $C(y)$ in G. Moreover, we may choose each coset representative in $C(x)$. Otherwise $C(y)g_i \subseteq G - C(x)$, which means that $G - C(x) = C(x)g_i$ since $[G:C(x)] = 2$. Therefore $C(y)g_i \subseteq C(x)g_i$ and so $C(y) \subseteq C(x)$, a contradiction. The conclusion follows because the mapping $y^{g_i} \to xy^{g_i}$ embeds y^G in $(xy)^G$.

Lemma 2. *If at least $3 \cdot |Z|$ elements of G have centralizers of index at least 3, then $Pr_2(G) \leq 1/2$.*

Proof: Observe that

$$\begin{aligned}
|Rw_2(G)| &= k \cdot |G| \leq (|X|/3 + |Y|/2 + |Z|) \cdot |G| \\
&= (|Z| + (|X| - 3 \cdot |Z|)/3 + |Y|/2 + |Z|) \cdot |G| \\
&\leq (|Z| + (|X| - 3 \cdot |Z|)/2 + |Y|/2 + |Z|) \cdot |G| \\
&= (|X| + |Y| + |Z|) \cdot |G|/2 \\
&= |G|^2/2.
\end{aligned}$$

Thus $Pr_2(G) \leq 1/2$ as claimed.

Lemma 3. *If G is not 3-rewriteable, then $[G:Z] \geq 6$.*

Proof: If $[G:Z]$ is 1, 2, 3 or 5, then G is abelian since G/Z is cyclic. If $[G:Z] = 4$ and x is a non-central element, then $Z \subset C(x) \subset G$ implies $[G:C(x)] = 2$; i.e., G is 3-rewriteable.

It isn't necessary to invoke the centralizer characterization of 3-rewriteability to complete the proof of Lemma 3. If $[G:Z] = 4$, then $G/Z \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$. Thus $G = Z \cup xZ \cup yZ \cup xyZ$. The only triple products from G whose 3-rewriteability we might question have form $(xz_1)(yz_2)(xyz_3)$ or $(xz_1)(xyz_2)(yz_3)$. But, notice that $(xz_1)(yz_2)(xyz_3) = (xyz_3)(xz_1)(yz_2)$ and that $(xz_1)(xyz_2)(yz_3) = (yz_3)(xz_1)(xyz_2)$ because $x^2 \in Z$. This proof makes Lemma 3, which is an analogue of the fact that $[G:Z] \geq 4$ for nonabelian G, an appealing student exercise.

Now we can weave that elementary proof we promised. Note that $X \neq \varnothing$ since G is not 3-rewriteable. Choose $g \in X$ and set $n = [G:C(g)]$. Then $Z \cup Zg \subseteq C(g)$ and $(Z \cup Zg) \cap Y = \varnothing$. Thus $|C(g) \cap Y| \leq |G|/n - 2|Z|$ and so $|(G - C(g)) \cap Y| \geq |Y| - |G|/n + 2 \cdot |Z|$. If $x \in (G - C(g)) \cap Y$, then $[G:C(x)] = 2$ and $C(g) \cap (G - C(x)) \neq \varnothing$ implies, by Lemma 1, that $[G:C(xg)] \geq [G:C(g)] \geq 3$.

Therefore $(G - C(g)) \cap Y \subseteq X$; in fact $(G - C(g)) \cap Y \subseteq X - Zg$ as $Zg \subseteq X \cap C(g)$. Thus $|X| - |Z| = |X - Zg| \geq |(G - C(g)) \cap Y| \geq |Y| - |G|/n + 2 \cdot |Z|$; i.e.,

$$|X| \geq |Y| - |G|/n + 3 \cdot |Z|. \tag{6}$$

In view of Lemma 2 and (6) we are done if $|Y| \geq |G|/3$, so assume $|Y| < |G|/3$. In this case Lemma 3 implies that $|X| > |G|/2$ and, therefore, that $|X| > 3 \cdot |Z|$. The theorem is proved.

Corollary. *If G is not 3-rewriteable, then at least $|G| \cdot (n-1)/2n + |Z|$ elements of G have centralizers of index at least 3 where n is the greatest centralizer index among the elements of G. In particular, more than $1/3$ of the elements of G have centralizers of index at least 3.*

Proof: This follows directly from (6) by substituting $|G| - |X|$ for $|Y| + |Z|$.

The $1/2$ bound for 3-rewriteability is sharp in two senses.

i) *$Pr_2(G) = 1/2$ if, and only if, $G/Z \cong S_3$.* Our opening example suggests the involvement of S_3. That $Pr_2(G) = 1/2$ implies $G/Z \cong S_3$ is a straight forward application of Lemma 3 and the Corollary. The converse follows since $|X| = 3 \cdot |Z|$ and $|Y| = 2 \cdot |Z|$ for groups satisfying $G/Z \cong S_3$.

ii) *There exists a sequence, $\{G_n\}$, of 3-rewriteable groups such that $Pr_2(G_n) \downarrow 1/2$.* But where? A result of Ito **[9]** says that groups in which each conjugacy class is of order one or p, for a fixed prime p, must be the direct product of a p-group (a group whose order is a power of p) with this property and an abelian group. Thus, if G is 3-rewriteable we may write $G \cong T \times A$, where T is a 3-rewriteable 2-group and A is abelian. Conjugacy classes in direct products are direct products of conjugacy classes, so

$$Pr_2(G) = Pr_2(T \times A) = Pr_2(T) \cdot Pr_2(A) = Pr_2(T).$$

Net result: we may restrict our attention to 2-groups.

The quaternion group of order eight, mentioned in conjunction with the $5/8$ bound, is worth a look:

$$Q = \langle x, y, z | x^2 = y^2 = z^2 = x^{-1}y^{-1}xy = x^{-1}z^{-1}xz = e, y^{-1}z^{-1}yz = x \rangle.$$

The relevant facts are;

$$|Q| = 8 = 2^3,$$
$$Z = Q' = \{e, x\},$$
$$k = 5 = |Z| + (|G| - |Z|)/2 = (|G| + |Z|)/2,$$
$$Pr_2(Q) = 5/8 = 1/2 + |Z|/(2 \cdot |G|).$$

We generalize by taking G_n to be (an extra-special 2-group **[12]**) generated by $x_1, x_2, \ldots, x_{2n+1}$ subject to the relations

$$x_i^2 = e \text{ for } 1 \leq i \leq 2n+1,$$
$$x_i^{-1}x_j^{-1}x_ix_j = \begin{cases} x_1 & \text{for } i \text{ even and } j = i+1, \\ e & \text{otherwise.} \end{cases}$$

Then $|G_n| = 2^{2n+1}$ and $Z = G_n' = \{e, x_1\}$ so that $Pr_2(G_n) = k/|G_n| = 1/2 + 1/2^{2n+1}$.

A PROBLEM. We encourage study of the problem of determining bounds for $Pr_n(G; S)$. The following lemma generalizes (3) and prompts a conjecture.

Lemma 4. *If $n \geq 2$ and $\sigma \in S_n - \{id\}$, then $|Rw_n(G;\{\sigma\})| \leq k \cdot |G|^{n-1}$.*

Proof: The proof is by induction on n. The case for $n = 2$ was made in (2). Now assume the result holds for $n - 1$.

If $\sigma(n) = n$, then $x_1 x_2 \cdots x_n = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$ if, and only if, $x_1 x_2 \cdots x_{n-1} = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n-1)}$. Therefore $|Rw_n(G;\{\sigma\})| = |Rw_{n-1}(G;\{\hat{\sigma}\})| \cdot |G|$ where $\hat{\sigma}$ is σ restricted to $\{1,2,\dots,n-1\}$. The induction hypothesis yields the result.

If $\sigma(n) < n$, say $\sigma(n) = m$, then $x_1 x_2 \cdots x_n = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$ if, and only if, $x_n^{-1} x_{\sigma(j-1)}^{-1} \cdots x_{\sigma(1)}^{-1} x_1 x_2 \cdots x_n = x_{\sigma(j+1)} x_{\sigma(j+2)} \cdots x_m$ where $\sigma(j) = n$. Let $g = x_{\sigma(j-1)}^{-1} x_{\sigma(j-2)}^{-1} \cdots x_{\sigma(1)}^{-1} x_1 x_2 \cdots x_{n-1}$ and $h = x_{\sigma(j+1)} x_{\sigma(j+2)} \cdots x_m$. Notice that $|\{x_n | x_n^{-1} g x_n = h\}|$ is $|C(g)|$ or 0 for fixed $x_1, x_2, \cdots x_{n-1}$, and that g varies over G as x_m varies over G. Thus

$$\begin{aligned}
|Rw_n(G;\{\sigma\})| &\leq \sum_{x_1} \cdots \sum_{x_m} \cdots \sum_{x_{n-1}} |C(g)| \\
&= \sum_{x_1} \cdots \sum_{x_{n-1}} \left(\sum_{x_m} |C(g)| \right) \\
&= \sum_{x_1} \cdots \sum_{x_{n-1}} \left(\sum_{g} |C(g)| \right) \\
&= \sum_{x_1} \cdots \sum_{x_{n-1}} (k \cdot |G|) \\
&= k|G|^{n-1} \text{ as claimed.}
\end{aligned}$$

It follows from (3) and Lemma 4 that

$$\begin{aligned}
Pr_n(G;S) &= |Rw_n(G;S)|/|G|^n \leq |S| \cdot k/|G| = |S| \cdot Pr_2(G) \\
&\leq |S| \cdot (p_s^2 + p_s - 1)/p_s^3. \qquad (7)
\end{aligned}$$

Since $(p_s^2 + p_s - 1)/p_s^3 \downarrow 0$ as $p_s \to \infty$ we may use (7) to conclude that, for $|S|$ fixed and sufficiently large p_s, a "5/8-like" bound exists for $Pr_n(G;S)$. Random sampling (using CAYLEY **[8]**) of the "S-rewriteability hypercube" of various groups suggests such bounds exist independent of p_s.

Conjecture. *If G is not S-rewriteable then there exists $\rho_n(S) < 1$, independent of G, such that $Pr_n(G;S) \leq \rho_n(S) < 1$.*

Specifically, if $p_s \geq 7$, then $Pr_3(G; S_3 - \{\text{id}\}) \leq 275/343$. However, CAYLEY suggests $Pr_3(G; S_3 - \{\text{id}\}) \leq 17/18$. Thus for 3-rewriteability our conjecture is:

If G is not 3-rewriteable, then $Pr_3(G; S_3 - \{\text{id}\}) \leq \rho_3(S_3 - \{\text{id}\}) = 17/18$.

If this conjecture proves to be true, then the 17/18 bound is sharp because $Pr_3(S_3; S_3 - \{\text{id}\}) = 17/18$.

We conclude by observing that if G is a non-abelian finite simple group then $Pr_3(G, S_3 - \{\text{id}\}) \leq 5/12$. This follows from (7) because $Pr_2(G) \leq Pr_2(A_5)$ **[5]** and $Pr_2(A_5) = 1/12$. It seems likely that the bound is actually 27/100 because CAYLEY shows $Pr_3(A_5, S_3 - \{\text{id}\})$ to be 27/100.

ACKNOWLEDGMENT. The authors thank the referees for their suggestions.

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The beginning of wisdom is the definition of terms.

—*Socrates* (*470?–399* B.C.)

How Not to Land at Lake Tahoe!

Richard Barshinger

The following problem gives a simplified model of landing an airplane. It is adapted and extended from Trim [1] and is regularly presented in first semester calculus at my campus, where it is unanimously enjoyed and wins some converts to the methods of calculus.

Problem. An aircraft landing approach pattern is shaped generally as in Figure 1 below. The following conditions are imposed:

a) The cruising altitude is h when descent begins at a horizontal distance L from the airstrip.
b) A constant horizontal airspeed U must be maintained throughout descent (somewhat unrealistic).
c) At no time must the vertical component of acceleration exceed (in absolute value) some fixed constant k, $0 \leq k \ll g$, where g is the acceleration constant for gravity; i.e., $g = 32\ \text{ft}/\text{sec}^2$ (English units).

Model the plane's approach path by means of a cubic polynomial, using a coordinate system with origin at the beginning of the runway, so that descent starts at the point $(x, y) = (-L, h)$, in units of your choice. Impose suitable conditions at the beginning of descent and at touchdown. Discuss the implications of condition c) above, in the cases: 1) transcontinental flight; and, 2) peculiar airport situations (such as at South Lake Tahoe, CA).

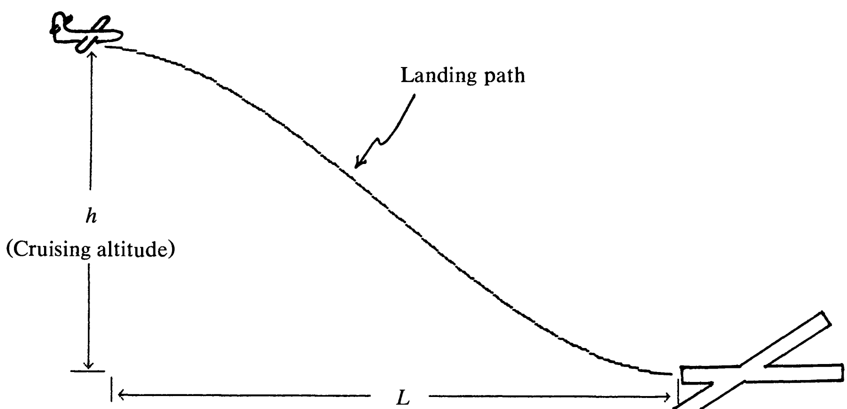

Figure 1

Solution. We let the landing path have the form:

$$y(x) = ax^3 + bx^2 + cx + d.$$

The following reasonable conditions are imposed:

$$\left.\begin{array}{lll} y(0) & = 0 & (\text{touchdown}) \\ \left.\dfrac{dy}{dx}\right|_{x=0} & = 0 & (\text{no crash}) \end{array}\right\} \text{imply } c = d = 0;$$

$$\left.\begin{array}{lll} y(-L) & = h & (\text{descent}) \\ \left.\dfrac{dy}{dx}\right|_{x=-L} & = 0 & (\text{no dive}) \end{array}\right\} \text{imply } \begin{array}{l} a = 2h/L^3 \\ b = 3h/L^2. \end{array}$$

Thus these conditions give:

$$y(x) = h\{2(x/L)^3 + 3(x/L)^2\},$$

where x/L is a dimensionless coordinate.

By using the chain rule (with the simplification of constant horizontal airspeed component $dx/dt = U$), we obtain:

$$v_y = \frac{dy}{dt} = \frac{6Uh}{L}\{(x/L)^2 + (x/L)\}$$

and

$$a_y = \frac{d^2y}{dt^2} = \frac{6U^2h}{L^2}\{2(x/L) + 1\}.$$

Now,

$$(a_y)_{\max(\min)} = \overset{+}{(-)}\,\frac{6U^2h}{L^2},$$

which occur at $(0,0)$ and $(-L,h)$, respectively. [Hence the airport approach resembles a ride in an elevator, where we "feel" the motion only at the top and bottom of descent.] Since we want $|a_y|_{\max} \leq k \ll g$, we have:

$$\frac{6U^2h}{L^2} \leq k.$$

Implications. 1) Los Angeles to New York (LAX to JFK) transcontinental flight aboard a jumbo ("heavy") jet.

$$L \geq \sqrt{\frac{6U^2h}{k}}.$$

If U and h are large, while k is small, L (the distance from the airport where descent begins) must be relatively big. On such a flight, with an airspeed of $U = 600$ mph and a cruising altitude of $h = 37{,}000$ ft, the author discovered, from his own experience, that descent began at his home near Scranton, PA, about 130 miles from New York! This will make the value of k, which is given by:

$$k = \frac{6U^2h}{L^2\cdot(3600)^2},$$

come out to $k = 0.36$ ft/sec^2. [The value $(3600)^2$ converts k from ft/hr^2 to ft/sec^2, since a mix of units such as mph and ft is actually in use by airlines (as opposed to mathematicians?)!]

2) San Francisco to South Lake Tahoe. Here we solve for U and obtain:

$$U \leq \sqrt{\frac{kL^2}{6h}}.$$

If L and k are small but h is relatively large, and if we don't want our coffee or the flight attendant to go floating about the cabin, then the airspeed must be kept low.

A few years ago the author had occasion to visit his two sisters-in-law (who are both in applied mathematics, dealing blackjack in the casinos) at Lake Tahoe. As our "gamblers's special" aircraft crossed the last peak of the Sierra Nevada mountains ($h = 11{,}000$ ft), there was the airport, seemingly directly below us ($L = 20$ mi), and we almost dove into a landing (see Figure 2)!

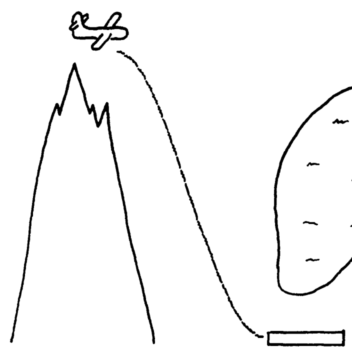

Figure 2. Landing at Lake Tahoe!
(Vertical scale exaggerated)

Our plane was, in fact, a two engine prop plane with an airspeed of about $U = 175$ mph. With the above values for U, L, and h, $k = 0.39\ \text{ft}/\text{sec}^2$, not much different from the value of k for the transcontinental flight discussed above! Parenthetically, because of noise restrictions aircraft are not allowed to land from over the lake to the north of the airport, and, consequently, jets cannot land at the airport at Lake Tahoe.

[Actually, I fudged a bit on the values for L and h in the example above, for the descent was somewhat more harrowing than I made it out to be. So therein lies a research project for the calculus class: to write letters and contact flight engineers at TWA and Golden West Airways for more accurate values of L, h, and U for the flights discussed.]

In practice, aircraft decrease their airspeed when landing and often engage in a banked loop around the airport in order to slow down further before touchdown. Nevertheless, the above simplistic model for the approach pattern qualitatively agrees with actual flying experience.

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Stenger's Conjecture on Independent Events

R. J. Gregorac and Robert Meany

In probability two events A and B are independent, if

$$P(A)P(B) = P(A \cap B), \tag{1}$$

where $P(A)$ denotes the probability of event A. Independence captures the idea that A and B are "probabilistically unrelated." Thus, when one finds independent events in one situation, one might guess the "same" events in another closely related situation would again be independent. William Stenger **[1]** illustrated how erroneous this idea can be with the following example. Flip n coins, $n \geq 2$. Let A be the event that the same side turns up on all coins and B the event that at most one head occurs. If all coins are fair ($p = \frac{1}{2}$), A and B are dependent for $n = 2$, independent for $n = 3$ and dependent for all $n \geq 4$.

What happens if instead, n is fixed but p is allowed to vary? In the special cases $p = 0$ and 1 the events A and B are independent for all n, so suppose then that the coins have probability of tails p, $0 < p < 1$. Stenger computed cases of independence for $n = 4$ ($p = 0.4113\ldots$), $n = 5$ ($p = 0.3709\ldots$) and conjectured for each $n \geq 3$ a probability p exists giving independent events.

We here show that a simple transformation changes this conjecture to an equivalent one which is easy to answer. Moreover, the same transformation can always be applied to any two events based on Bernoulli trials.

In Stenger's example $P(A \cap B) = p^n$, $P(A) = (1-p)^n + p^n$ and $P(B) = n(1-p)p^{n-1} + p^n$, and p must be found such that $P(A \cap B) = P(A)P(B)$. By substituting x for p and cancelling a factor this is equivalent to Theorem 1.

Theorem 1. *The polynomial*

$$((1-x)^n + x^n)(n(1-x) + x) - x$$

has exactly one root in $(0,1)$ *for* $n \geq 3$.

Proof: Substituting $x = (1+y)^{-1}$, $0 < y < \infty$, in the above polynomial yields the equivalent problem of showing

$$q_n(y) = (y^n + 1)(ny + 1) - (1+y)^n = 0$$

has exactly one positive solution. Note that if $n \geq 3$, then a y^2 term in $(1+y)^n$ will not be cancelled by any term in $(y^n + 1)(ny + 1) = ny^{n+1} + y^n + ny + 1$. Thus there is exactly one sign change in $q_n(y)$, so there is exactly one positive real root by Descartes rule of signs. ∎

It is clear that this transformation extends to arbitrary binomial examples based on n Bernoulli trials as follows.

The probability of an arbitrary event A can be expressed as $P(A) = \sum_{r+s=n} c_{rs}(1-p)^r p^s$ for some constants $c_{rs} \geq 0$. Replacing p by $(1+y)^{-1}$ changes $P(A)$ to $g(y)/(1+y)^n$, where $g(y)$ is a polynomial with nonnegative coefficients. Thus, replacing p by $(1+y)^{-1}$ in equation (1) and clearing the denominators by multiplying by an appropriate power of $1+y$, one can change equation (1) to the difference of two polynomials with nonnegative coefficients. Following the idea in the proof of Theorem 1, if these polynomials are identical in y, then all p will give independent events. This might occur, for example, when $P(A) = 1$.

If the polynomial corresponding to, say, $P(A \cap B)$ is cancelled by a proper part of the polynomial associated with $P(A)P(B)$, there will be no sign changes, so no p $(p \neq 0, 1)$ exists giving independent events. Simple examples of this kind can be found such that $A \cap B = \phi$ so $P(A \cap B) = 0$, but $P(A) \neq 0 \neq P(B)$. Another example is the case $n = 2$ in Theorem 1; $q_2(y) = [2y^3 + (1+y)^2] - (1+y)^2$.

Finally, in the remaining cases where neither of the above occurs, if there is an odd number of sign changes in the difference of these polynomials, then there will be at least one p yielding independent events A and B where $0 < p < 1$.

If one has events based on a multinomial distribution which is the union of events of probabilities like $cp_1^{n_1} \dots p_r^{n_r}$, then one can vary p_{r-1} while keeping $p_1, \dots, p_{r-2}$ fixed, so that $a = 1 - p_1 - \cdots - p_{r-2} > 0$. The substitution $p_{r-1} = (a+y)^{-1}$ can be used in the manner above and if one p_{r-1} is found that gives independent events, then the $p_1, \dots, p_{r-2}$ can be varied subject to $p_1 + \cdots + p_{r-2} = 1 - a$ and will give a family of solutions.

The numerical evidence in Stenger's example suggests the following result.

Theorem 2. *If $n > 3$, then*

$$\frac{1}{n+2} < p_{n+1} < p_n < \frac{1}{2},$$

where p_n is a root of the $(n+1)$-st degree polynomial in Theorem 1.

Proof: Let $\omega_n = \omega$ be the unique positive root of $q_n(y) = (y^n + 1)(ny + 1) - (1+y)^n$ where $n > 3$. Note

$$q_n(y) = y^n\left[\left(1 + \frac{1}{y^n}\right)(ny+1) - \left(1 + \frac{1}{y}\right)^{y \cdot n/y}\right].$$

Thus, for $y > 1$,

$$q_n(y) > y^n\left[(ny+1) - e^{n/y}\right]. \tag{2}$$

Now observe $1 < \omega < n$, for $q_n(1) = 2(n+1) - 2^n < 0$ and $q_n(n) > n^n[n^2 + 1 - e] > 0$. One sees $q_{n+1}(y)$ and $q_n(y)$ are related by

$$q_{n+1}(y) = yq_n(y) + y^{n+2} + 1 - (1+y)^n + ny(1-y)$$

so

$$q_{n+1}(\omega) = \omega^{n+2} + 1 - (1+\omega)^n + n\omega(1-\omega).$$

Since $n > \omega$, $n\omega > \omega^2$, so

$$0 = q_n(\omega) > (\omega^n + 1)(\omega^2 + 1) - (1+\omega)^n > \omega^{n+2} + 1 - (1+\omega)^n.$$

Because $\omega > 1$, $n\omega(1-\omega) < 0$. Therefore $q_{n+1}(\omega) < 0$. Since $q_{n+1}(n+1) > 0$, we see $\omega = \omega_n < \omega_{n+1} < n+1$, proving $1/(n+2) < p_{n+1} < p_n < \frac{1}{2}$ as claimed. (The lower bound can be improved by considering $ny + 1 - e^{n/y} \geq 0$ in (2).) ■

A few further examples of p_n are

$$p_6 = 0.3449\ldots, \quad p_{12} = 0.2574\ldots, \quad p_{24} = 0.1783\ldots.$$

A final comment seems required, since this was written after Robert Meany's death last year. The first author had sent a proof of these results to Professor Stenger shortly after seeing his conjecture. Meany then found the delightfully simple proof of Theorem 1 given here and we kept this example for classroom use.

It was only now that the first author realized how general Meany's argument is and thought these comments would be of interest to others.

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> It is with mathematics not otherwise than it is with music, painting or poetry. Anyone can become a lawyer, doctor or chemist, and as such may succeed well, provided he is clever and industrious, but not every one can become a painter, or a musician, or a mathematician: general cleverness and industry alone count here for nothing.
>
> —*P.J. Moebius*

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as an operations research consultant for Milliken Textile while a Lilly Endowment Fellow) and administration (Chair: 1981–87). He likes teaching and pure mathematics best. The Inland-Ryerson Foundation and Rose-Hulman have given him two teaching awards and a scholarship award over the years. His current passions include counting problems in finite groups, directing an NSF-REU and bicycle racing. As a mathematician, he's an awfully good criterium rider.

Mark Walker received his B.S. (with honors) in mathematics from New Mexico State University in 1990. During the summer of 1989 he participated in an NSF-REU at Rose-Hulman (some of his work from that summer is reflected in this paper and in [**15**]). In 1990 he was awarded an NSF Graduate Fellowship. He is working on a Ph.D. in mathematics at the University of Illinois-Urbana.

Richard Barshinger earned his Ph.D. (part time) in mathematical sciences at SUNY-Binghamton (1981), while continuing to hold fulltime academic employment at Penn State-Scranton. [He would not recommend this method of approach to anyone.] His thesis advisor was Jim Geer, and he continues to work in the field of uniform asymptotics. In addition, he is a professional organist/harpsichordist, having studied antique Flemish harpsichords in Antwerp, while on a grant from the government of Belgium.

Robin Hartshorne was educated at Harvard, and spent the early part of his professional career there. He then moved to Berkeley where he is a professor at the University of California. He has written an introductory text "Algebraic Geometry" (Springer 1977) and has received the Steele Prize of the American Mathematical Society. Besides mathematics, he has ongoing interests in mountaineering and in music.

Bob Scher graduated from Harvard with a degree in philosophy. Primary interests are number theory and mathematics education. He taught at Dominican College, receiving a grant to continue exploring new pedagogical approaches in middle and upper levels. He has interests outside mathematics and has written *How to Tie Your Shoes*, (Harmony Books, Fall, 1992), and *The Fear of Cooking* (Houghton Mifflin, 1984), and has also published a game, *Byzantium—Beauty vs. Ugly in a Game of Design*. He is Director of Communications at PeerLogic, a pioneer in advanced distributed computing software.

It is the man not the method that solves the problem.

—*H. Maschke*

PROBLEMS AND SOLUTIONS

Edited by:
Richard T. Bumby, Fred Kochman and Douglas B. West

Proposed problems should be sent to the MONTHLY PROBLEMS address given on the inside front cover. Please include solutions, relevant references, etc. Three copies are requested.

Solutions of published problems should arrive before October 31, 1992 at the MONTHLY PROBLEMS address given on the inside front cover. Solutions should be typed with double spacing, including the problem number and the solver's name and mailing address. Two copies suffice. A self-addressed postcard or label should be included if an acknowledgement is desired.

*An asterisk (*) after the number of a problem, or part of a problem, indicates that no solution is currently available. Partial solutions will be useful in such cases. Otherwise, the published solution is likely to be based on a solution which is complete and correct. Of course, an elegant partial solution or a method leading to a more general result is always useful and welcome. In addition, references to other appearances of MONTHLY problems or to solutions of these problems in the literature are also solicited.*

PROBLEMS

10220. *Proposed by Solomon W. Golomb, University of Southern California, Los Angeles, CA.*

Suppose ε is a given positive number. A positive integer n will be called *ε-squarish* if and only if it has a factorization $n = ab$ with $1 \leq a < b < (1 + \varepsilon)a$. Prove that there are infinitely many occurrences of six consecutive ε-squarish numbers.

10221. *Proposed by Raphael M. Robinson, University of California, Berkeley, CA.*

Let α and β be conjugate algebraic numbers with $|\alpha| = 1$.
(a) Show that if $|\beta| \neq 1$, then $|\beta|^2$ must be irrational.
(b) Show that the possible values of β are everywhere dense in the complex plane.

10222. *Proposed by Gerry Myerson, Macquarie University, North Ryde, NSW, Australia.*

(a) Let h be a strictly increasing convex function on $[0,1]$. Let n be a positive integer. Assume that $0 \leq a_1 \leq \cdots \leq a_n \leq 1$ and $0 \leq x_1 \leq \cdots \leq x_n \leq 1$. Prove that

$$\sum_{j=1}^{n} h(|x_j - a_j|) \leq \max\left(\sum_{j=1}^{n} h(a_j), \sum_{j=1}^{n} h(1 - a_j)\right).$$

(b) Let n be a positive integer and let $a_j = (2j-1)/2n$ for $1 \leq j \leq n$. Assume that $0 \leq x_1 \cdots \leq x_n \leq 1$. Let h be a strictly increasing, but not necessarily convex, function on $[0,1]$. Prove that

$$\sum_{j=1}^{n} h(|x_j - a_j|) \leq \sum_{j=1}^{n} h(a_j).$$

10223. *Proposed by Julio Kuplinsky, Amherst, NY.*

For $p \in \mathbb{R}$, $q = 1 - p$, and positive integers n, prove

$$\sum_{k=n}^{2n-1} \binom{k-1}{n-1}\left[p^n q^{k-n} + p^{k-n} q^n\right] = 1.$$

10224. *Proposed by Yves Nievergelt, Eastern Washington University, Cheney, WA.*

Consider all 2 by 2 real matrices $A = (a_{i,j})$ having non-negative determinant, all entries positive, and $a_{1,1} = a_{2,2}$. Also, for each positive integer p, denote by $a_{i,j}^{(p)}$ the entries of the power A^p. Prove that

$$\lim_{p\to\infty} \frac{a_{1,1}^{(p)}}{a_{2,1}^{(p)}} = \sqrt{\frac{a_{1,2}}{a_{2,1}}} = \lim_{p\to\infty} \frac{a_{1,2}^{(p)}}{a_{1,1}^{(p)}}.$$

10225. *Proposed by Paul R. Chernoff, University of California, Berkeley, CA.*

Suppose that ϕ: $[0,\infty) \to [0,\infty)$ is a strictly increasing, strictly concave function with $\phi(0) = 0$. Let m^* be Lebesgue outer measure on the unit interval $I = [0,1]$. For $E \subset I$, define $n^*(E) = \phi(m^*(E))$. Show that n^* is an outer measure and determine the n^*-measurable sets.

10226. *Proposed by Chu Wenchang, Academica Sinica, Beijing, China.*

Consider the functional equation

$$f(a-b)f(a-c)f(a-d)f(a-e) - f(b)f(c)f(d)f(e)$$
$$= q^b f(a)f(a-b-c)f(a-b-d)f(a-b-e)$$

with parameter q, where the variables a, b, c, d and e are related by

$$b + c + d + e = 2a.$$

(a) When $q = 1$, show that

$$f(\alpha) = \sin(k\alpha),$$

for any k, is a solution.

(b) When $0 < q < 1$, show that

$$f(\alpha) = \left(\Gamma_q(\alpha)\Gamma_q(1-\alpha)\right)^{-1}$$

is a solution.

10227. *Proposed by Antonio Montes, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Suppose that α is a convex simple closed curve in the plane which is piecewise C^1 and suppose that the origin lies inside α.

(a) Show that the length of α is given by

$$-\oint_\alpha \vec{\rho} \cdot d\vec{\tau},$$

where $\vec{\tau}$ is the unit tangent vector in a counterclockwise direction and $\vec{\rho}$ is the vector from the origin to the curve.

(b) If, in addition, α is C^1 and piecewise C^2, show that the length of α is given by

$$\int_0^{2\pi} r(\theta)^2 \kappa(\theta)\, d\theta,$$

where $r(\theta)$ indicates the distance from the origin to the point on α with polar angle θ and $\kappa(\theta)$ denotes the curvature of α at the point $(\theta, r(\theta))$.

10228. *Proposed by Ernesto Bruno Cossi and Marcos Antonio Sebastiani, Universidade Federal do Rio Grande do Sul, Porto Alegre, Brazil.*

(a) Let χ be a Banach space, and let B be a bounded, nonempty subset of $\mathscr{X}$ such that, for any pair of points x and y in B, there is an open ball U such that $U \subset B$, $x \in U$ and $y \in U$. Show that B is an open ball.

(b) Show that the result of part (a) does not generalize to the case in which $\mathscr{X}$ is only assumed to be a complete metric space.

NOTES

(10222) A "convex function" h is one for which $h(\lambda x + (1-\lambda)y) \le \lambda h(x) + (1-\lambda)h(y)$ for every x and y in its domain and every λ with $0 \le \lambda \le 1$. Note that there is no assumption that the function h in this problem is differentiable. **(10223)** If $0 \le p \le 1$, the proposer provides the following probabilistic interpretation. "Peter and Mary are playing a series of games. Peter wins each game with probability p and loses to Mary with probability q. The winner of the series is the first to win n games. The term of the sum with index k is the probability that the series ends with the kth game." (This probabilistic model appeared in problem E3386 [1990, 427; 1992, 272].) What is desired here is a proof which does not use the probabilistic model and is immediately valid for all $p \in \mathbb{R}$. **(10224)** As an example, take $A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$. Then

$$A^{16} = \begin{pmatrix} 886731088897 & 1254027132096 \\ 627013566048 & 886731088897 \end{pmatrix}$$

and $a_{1,1}^{(16)}/a_{2,1}^{(16)} = 88673108897/62701356048 = 1.41421356237\ldots \approx \sqrt{4/2} = \sqrt{2}$ to all displayed digits. **(10225)** The function $\phi(x) = \sqrt{x}$ is an example. It might be helpful to note that the hypotheses on ϕ guarantee that it is continuous and

subadditive (i.e. $\phi(x+y) \leq \phi(x) + \phi(y)$). **(10226)** The identity in part (b) may be interpreted as the q-extension of the trigonometric identity in part (a) since the limit as $q \to 1^-$ of the indicated solution in part (b) is of the form given in part (a). The q-gamma function in the statement of the problem is defined by

$$\Gamma_q(\alpha) = \frac{(q;q)_\infty}{(q^\alpha;q)_\infty}(1-q)^{1-\alpha}$$

where $(x;q)_\infty = \prod_{n=0}^{\infty}(1-xq^n)$. Further details may be found in the books of N. J. Fine, *Basic Hypergeometric Series and Applications* (reviewed in this MONTHLY 97 (1990), pp. 82–88), and G. Gasper and M. Rahman, *Basic Hypergeometric Series* (reviewed in this MONTHLY, 98 (1991), pp. 282–285). **(10227)** Since the assumptions in part (a) are so weak, some care needs to be exercised in interpreting the integral. The additional smoothness in part (b) allows θ to be taken as a parameter for describing α. **(10228)** The property studied in this problem appears to be already interesting when $\mathscr{X}$ is the Euclidean plane. One might approach this problem by starting by analyzing this example and then attempting to find the appropriate level of generality of the methods employed. The statement given here uses the following terminology. A "metric space" is a set with a real-valued distance function $d(x,y)$ satisfying: (i) $d(x,y)=0$ iff $x=y$; (ii) $d(x,y)=d(y,x)$; (iii) $d(x,z) \leq d(x,y)+d(y,z)$. An "open ball" is a set of the form $B(x,r) = \{y: d(x,y) < r\}$. A metric space is called "complete" if every Cauchy sequence converges. A complete metric space χ is a "Banach space" if it is a vector space over $\mathbb{R}$ in which $d(x,y) = d(0, y-x)$ for all $x, y \in \mathscr{X}$ and $d(0, \alpha x) = |\alpha| x$ for all $x \in \mathscr{X}$ and $\alpha \in \mathbb{R}$.

SOLUTIONS

Matrices with Product Zero

E3382 [1990, 343]. *Proposed by Geoffrey R. Robinson, UMIST, Manchester, England.*

Suppose that R is a commutative ring with identity, and that A is an n by n matrix with entries in R. If det A is a zero divisor in R, show that there is an n by n matrix B with entries in R such that B is not the zero matrix O, but $AB = BA = O$. More specifically, show that B may be represented as a polynomial in A, i.e. as a finite linear combination of $I, A, A^2, \ldots$ with coefficients from R.

Composite solution by the proposer and O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands.

Special case: det $A = 0$.

Let $f(x)$ be the monic polynomial of least degree with $f(A)$ nilpotent, and let k be such that $f(A)^k = 0$. (The Cayley–Hamilton theorem gives a monic polynomial $h(x)$ with $h(A) = 0$, which is certainly nilpotent, so there are such polynomials.) Write $f(x) = xg(x) + \gamma$, where γ is the constant term of $f(x)$, and note that

$$(\gamma I)^k = (f(A) - Ag(A))^k = (f(A))^k + AM = AM$$

for some matrix M. Taking determinants,

$$\gamma^{kn} = \det(AM) = \det A \cdot \det M = 0.$$

Now, γI and $f(A)$ are commuting nilpotent matrices and so $Ag(A) = f(A) - \gamma I$ is also nilpotent; say, $0 = (Ag(A))^l = A^l g(A)^l$. Since $g(x)$ has smaller degree than $f(x)$, $g(A)^l \neq 0$. Hence there is a value of j with $0 < j < l$ such that $A^j g(A)^l \neq O$ and $A^{j+1} g(A)^l = O$. Define $B = A^j g(A)^l$.

General case: $\det A = \alpha \neq 0$, and $\alpha\beta = 0$ for some $\beta \neq 0$.

Let $J = \{c \in R\colon c\beta = 0\}$; note that J is an ideal of R. Let $\bar{A}$ be the matrix obtained from A by reducing entries to R/J. Since $\det A = \alpha \in J$, we have $\det \bar{A} = 0$ in R/J. By the case discussed above, there is a monic polynomial $\bar{h}$ such that $\bar{h}(\bar{A})$ is nonzero and $\overline{A}\bar{h}(\bar{A}) = O$ in R/J. Let h be a monic polynomial with coefficients in R which reduces to $\bar{h}$ in R/J. Take $B = h(A)$ and let $\bar{B}$ denote the result of reducing the entries of B to R/J. Then $\bar{B} = \bar{h}(\bar{A}) \neq O$, and hence $\bar{B}$ has at least one entry which is not annihilated by β. Thus $\beta B \neq O$, and $A(\beta B) = \beta(AB) = O$ since AB has all entries in J because $\overline{AB} = O$ in R/J. Thus βB is the desired matrix.

Editorial comment. Maki Iisaka, David G. Robinson, and William P. Wardlaw noted that the result of the problem is proved in N. H. McCoy, *Rings and Ideals*, Carus Mathematical Monographs No. 8, 1948, pp. 176–178. The above proof is different. The problem had a high fraction of incorrect solutions because of the mistaken assumption that the matrix obtained by a straightforward application of the Cayley–Hamilton Theorem was nonzero.

Also solved by S. Chen, D. R. Estes, M. Iisaka, D. G. Robinson, and W. P. Wardlaw. Three incorrect solutions were received.

Orthogonal Vectors of Motion

E3390 [1990, 428]. *Proposed by Robert B. Israel, University of British Columbia, Vancouver, BC, Canada.*

Let $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$ be the position, velocity, and acceleration vectors of a particle at time t. Suppose the particle moves so that $\mathbf{a}$ is always perpendicular to both $\mathbf{r}$ and $\mathbf{v}$.

(a) Show that $\mathbf{v}_\infty = \lim_{t\to\infty} \mathbf{v}$ exists and show that $t\mathbf{v}_\infty - \mathbf{r}$ is bounded.

(b) Show that if $\int_0^\infty t|\mathbf{a}(t)|dt < \infty$, then $\lim_{t\to\infty}(t\mathbf{v}_\infty - \mathbf{r})$ exists.

Solution by Michael Golomb, Purdue University, West Lafayette, IN. We use $\cdot$ for inner product and $|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$ for norm. From the assumption $\mathbf{v} \cdot \mathbf{a} = 0$, we conclude that $\mathbf{v} \cdot \mathbf{v}$ (and hence $|\mathbf{v}|$) is constant as a function of t. If this constant is 0, then the particle remains fixed and the assertions are trivially true. Hence we may assume $\mathbf{v} \neq \mathbf{0}$, and we may choose the time scale so that $\mathbf{v} \cdot \mathbf{v} = 1$.

Now the assumption $\mathbf{r} \cdot \mathbf{a} = 0$ implies $(d/dt)(\mathbf{r} \cdot \mathbf{v}) = \mathbf{v} \cdot \mathbf{v} = 1$, and hence $\mathbf{r} \cdot \mathbf{v} = t + c$. We may choose the time origin so that $c = 0$, and hence $\mathbf{r} \cdot \mathbf{v} = t$. From this we have $\mathbf{r} \cdot \mathbf{r} = t^2 + |\mathbf{r}(0)|^2$. If $\mathbf{r}(0) = \mathbf{0}$, then $|\mathbf{r}(t)| = t$ for all $t > 0$. This yields $\mathbf{r} \cdot \mathbf{v} = t = |\mathbf{r}||\mathbf{v}|$, which implies $\mathbf{r} = t\mathbf{v}$. But now $\mathbf{a} = \mathbf{0}$ and the assertions are again trivially true with a constant velocity vector. Hence we may assume $\mathbf{r}(0) \neq \mathbf{0}$, and we may choose the distance scale so that $|\mathbf{r}(0)| = 1$. This produces the following two equations of motion:

$$|\mathbf{r}(t)|^2 = 1 + t^2, \qquad |\mathbf{v}(t)|^2 = 1. \tag{1}$$

In order to study the limiting behavior for large t, we assume $t \geq t_0 > 0$, and we set $\mathbf{r}(t) = t\mathbf{y}(t)$. Using $\dot{\mathbf{y}}$ for $(d/dt)\mathbf{y}$, the equations of (1) become

$$\mathbf{y} \cdot \mathbf{y} = 1 + t^{-2} \tag{2}$$

$$t^2(\dot{\mathbf{y}} \cdot \dot{\mathbf{y}}) + 2t(\mathbf{y} \cdot \dot{\mathbf{y}}) + (\mathbf{y} \cdot \mathbf{y}) = 1 \tag{3}$$

Differentiating (2) yields $\mathbf{y} \cdot \dot{\mathbf{y}} = -t^{-3}$, and then (3) reduces to $|\dot{\mathbf{y}}| = t^{-2}$. Since $\dot{\mathbf{y}} = t^{-1}\mathbf{v} - t^{-2}\mathbf{r}$, we conclude $|t\mathbf{v} - \mathbf{r}| = 1$. This implies

$$t\mathbf{v}(t) - \mathbf{r}(t) = \mathbf{u}(t), \quad \text{where } |\mathbf{u}(t)| = 1. \tag{4}$$

The linear equation (4) has the solution $\mathbf{r}(t) = (t/t_0)\mathbf{r}(t_0) + t\int_{t_0}^{t} s^{-2}\mathbf{u}(s)\,ds$ for $t \geq t_0$, from which differentiation yields $\mathbf{v}(t) = t_0^{-1}\mathbf{r}(t_0) + \int_{t_0}^{t} s^{-2}\mathbf{u}(s)\,ds + t^{-1}\mathbf{u}(t)$. Since $\int_{t_0}^{\infty} s^{-2}\mathbf{u}(s)\,ds$ is convergent and $|\mathbf{u}| = 1$, we conclude that $\mathbf{v}(t)$ approaches a limit vector $\mathbf{v}_\infty = t_0^{-1}\mathbf{r}(t_0) + \int_{t_0}^{\infty} s^{-2}\mathbf{u}(s)\,ds$, as desired for (a). Furthermore,

$$t\mathbf{v}_\infty - \mathbf{r}(t) = t\int_t^\infty s^{-2}\mathbf{u}(s)\,ds. \tag{5}$$

Since $|\mathbf{u}| = 1$, the norm of this is bounded by $t\int_t^\infty s^{-2}\,ds = 1$, which completes the proof of (a).

Now consider (b), where we assume $\int_0^\infty s|\mathbf{a}(s)|ds$ is bounded. Using integration by parts, we have $\int_{t_0}^{t} s\mathbf{a}(s)\,ds = \mathbf{u}(t) - \mathbf{u}(t_0)$. By the assumption, this implies that $\mathbf{u}(t)$ converges to a vector $\mathbf{u}_\infty$. Using the relationship $t\mathbf{a} = \dot{\mathbf{u}}$ that arises by differentiating (4), the result of integrating by parts in (5) is

$$t\mathbf{v}_\infty - \mathbf{r}(t) = \mathbf{u}(t) + t\int_t^\infty \mathbf{a}(s)\,ds.$$

Since $|t\int_t^\infty \mathbf{a}(s)\,ds| \leq \int_t^\infty |s\mathbf{a}(s)|\,ds$, we obtain $\lim_{t\to\infty}[t\mathbf{v}_\infty - \mathbf{r}(t)] = \mathbf{u}_\infty$, which proves (b).

The arguments hold without change in n-dimensional Euclidean space and in Hilbert space.

Editorial comment. The proposer notes that the special case $\mathbf{a} = c\mathbf{v} \times \mathbf{r}/|\mathbf{r}|^3$ describes the motion of a charged particle in the field of a magnetic monopole (see A. D. Jette, this MONTHLY 76(1960), 164–167). This satisfies the condition of (b), since $|\mathbf{a}| \sim c'/t^3$ for some constant c' as $t \to \infty$. He also proves a converse to (b): for any nonnegative continuous function $m(t)$ such that $\int_0^\infty sm(s)\,ds = \infty$, he constructs a trajectory with $|\mathbf{a}(t)| \leq m(t)$ such that $\lim_{t\to\infty}(t\mathbf{v}_\infty - \mathbf{r})$ does not exist.

Solved also by M. Falkowitz (Israel), E. A. Herman, M. E. Kuczma (Poland), O. P. Lossers (The Netherlands), S. L. Paveri-Fontana (Italy), O. Saleh & T. Walters, K. Schilling, J. H. Steelman, R. Stong, and the Proposer. One incorrect solution was received.

Convergence-Preserving Functions

E3404 [1990, 847]. *Proposed by the editors. (A modification of a problem proposed by the late Reuven Gurevič.)*

Suppose f is a function from $\mathbb{R}$ to $\mathbb{R}$ such that $\Sigma f(a_n)$ is convergent whenever Σa_n is a convergent series of real terms. Prove that f is differentiable at the origin.

Solution by Yoav Benyamini, Technion, Haifa, Israel. We prove a stronger result in a more general setting. Let X and Y be normed spaces, and assume that f is a function from X into Y such that the series $\Sigma f(a_n)$ converges in Y for every

convergent series Σa_n in X. Then there is an $\varepsilon > 0$ and a continuous linear operator T from X into Y so that $f = T$ in the ball $B_X(\varepsilon) = \{x \in X: \|x\| < \varepsilon\}$. It follows that in the special case when $X = Y = \mathbb{R}$, there is a $\lambda \in \mathbb{R}$ such that $f(x) = \lambda x$ in some neighborhood of 0. In particular, f is differentiable at 0.

The proof requires two assertions:

I. There is an $\varepsilon_1 > 0$ such that $f(x + z) = f(x) + f(z)$ whenever x, z in X satisfy $\|x\|, \|z\| < \varepsilon_1$.

II. There is an $\varepsilon_2 > 0$ and a constant K such that $\|f(x)\| \leq K\|x\|$ for all $x \in X$ with $\|x\| < \varepsilon_2$.

Once we have I and II, we can take $\varepsilon = \frac{1}{2}\min\{\varepsilon_1, \varepsilon_2\}$. If $\|x\|, \|z\| < \varepsilon$, then $\|x - z\| < \min\{\varepsilon_1, \varepsilon_2\}$, and I and II imply $\|f(x) - f(z)\| = \|f(x - z)\| \leq K\|x - z\|$. Thus f is continuous on the ball $B_X(\varepsilon)$. The additivity implies that $f(\lambda x) = \lambda f(x)$ for rational λ's such that $x, \lambda x \in B_X(\varepsilon)$, and continuity implies that this holds for irrational λ as well. Hence on $B_X(\varepsilon)$, f is the restriction of a continuous linear operator.

Proof of I. We first show that f is an odd function in some neighborhood of zero. Indeed, if this were false, choose a sequence x_n converging to zero so that $y_n = f(x_n) + f(-x_n) \neq 0$. Choose integers N_n so that $N_n\|y_n\| > 1$, and define a sequence $\{a_j\}$ by blocks: the nth block will have $2N_n$ terms obtained by repeating the two elements x_n and $-x_n$, N_n times. As x_n converges to zero, the series Σa_j converges to zero. But $\Sigma f(a_j)$ does not satisfy the Cauchy criterion for convergence since the sum of the nth block is $N_n\|y_n\|$, which does not converge to zero.

Now assume I is false, and find two sequences x_n and z_n converging to zero in X such that $y_n = f(x_n + z_n) - f(x_n) - f(z_n) \neq 0$ in Y. Choose integers $N_n \geq \|y_n\|^{-1}$, and define a sequence $\{a_j\}$ in X in blocks as follows: the nth block as $3N_n$ terms and is obtained by repeating the three terms $x_n + z_n, -x_n, -z_n$ a total of N_n times. Since x_n and z_n converge to zero, and the sum of each triplet is zero, the series Σa_j converges to zero. However, $\Sigma f(a_j)$ does not satisfy the Cauchy condition for convergence, because the sum of the nth block is $N_n\|y_n\|$, which by the choice of N_n does not converge to zero.

Proof of II. Assume II is false, and find a sequence $x_n \in X$ such that $\|x_n\| < 2^{-n}$, while $\|f(x_n)\| > 2^n\|x_n\|$. Note that since $f(0) = 0$, none of the x_n's is 0. Again we define a sequence $\{a_j\}$ in blocks: the nth block has $2N_n$ terms, where N_n is chosen so that $2^{-n-1} \leq N_n\|x_n\| < 2^{-n}$, and consists of N_n copies of x_n followed by N_n copies of $-x_n$. The series Σa_j then convergers to zero, but $\Sigma f(a_j)$ again does not satisfy the Cauchy condition: the norm of the sum of the first N_n terms of the nth block satisfies $\|f(x_n)\|N_n > 2^n\|x_n\|N_n \geq 2^n 2^{-n-1} = \frac{1}{2}$ and hence does not converge to zero.

Note: the reason for including the $-x_n$ terms in the last construction is to ensure that Σa_j converges to zero. Since X is not assumed to be complete, absolute convergence does not imply convergence, and we must be careful that the sum of the constructed series really converges to an element in X.

Editorial comment. Gerald Wildenberg proved earlier in this MONTHLY [95(1988) 544–544] that the stated conditions of the problem imply $f(x) = kx$ in a neighborhood of 0.

Solved also by M. Cook, B. G. Dearden & M. B. Gregory, M. Golomb, R. Gurevič & V. Ja. Kreinovič, R. B. Israel (Canada), K. S. Kedlaya (student), M. E. Kuczma, H. C. Morris, A. Müller (Switzerland), A Nijenhuis, A. Riese, K. Schilling, R. Stong, A Tissier (France), G. Wildenberg, and M. Zelený (student, Czechoslovakia). Four incorrect or incomplete solutions were received.

A Problem on Graph Coloring

E3409 [1990, 916]. *Proposed by Ioan Tomescu, University of Bucharest, Bucharest, Romania.*

Suppose G is a connected k-chromatic graph which is neither a complete graph nor a cycle on m vertices with $m \equiv 3 \pmod 6$. Prove that in any k-coloring of G there exist two vertices of the same color having a common neighbor.

Solution by R. J. Chapman, University of Exeter, United Kingdom. Let G be k-chromatic and suppose G is neither a complete graph nor an odd cycle. Then by Brooks's Theorem $k \le \Delta$, where Δ is the maximum degree of G. Choose a vertex v of degree Δ. The Δ vertices adjacent to v must be colored with at most $k - 1 < \Delta$ colors. Hence there is a pair of vertices of the same color both adjacent to v.

It remains only to check the result for G an m-cycle, where m is odd and not divisible by 3. The graph G is clearly 3-chromatic, so suppose that G is 3-colored with no vertex adjacent to two vertices of the same color. Then each vertex and its two neighbors must be colored with all three colors. It follows that it we traverse the cycle in the appropriate direction, then the colors of the vertices are 1231231... . But this is impossible as m is not divisible by 3.

Editorial comment. Brooks's Theorem may be found in any of the following references: B. Bollobás, *Graph Theory*, Springer-Verlag, 1979, p. 91; J. A. Bondy & U. S. R. Murty, *Graph Theory and Its Applications*, North Holland, 1976, p. 122; R. L. Brooks, On coloring the nodes of a network, *Proc. Cambridge Philos. Soc.* 37(1941) 194–197.

Solved also by J. Balogh (student, Hungary), D. Callan, J. A. de-Loera, R. B. Maddox, A. Pedersen (Denmark). S. G. Penrice, D. F. Rall (Canada), R. Stong, P. Tracy, the Anchorage Math Solutions Group, and the proposer.

A Monotonic Function

6643 [1990, 929]. *Proposed by Gérard Letac, Université Paul Sabatier, Toulouse, France.*

For positive real t put

$$\phi(t) = \frac{1}{\Gamma(t)} \int_t^{+\infty} x^{t-1} e^{-x}\, dx.$$

Prove that ϕ increases from 0 to $\frac{1}{2}$ when t varies from 0 to $+\infty$.

Solution by Tom Paine, Southern Illinois University at Carbondale, Illinois. Set $f_t(x) = 0$ for $x \le 0$ and $f_t(x) = x^{t-1}e^{-x}/\Gamma(t)$ for $x > 0$. Then

$$\frac{d\phi}{dt} = -f_t(t) + \int_t^{\infty} \frac{\partial f_t}{\partial t}(x)\, dx. \tag{1}$$

From the well-known convolution equality

$$f_{t+\varepsilon}(x) = \int_0^x f_\varepsilon(u) f_t(x-u)\, du \tag{2}$$

(an immediate consequence of the formula expressing the beta function in terms of

gamma functions) we have

$$\frac{\partial f_t}{\partial t}(x) = \lim_{\varepsilon \to 0} \int_0^\infty \frac{f_\varepsilon(u)}{\varepsilon}(f_t(x-u) - f_t(x))\,du$$

$$= \int_0^\infty \frac{e^{-u}}{u}(f_t(x-u) - f_t(x))\,du. \tag{3}$$

Set $F(u) = \int_u^\infty (e^{-y}/y)\,dy$ and use integration by parts to show the final integral in (3) is equal to

$$-\int_0^\infty F(u) f_t'(x-u)\,du.$$

Then substitute it into (1); a routine interchange of the order of integration then yields

$$\frac{d\phi}{dt} = -f_t(t) + \int_0^t F(u) f_t(t-u)\,du. \tag{4}$$

The identity $f_t(t) = \int_0^t e^{-u} f_t(t-u)\,du$ gives

$$\frac{d\phi}{dt} = \int_0^t (F(u) - e^{-u}) f_t(t-u)\,du. \tag{5}$$

Putting

$$g(t) = e^t t^{-t} \Gamma(t) \frac{d\phi}{dt}$$

and making the change of variable $u = zt$, we obtain

$$g(t) = \int_0^1 (e^{tz} F(tz) - 1)(1-z)^{t-1}\,dz. \tag{6}$$

Thus

$$\frac{dg}{dt} = \int_0^1 \left(z e^{tz} F(tz) - \frac{1}{t} \right)(1-z)^{t-1}\,dz$$

$$+ \int_0^1 (e^{tz} F(tz) - 1)(1-z)^{t-1} \ln(1-z)\,dz. \tag{7}$$

Integration by parts allows the second integral in (7) to be written as

$$\int_0^1 \frac{(1-z)^t}{t} \left[\left(t e^{tz} F(tz) - \frac{1}{z} \right) \ln(1-z) - (e^{tz} F(tz) - 1)(1-z)^{-1} \right] dz.$$

Thus

$$\frac{dg}{dt} = \int_0^1 \left(\frac{z t e^{tz} F(tz) - 1}{t} \right) \left[1 + \frac{1-z}{z} \ln(1-z) \right] (1-z)^{t-1}\,dz$$

$$- \int_0^1 (e^{tz} F(tz) - 1) \frac{(1-z)^{t-1}}{t}\,dz. \tag{8}$$

The integrand in the first integral on the right-hand side of (8) is easily shown to be negative (since $F(u) < u^{-1} e^{-u}$) and the second integral is $g(t)/t$. Thus $tg' + g < 0$, i.e. $(tg)' < 0$, and for $s < t$ we have

$$tg(t) < sg(s). \tag{9}$$

Thus if there exists $t^* > 0$ such that $g(t^*) < 0$, then $g(t)$ and hence $d\phi/dt$ will be negative for all $t \geq t^*$. However, integration by parts gives

$$\phi(t) = e^{-x}\frac{x^t}{t\Gamma(t)}\bigg|_t^\infty + \left(\int_t^{t+1} + \int_{t+1}^\infty\right) e^{-x}\frac{x^t}{t\Gamma(t)}\,dx$$

$$= -f_{t+1}(t) + \int_t^{t+1} f_{t+1}(x)\,dx + \phi(t+1). \tag{10}$$

Since $f_{t+1}(x)$ is decreasing in x on $x > t$ we have $\phi(t) < \phi(t+1)$. It follows from the Mean Value Theorem that there is an increasing sequence $\{t_j\}_{j=1}^\infty$ with $t_j \to \infty$ such that $\phi'(t_j) > 0$. This contradicts the existence of t^*, and the result follows.

Evaluation of the limiting values of $\phi(t)$ by the editors. First, we have

$$\Gamma(t+1)\phi(t) = t\int_t^\infty x^{t-1}e^{-x}\,dx$$

$$= -t^t e^{-t} + \int_t^\infty x^t e^{-x}\,dx$$

$$= -1 + \int_0^\infty e^{-x}\,dx + o(1).$$

as $t \to 0+$. Also, $\Gamma(t+1) \to 1$ as $t \to 0$, so $\phi(t) \to 0$ as $t \to 0$.

Next, we estimate $\phi(t)$ for large t by first substituting $x = t(1+s)$ in the integral defining $\phi(t)$. We get

$$\phi(t) = \frac{t^{t+1}e^{-t}}{\Gamma(t+1)}\{I_1(t) + I_2(t)\}, \tag{11}$$

where

$$I_1(t) = \int_0^1 (1+s)^{t-1}e^{-st}\,ds, \qquad I_2(t) = \int_1^\infty (1+s)^{t-1}e^{-s}\,ds.$$

By Stirling's formula

$$t^{t+1}e^{-t}/\Gamma(t+1) = \{1 + O(1/t)\}\sqrt{t/(2\pi)},$$

so it suffices to estimate $I_1(t)$ and $I_2(t)$ for large t. Since $(1+s)e^{-s}$ is decreasing for $s > 0$, we have

$$I_2(t) = \int_1^\infty \left(\frac{1+s}{e^s}\right)^{t-1}\frac{ds}{e^s} < \int_1^\infty \left(\frac{2}{e}\right)^{t-1}\frac{ds}{e^s} < \left(\frac{2}{e}\right)^{t-1}. \tag{12}$$

Thus I_2 is exponentially small for large t.

To estimate $I_1(t)$ we note that'

$$e^{-s^2/2} < (1+s)e^{-s} < e^{-s^2/2+s^3/3}$$

for positive s. Thus

$$\int_0^1 (1+s)^{-1}e^{-s^2t/2}\,ds < I_1(t) < \int_0^1 e^{-s^2t/2+s^3t/3}\,ds. \tag{13}$$

Letting L denote the left-hand integral in (13) and R the right-hand integral, we get first

$$L > \int_0^\infty (1-s)e^{-s^2t/2}\,ds = \sqrt{\frac{\pi}{2t}} - \frac{1}{t}. \tag{14}$$

Next, for $\lambda \in (0, 1/2)$, to be specified later, we have

$$R = \left\{ \int_0^\lambda + \int_\lambda^1 \right\} e^{-s^2 t/2 + s^3 t/3}\, ds. \tag{15}$$

The first integral in (15) is at most

$$e^{\lambda^3 t/3} \int_0^\infty e^{-s^2 t/2}\, ds = e^{\lambda^3 t/3} \sqrt{\pi/(2t)}\,. \tag{16}$$

For $\lambda \le s \le 1$, the minimum of the quadratic function $s/2 - s^2/3$ is taken at $s = \lambda$ and has the value $\lambda/2 - \lambda^2/3 > \lambda/3$. Thus, the second integral in (15) is at most

$$\int_\lambda^1 e^{-st\lambda/3}\, ds < 3/(\lambda t). \tag{17}$$

Combining (15), (16), and (17) and setting $\lambda = t^{-3/8}$ we get

$$R < e^{(1/3)t^{-1/8}} \sqrt{\pi/(2t)} + 3t^{-5/8} = \sqrt{\pi/(2t)} + O(t^{-5/8}). \tag{18}$$

Now (13), (14), and (18) give

$$I_1(t) = \sqrt{\pi/(2t)} + O(t^{-5/8}). \tag{19}$$

Finally, (11), (12), and (19) yield

$$\phi(t) = \left\{ \sqrt{\frac{t}{2\pi}} + O\left(\frac{1}{\sqrt{t}}\right) \right\} \{ I_1(t) + I_2(t) \} = \frac{1}{2} + O(t^{-1/8}).$$

Thus $\phi(t) \to \frac{1}{2}$ as $t \to \infty$.

Editorial comment. Both Paine and Hans U. Gerber (Switzerland) formulated more general problems in probabilistic terms. Gerber let $\{X(t)\}$, $t \ge 0$, be a process with independent, stationary, and nonnegative increments, and considered $\phi(t)$, the probability that $X(t)$ is greater than its expectation (assumed finite). Then $\phi(\infty) = \frac{1}{2}$ follows readily from the central limit theorem. To see that $\phi(0) = 0$, assume $\{X(t)\}$ is a jump process with $E[X(t)] = t$. The number of jumps with size greater than x is an interval of length 1 is a Poisson process with parameter $tQ(x)$ where $-dQ$ is Levy measure. For small t the function $\phi(t)$ behaves like $tQ(t)$, which tends to 0, even if $Q(0) = \infty$, since

$$E(X(1)) = \int_0^\infty Q(x)\, dx = 1.$$

The present result than asserts monotonicity for the gamma process. It is also true for the inverse Gaussian process where

$$\phi(t) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp(-y^2/2 - 2y\sqrt{t})\, dy,$$

but not for the Poisson process where $\phi(t)$ has discontinuities downwards. Paine assumed $\{X(t)\}$ infinitely divisible with density p_t, $E\{X(t)\} = t$, and that its Levy measure has Radon–Nikodym derivative f_0. He obtained (compare (5) of his solution)

$$\frac{d}{dt}(P(X(t) \ge t)) = \int_0^t (F(u) - f_0(u)) p_t(t-u)\, du$$

where $F(u) = \int_u^\infty f_0(y)\, dy$, but could not establish positivity for the last integral. The proposer's solution also used the language of probability. It also used the

identity

$$\frac{1}{2} - \phi(t) = \sum_{n=0}^{\infty} g(t+n)h(t+n),$$

where both

$$g(t) = \frac{t^t e^{-t}}{\Gamma(t+1)} \quad \text{and} \quad h(t) = \int_0^1 \left\{1 - \left(1 + \frac{u}{t}\right)^t e^{-u}\right\} du$$

were shown to be positive and decreasing.

Armido R. DiDonato began with (see **[1]** and **[2]**)

$$1 - \phi(t) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \exp[t(z - 1 - \ln z)] \frac{dz}{z-1}$$

and applied the method of steepest descent. His analysis proved much more than the problem's assertion, e.g.

$$(-1)^n \left[\frac{d^n\phi(t)}{dt^n}\right] < 0, \qquad n \geq 1$$

and

$$\phi(t) = \frac{1}{2} - \frac{1}{3\sqrt{2\pi t}} + o\left(\frac{1}{\sqrt{t}}\right), \qquad t \to \infty.$$

(The latter assertion is a special case of Problem 210, Part II, Volume I of G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis.*) On the other hand, Rolf Richberg (Germany) provided a fairly elementary (though somewhat lengthy) real variable solution to the problem. A partial solution was also received.

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1. N. M. Temme, Uniform asymptotic expansions of the incomplete gamma functions and the incomplete beta function, *Math. Comp.* 29(1975), 1109–1114.
2. A. R. DiDonato and A. H. Morris, Computation of the incomplete gamma function ratios and their inverse, *ACM Trans. on Math. Software*, 12(1986), 377–393.

Convergence of a Parametrized Sum

E3416 [1991, 54]. *Proposed by A. Zeifman, Vologda State Pedagogical Institute, Vologda, USSR.*

Suppose that $a_1, a_2, \ldots$ is a given sequence of positive numbers. For positive x and positive integral N put

$$S_N(x) = \sum_{n=1}^{N} \frac{a_1 a_2 \cdots a_{n-1}}{(a_1 + x)(a_2 + x) \cdots (a_n + x)}$$

(The first term of the sum is understood to be $1/(a_1 + x)$.) It is easy to prove that $S_N(x) < 1/x$ for all x and N. Prove that $\lim_{N \to \infty} S_N(x) = 1/x$ if and only if $\Sigma 1/a_n$ diverges.

Solution by Luiz Felipe Martins, Brown University, Providence, RI. Let $b_n = 1/a_n$. Then

$$xS_N(x) = \sum_{n=1}^{N} \frac{b_n x}{(1+b_1x)\cdots(1+b_nx)}$$

$$= \sum_{n=1}^{N} \left(\frac{1}{(1+b_1x)\cdots(1+b_{n-1}x)} - \frac{1}{(1+b_1x)\cdots(1+b_nx)} \right)$$

which telescopes to give

$$S_N(x) = \frac{1}{x}\left(1 - \frac{1}{(1+b_1x)\cdots(1+b_nx)}\right)$$

To complete the proof, let $N \to \infty$ and recall that $\prod(1+b_nx)$ and diverges if and only if Σb_n diverges.

Also solved by the proposer and 38 others.

Multiple Tangents to Polynomials

E3423 [1991, 158]. *Proposed by Alan Horwitz, Pennsylvania State University, Media, PA.*

For $n \leq 2$ let $M(n)$ be the maximum number of multiple tangents that an nth degree polynomial with real coefficients can have. (By multiple tangent of a real polynomial we mean a line tangent to the graph of the polynomial at more than one point. For example, the x-axis is a multiple tangent of $x^4 - 2x^2 + 1$.)

(a) Prove that $M(n) \leq \binom{n-2}{2}$.

(b)* Prove or disprove: $M(n) = \binom{n-2}{2}$.

Solution by Richard Strong, University of California, Los Angeles, CA. We will show that equality holds. Let $f(x)$ be a real polynomial of degree n, let $x_1, \ldots, x_k$ be the points where $f''(x)$ changes sign (hence $k \leq n-2$). Let $x_0 = -\infty$ and $x_k + 1 = \infty$. Define C^1-curves F_i by

$$F_i = \begin{cases} \text{the tangent line to } y = f(x) \text{ at } x_{i-1} & \text{if } x \leq x_{i-1} \\ f(x) & \text{if } x_{i-1} \leq x \leq x_i, \\ \text{the tangent line to } y = f(x) \text{ at } x_i & \text{if } x \geq x_i \end{cases}$$

and let C_i be the convex region below F_i if $f''(x) < 0$ on (x_{i-1}, x_i) and above F_i otherwise. Every tangent to f is tangent to some C_i. We first prove that for $i \neq j$, there is at most one common tangent to C_i and C_j. If L_1 and L_2 are distinct tangents to C_i, then these lines must be tangent to C_i at points of the form $(x, f(x))$ for $x \in [x_{i-1}, x_i]$. Since $f'(x)$ is monotone on this interval, the intersection point of L_1 and L_2 must have x-coordinate in the interval (x_{i-1}, x_i). Since the intersection point of two tangents to C_j must have x-coordinate in (x_{j-1}, x_j), L_1 and L_2 cannot also both be tangent to C_j.

If $j = i+1$, then the unique common tangent to C_i and C_j is the tangent to $y = f(x)$ at x_i, which is not a multiple tangent. Therefore, the number of multiple tangents is at most the number of pairs i, j such that $1 \leq i < j \leq k+1$ and i, j differ by at least two. There are $\binom{n-2}{2}$ such pairs, which completes the proof of (a). Equality hold if every such pair of regions has a common tangent and these tangents are distinct.

The polynomials of degree n form an $(n+1)$-dimensional manifold. We will show that those in an open set, except for a lower-dimensional set, have $\binom{n-2}{2}$ multiple tangents. The exceptions come from triple or (higher) tangents; a polynomial with a triple tangent $y = ax + b$ has the form $g(x)(x-c)_2(x-d)_2(x-e)_2 + ax + b$. For polynomials of degree n, $g(x)$ has degree $n-6$, hence $n-5$ coefficients, and this is a lower dimensional set. Therefore, it suffices to show that in an open neighborhood of the Chebyshev polynomial $T_n(x)$, every pair of non-adjacent C_i's have a common tangent.

Given a polynomial $f(x)$, let R_a be the (positive) ray tangent to $y = f(x)$ at $x = a$; i.e., the part of the line tangent at $x = a$ with x-coordinate at least a. Let $B(a, b)$ be the union of the rays R_x for x between a and b inclusive. This leads us to sufficient condition for C_i, C_j to have a common tangent:

$(*)$ If $i < j - 1$ and there are tangent rays R_a and R_b to C_i such that R_a intersects the interior of C_j, R_b does not intersect C_j, $C_j \cap B(a, b)$ is compact, and C_j does not intersect the graph of $y = f(x)$ for x between a and b, then C_i and C_j have a common tangent.

To prove this, let c be the largest or smallest value between a and b such that R_c intersects $C_j \cap B(a, b)$. Then the tangent to C_i at $x = c$ is also tangent to C_j.

If $(*)$ holds for $f(x)$ and all $i < j - 1$, then it holds for all polynomials sufficiently close to $f(x)$. Thus we need only check that it holds for $T_n(x)$. Each F_i has a unique local extremum. If F_i and F_j both have local maxima (or both local minima), then let a and b be slightly after and slightly before the extremum in F_i. Since $y = 1$ is tangent at all the maxima (and $y = -1$ at all the minima) of $T_n(x)$, these rays satisfy $(*)$.

If C_i and C_j contain extrema of opposite types, let a, be the extremum in C_i and let $b = x_i$ (the inflection point). The tangent ray to $y = T_n(x)$ and $x = b$ lies below the graph of $y = T_n(x)$ for $-1 \leq x \leq 1$ if F_i has a maximum and above it if F_i has a minimum. There, this ray cannot intersect C_j, and again $(*)$ is satisfied.

Also solved by J. H. Lindsey II, O. P. Lossers (The Netherlands), and M. D. Meyerson. Partially solved by M. Dindos (Czechoslovakia), R. High, H. Kipman, L. Zsilinszky (Czechoslovakia), and the proposer.

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UNSOLVED PROBLEMS

Edited by **Richard Guy**

In this department the MONTHLY presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial or related results. Typescripts should be sent to Richard Guy, Department of Mathematics & Statistics, The University of Calgary, Alberta, Canada T2N 1N4.

Perfect Sums

Bob Scher

Since the discovery of the Pythagorean Theorem, the elegance of integral sums of equal powers has held an important place as well as a fascination for mathematicians, professional and amateur alike. In the 17th century, these equations were given a powerful impetus by Fermat through his letters and annotations. His celebrated Last Theorem (FLT) states that the Pythagorean Theorem cannot be generalized in one of the most obvious directions: if x, y, z are non-zero integers and n an integer > 2, then $x^n + y^n = z^n$ is impossible. Although FLT remains a conjecture, it has been established for all $n \leqslant 125{,}000$.

The theorem presented below places a new and rather strict congruence condition on certain integral sums of cubes and of fifth powers. Aside from the general interest in diophantine equations composed of the sums of equal powers, this theorem has particular significance since it applies also to a conjecture of Euler, which was intended specifically as a generalization of FLT and thus hypothesizes an even more sweeping generalization of the Pythagorean Theorem, namely: if a sum of s positive integer nth powers equals an nth power, then $s \geqslant n$. See [**1**, pp. 79–81]. When Euler proposed it, the conjecture was known to be valid only for $n \leqslant 3$, since, e.g., $3^3 + 4^3 + 5^3 = 6^3$, and Euler—and probably Fermat—had already proved $s = 2$ impossible when $n = 3$.

Euler's conjecture stood unchallenged for almost 200 years until 1966 when, by computer search, Lander and Parkin discovered the first counterexample: for $n = 5$, $s = 4$ [**2**]. No other primitive counterexamples have been found for fifth powers or for any other prime exponent. In 1988, Noam Elkies, employing the theory of elliptic curves, obtained parametric solutions for $n = 4$, $s = 3$ [**3**].

The equations addressed in Euler's hypothesis are bound by the added condition that, when set equal to zero by transposing the single term, only one of the terms may differ in sign from the rest. Let us expand Euler's Conjecture (EC) to include all admissible combinations of positive and negative terms and call this the *expanded Euler conjecture* (*EEC*). (Of course for FLT, no choice arises since, when set to zero, one term must always differ in sign from the other two.)

The congruence condition embodied in the theorem below seems to have gone unnoticed and may prove useful as an aid in searches for additional solutions—one hopes, non-singular ones—as well as for solutions or proofs of impossibility for sums of any possible combination of positive and negative integer fifth powers, up to the maximum number of terms designated. Since the cases for $n = 2, 3$, and 4 are settled for EC and EEC, 5 is least exponent not fully accounted for. Evidence in Section 2 suggests that fifth powers yield the only prime counterexamples to both conjectures.

Since the next section addresses only odd (prime) exponents, our treatment will be less cluttered if we use both positive and negative integers for the bases since the signs of their powers will be equivalent.

1. PERFECT SUMS. Consider the following sum, whose terms are not all units and have no common factor, with y_i, y_j integers, p a prime, and i, j, k, natural numbers, $k > 2$:

$$\sum_{i=1}^{k} y_i^p = 0. \tag{1}$$

Definition. *A zero sum of the form* (1) *is called perfect if for every y_i, there also exists in* (1) *a unique y_j, not necessarily distinct from y_i, such that $y_i + y_j \equiv 0$* (mod p). *A zero sum of the form* (1) *that contains at least one unmatched* (*non-zero*) *term is called imperfect.*

Theorem. *Every sum of the form* (1) *for $p = 3$, $k < 9$, and $p = 5$, $k < 7$ is perfect.* (2)

Proof: The proof is direct and depends on two observations:

I) Since the sum in (1) = 0, then clearly it must satisfy

$$\sum_{1}^{k} y_i^p \equiv 0 \pmod{p^2}. \tag{3}$$

II) Every integer pth power is congruent (mod p^2) to one of only p distinct least residues. (4)

(A proof of II using the Taylor series is in [**4**, pp. 96–97].) In the terms of our discussion, (4) states that $y_i^p \equiv r_i \pmod{p^2}$, $0 \leqslant |r_i| \leqslant (p^2 - 1)/2$, but with r_i selected only from $\pm r_m \pmod{p^2}$, $m = 0, 1, \ldots (p-1)/2$. From (3) we have

$$\sum_{1}^{k} r_i \equiv 0 \pmod{p^2}. \tag{5}$$

Denote by $[r_1, r_2, \ldots, r_i, \ldots, r_k]^p$ a zero *residue sum* that satisfies (5) and is constructed from the set of p least residues in (4), with r_i^t signifying t terms, each with value r_i. Residue sums of perfect sums are (zero and) *perfect*; those of imperfect sums are (zero and) *imperfect*. To prove (2), we work backwards, constructing, for a given p, an imperfect residue sum with the least possible k, say k^*. Then every zero residue sum with $k < k^*$ must be perfect, and thus every

integer pth power zero sum with $k < k^*$ is perfect (the trivial cases $k \leqslant 2$ are excluded).

To construct imperfect residue sums from the set of residues in (4), we use only unmatched terms, since by definition only such terms distinguish an imperfect sum. The sum of all the unmatched residues in a zero residue sum is clearly $\equiv 0 \pmod{p^2}$ since the sum of the matching residues, by definition, is $\equiv 0 \pmod{p^2}$ (in fact $= 0$, as they are least residues).

For $p = 3$, the cubic least residues (mod 9) are 0 and ± 1. For constructing an imperfect residue sum we have just the single unmatched term: $+1$ (or -1), and thus this sum must contain at least 9 terms: $[1^9]^3 \pmod 9$. So for $p = 3$, $2 < k < 9$, every sum that satisfies (1) has a perfect least residue sum and is therefore a perfect sum.

For $p = 5$, the fifth-power least residues (mod 25) are $0, \pm 1, \pm 7$. Using unmatched terms, we can verify directly that any imperfect residue sum (mod 25) contains at least 7 terms. The only possible such sum is $[1^4, 7^3]^5 \pmod{25}$ and its corresponding permutations of signs and exponents:

$$[\pm 1^4, \pm 7^3]^5 \quad \text{and} \quad [\pm 1^3, \mp 7^4]^5 \pmod{25}. \tag{7}$$

Thus for $p = 5$, $2 < k < 7$, every sum that satisfies (1) is perfect, and this establishes the theorem.

There is an obvious but useful corollary:

Corollary. *In a perfect sum, the number of terms divisible by the exponent, p, has the same parity as the total number of terms, k.* (8)

Example. Denote by $(y_1, y_2, \dots, y_i, \dots, y_k)^p$ examples of (1) with y_i^t signifying t terms, each with value y_i.

For $p = 5$, the only known solution of (1) for $k < 6$ is the perfect sum $(27, 84, 110, 133, -144)^5$ **[2]**, the counterexample to Euler's conjecture mentioned above.

Special Cases of Fifth Power Sums Equal to Zero.

$k = 3$. The sum becomes FLT for $p = 5$ and the reader may note that our theorem in (2) gives an immediate proof of FLT for $p = 5$, for the case when none of the terms is divisible by p, since this would have to be an—inadmissible—imperfect sum. (FLT is much more difficult to prove for $p = 5$ when p divides one of the terms.)

$k = 4$. An open question. There are no such sums known, nor is there yet a proof of its impossibility. From (8), we know that such sums would have to contain two terms, or no terms, divisible by p.

$k = 5$. No solutions are known other than the single counterexample already cited. By our theorem these sums must contain exactly one term, or three terms, divisible by p.

$k = 6$. Many non-singular solutions are known (see **[5]**).

$k = 7$. (7) represents the only valid imperfect residue sum (mod 25), but no actual imperfect sum is known. For $k = 8$, there exists an imperfect sum $(4, -6^2, 7^3, -8, -5)^5$ in **[5]** whose residue sum corresponds to (7) plus a zero term: $[-1^3, 7^4, 0]^5 \pmod{25}$.

Limitations. A result of Cauchy shows that the method of perfect sums cannot be applied to primes of the form $p \equiv 1 \pmod 6$. He first observed that if p is prime, a, an integer, $1 < a \leqslant p - 2$, and $a^2 + a + 1 \equiv 0 \pmod p$, then $p \equiv 1 \pmod 6$.

[This follows from the fact that $a^2 + a + 1$ is odd, and the square of an integer can be congruent only to 0 or 1 (mod 3).] Using a pre-established identity, he showed that when $p \equiv 1 \pmod 6$, $(a+1)^p - a^p - 1^p \equiv 0 \pmod{p^2}$, in fact, $\equiv 0 \pmod{p^3}$ [**6**].

This generates instances of imperfect sums with only three terms, and thus for $p \equiv 1 \pmod 6$ the above method cannot rule out imperfect sums for $k \geqslant 3$. For example, if $a = 2$ in $a^2 + a + 1 \equiv 0 \pmod p$, then $p = 7$. The seventh-power least residues (mod 49) are $0, \pm 1, \pm 18, \pm 19$, and this example yields the residue sum $[1, 18, -19]^7 \pmod{49}$.

For other $p > 5$, the situation is more complex. For instance, the 11th power residues (mod 121), $0 \pm 1 \pm 3 \pm 9 \pm 27 \pm 40$, can form an imperfect residue sum for $k = 4$, e.g., $[3^3, -9]^{11}$ or $[27^3, 40]^{11} \pmod{121}$. There are also primes, e.g., $p = 59$, for which $a^2 + a + 1 \not\equiv 0 \pmod p$, but $(a+1)^p - a^p - 1^p \equiv 0 \pmod{p^2}$ [**7**]. We have, for example, the two imperfect residue sums $[1, 298, -299]^{59}$ and $[1, 299, -300]^{59} \pmod{59^2}$.

2. EXPANDING THE INQUIRY. Using positive integers to accommodate composite exponents, we write the original Euler Conjecture (EC) and its expanded version (EEC).

EC and EEC. For b, n, k, positive integers, $n > 1$, $k > 2$, and with terms not all units and having no common factor:

$$\text{(EC)} \qquad \text{If } b_1^n + b_2^2 + \cdots - b_k^n = 0, \qquad \text{then } k > n.$$

$$\text{(EEC)} \qquad \text{If } \pm b_1^n \pm b_2^n \pm \cdots \pm b_k^n = 0, \text{ then } k > n.$$

If we let $k^*_{(n)}$ represent the smallest known k for a given n for which EC holds, we have the following table for $n \leqslant 10$, noting that for $n = 2, 3$, and 4, $k^*_{(n)}$ is the least possible:

(EC) n	2	3	4	5	6	7	8	9	10
$k^*_{(n)}$	3	4	4	5	8	9	12	16	24

We have the corresponding table for EEC where $k_{(n)}$ represents the smallest known k for a given n for which EEC holds, noting as before that for $n = 2, 3$, and 4, $k_{(n)}$ is the least possible:

(EEC) n	2	3	4	5	6	7	8	9	10
$k_{(n)}$	3	4	4	5	6	9	10	12	14

Both $k^*_{(n)}$ and $k_{(n)}$ appear to grow faster than n, but $k_{(n)}$ seems to follow a more "reasonable" succession than $k^*_{(n)}$, which, by our present knowledge, begins to increase much more sharply. This suggests that $k_{(n)}$, the more general function, may be the more tractable one. Data for $n > 10$ would be welcome.

The tables suggest that EC is negated only by fourth and fifth powers, and EEC, only by fourth, fifth, and sixth powers. (For $n = 6$, the only known form of counterexamples to EEC, in positive integers, is $a^6 + b^6 + c^6 = d^6 + e^6 + f^6$.)

Though it has not been explicitly noted in the literature, those who work in this domain have certainly noticed that for $n \leqslant 10$, $k_{(n)}$ always has a representation in positive and negative terms that are equal in number when k is even, and, except for $n = 5$, differ only by one when k is odd. For $n = 5$, only Lander and Parkin's counterexample is known, which has but one differing sign, so it would be particularly interesting to know if there exists $k_{(5)} = 5$ with two differing signs. One would also like to know if $k_{(7)} = 8$ exists, and if so, would one of its

representations contain four positive and four negative terms, thus preserving the common pattern. The reader may consult [5] for the numerical details that support the conjecture expressed in this paragraph, as well as the data for the two tables above.

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To Whom It May Concern

I apologize that what were offered as unsolved problems in the Jan. 1992 MONTHLY, pp. 74–75 are in fact well known results.

Many of the big names in combinatorial number theory are among those who have written to say that Matiyasevich's generalized harmonic numbers are essentially Stirling numbers of the first kind, and that his conjectures follow fairly easily from known properties. See especially Glaisher (1900), but also Nielsen (1906), Carlitz (1953), Olsen (1966) and Comtet (1974).

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LETTERS

Material Implications

In *Material Implication Revisited* (this MONTHLY, March 1989, 247–250) we addressed the classical problems with the definition of material implication. In a Letter to the Editor (this MONTHLY, Aug/Sept 1989, 602–603), Dan Velleman replies with the suggestion that many uses of "if... then" create difficulties only because they should really be represented in the predicate logic using a universal quantifier.

Hence, "If the sun is shining, then it must be between 2:00 and 3:00 P.M." is false, Velleman claims, because what is really intended by the speaker is "Whenever the sun is shining, it must be between 2:00 and 3:00 P.M." Unfortunately, this reading sheds no light on the conditional, "If the sun *is* shining [now], then it must be between 2:00 and 3:00 P.M." nor does it help with the many paradoxes of implication.

His second example, drawn from mathematical discourse, is similar to "If $x < 0$, then $x^2 > 0$ and $x^3 < 0$." This, indeed, is best translated with "whenever," but results in a universally general proposition, not an implication.

In summary, while Velleman is right that not all "if" 's result in implication, it is not right to assume, as he does, that this observation is relevant to a comprehensive discussion of material implication.

Joseph S. Fulda

Derangements

Professor Mourad Ismail has informed me that the main result of my article "Derangement, Permanents, and Christmas Presents" (August–September, 1991) appears in two papers, [1] and [2], coauthored by him. As your readers may recall, I cited an even earlier reference, [3], in my article.

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The Source of all Identities

I enjoyed reading Leslie's article on higher derivatives of reciprocals [3], recently published in the Monthly. Leslie makes an elegant use of a functional calculus for

matrices proposed by Ikebe and Toshiyuki [**1**] to obtain a functional identity that yields a rich harvest of combinatorial identities.

I would like to note that the functional calculus for matrices of Ikebe and Toshiyuki in [**1**] is identical with the *matrix Leibniz' rule* of Kalman and Ungar [**2**], already implicit in [**4**]. The matrix Leibniz' rule of Kalman and Ungar, in turn, is a special case of a more general result from which the binomial theorem [**7**], generalized trigonometric and hyperbolic functions [**5**, **6**], various addition theorems [**4**, **7**], and combinatorial identities [**2**] spring out.

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There is an astonishing imagination, even in the science of mathematics ... We repeat, there was far more imagination in the head of Archimedes than in that of Homer.

—*Voltaire*

REVIEWS

Edited by Darrell Haile

The Unreal Life of Oscar Zariski. By Carol Parikh. Academic Press Inc., Harcourt, Brace, Jovanovich, Publishers Boston, San Diego, 1991, xxvii + 264 pp.

Robin Hartshorne

Oscar Zariski rewrote the foundations of algebraic geometry. In Rome in the early 1920s he learned the "geometric" algebraic geometry of the Italian school from its three great masters, Guido Castelnuovo, Federigo Enriques, and Francesco Severi. Later after emigrating to the United States, he realized the need for more rigorous foundations to support the intuition of the Italians. He brought to bear the abstract algebra of the Göttingen school of Emmy Noether, B. L. van der Waerden, and Wolfgang Krull. He introduced these techniques into algebraic geometry in a series of fundamental papers in 1939 and 1940, and then spent the next forty years developing and applying them to a wide range of topics. He remained committed to research mathematics throughout his long life, and was still publishing new results up until a few years before his death at age 85.

The "real life" of this man who said "Geometry is the real life" (p. 76) is documented in nearly a hundred books and articles written over a period of 65 years. The editors of his *Collected Papers* [1] have written a series of introductory essays which survey his mathematical work and which are reprinted as a 60-page appendix to this book:

- "Zariski's topological and other early papers," by Michael Artin and Barry Mazur
- "Zariski's papers on the foundations of algebraic geometry and on linear systems," by David Mumford
- "Zariski's papers on holomorphic functions," by Michael Artin
- "Zariski's papers on resolution of singularities," by Heisuke Hironaka
- "Zariski's papers on equisingularity," by Joseph Lipman and Bernard Teissier.

Since Zariski's mathematical opus and its significance are amply described in these essays and in Zariski's own very readable preface to his *Collected Papers*, I would like to devote this review to the more unusual aspect of this book, namely, the insight it gives us into the man behind the mathematics.

Carol Parikh's perceptive narrative of the life of Oscar Zariski, the man, is based on his own recollections tape recorded a few years before his death, and on the author's extensive interviews with his family, colleagues, and students. Here we learn of his birth in a Jewish settlement in eastern Poland, high school in Russia, university in Rome, and maturity in the United States. We see how his love for

mathematics carried him safely through turbulent times. We see his awareness of himself as a Jew mirrored by the changing societies in which he moves. We see his development as a mathematician in the context of the people around him. We see his humanity in his love for his family and the care he devoted to his students. We see the pain of his personal losses in a life shaped by his commitment to mathematics. All this and more in Carol Parikh's prize-winning English prose make the book a delight to read.

Contemplating Zariski's life, a number of questions come to mind. What is the human significance of a life devoted to mathematics? What drives a man to devote so much energy to an arcane subject which can only be appreciated by a handful of other mathematicians? What is the relationship between his mathematics and the rest of his life?

David Mumford, in his "Foreword for Non-Mathematicians" (pp. xv–xxvii), addresses the perennial question of explaining to the layman what on earth mathematicians do and why they are so excited about it. This is a difficult question, similar to the problem faced by the mountain climber trying to explain his enthusiasm for climbing. Mumford's approach is to give some elementary illustrations of the interaction of algebra and geometry at the level of high-school mathematics. While these aptly convey Mumford's own infectious enthusiasm for his subject, this approach ultimately fails to convey the depth and importance of a mathematician's work, for the same reasons that a trip to the local rock-climbing area fails to explain why mountaineers are driven to climb mountains.

To understand why the mountaineer will brave extremes of physical hardship and danger to reach his summit, and why the mathematician will struggle with all his forces to prove his theorem—"I have gotten stuck on a hard point that I have not been able to overcome for weeks," Zariski writes to his wife, "I've been close to despair," (p. 51)—we must look for a deeper, inner reason. Noting that during the difficult wartime years "his increasing fatigue, like his back pain, seemed to lead him more deeply into his work" (p. 103), Parikh quotes Gian-Carlo Rota: "Of all escapes from reality, mathematics is the most successful ever" (p. 103). Yes, and similarly the climber in his mountains is far from the complexities of city life, but that is only half the story.

One mountaineer put it this way: "Mountaineering is a passion, and it is very akin to love. There are times of intense pleasure and satisfaction, and others of utter frustration and hurt." [**2**, p. 219]. Already at age 19, Zariski in his diary refers to mathematics as "that darling old lady," and goes on to describe his own passion: "The happiness one finds in letting one's self be carried by the current of one's thoughts!... You begin with some question and step by step you witness the wonderful functioning of your own intellect. To put it briefly: in mathematics I feel absolutely sure of myself" (p. 10). Passion also entails inner turbulence. Zariski's wife of 60 years, Yole, recalls "Oscar was a man of many moods, and his moods were always much affected by his work. He could only feel happy when his work was going well" (p. 31). In the last decade of his life, these moods often darkened into depression, as his creative powers waned and he could concentrate on mathematics only a few hours each day (p. 175).

Zariski's father died when he was only two years old. I also lost my father at an early age, and have often felt that mathematics, like the comfort of solid granite in the mountains, was an island of security where I could be sure of myself in an otherwise untrustworthy world. Did Zariski perhaps have similar feelings, as he traded one politically unstable country for another and "carried with him like a magic cloak his devotion to mathematics" (p. 15)?

There often comes a period in midlife when the settled patterns of career and family lose their attraction. We ask: "Where am I and why am I here?" For a mathematician it might be "Why am I working so hard on one more technical generalization of so-and-so's theorem? Am I too old to do significant mathematics? Am I missing out on something else in life?" A serious encounter with these questions often leads to a major shift in life orientation. If Zariski ever asked himself these questions, there is no evidence of it here. But something very interesting did happen to him at about age forty. Having absorbed all there was to learn in the Italian school of algebraic geometry, and having spent ten years in the study of topological properties of algebraic varieties using the analysis situs of Solomon Lefschetz, he set out to write the definitive account of algebraic geometry to date, which in those days meant the theory of algebraic surfaces [**3**]. The effect of this effort is quoted (p. 68) from Zariski's own preface to his collected papers:

> In my Ergebnisse monograph I tried my best to present the underlying ideas of the ingenious geometric methods and proofs with which the Italian geometers were handling these deeper aspects of the whole theory of surfaces, and in all probability I succeeded, but at a price. The price was my own personal loss of the geometric paradise in which I had so happily been living.

It was after this that he began rewriting the foundations of algebraic geometry using modern algebra in what was to be his major life work. Happily for the field of algebraic geometry, this midlife transition led not to a turning away from mathematics but to a radically different conception of his work, and a renewed commitment to research which lasted for the rest of his life. Yet I wonder at the personal cost of this commitment when I read of "the unusual amount of personal loss" his moves from one country to another must have entailed (p. 83), and hear of his inability to speak of the deaths of his mother and older brother in the Nazi occupation of Poland (p. 111).

Zariski's Jewishness runs like a counterpoint throughout the narrative. The son of a Talmudic scholar, he spent his first eleven years in a traditional, almost exclusively Jewish society (p. 1). Yet by the age of high school he had become an atheist (p. 7), and when he first came to America he considered himself more Russian than Jewish (p. 133). On the other hand, speaking of his connection with Lefschetz in Princeton, he wrote "We are both European and more especially Russian, and even more especially both of us are Jews. This creates a communion of ideas and a possibility of frank discussion that neither one of us can have very often in the American milieu" (p. 52). In the 1930s Zariski faced an atmosphere of prejudice in which Harvard's influential Professor G. D. Birkhoff could oppose Lefschetz's nomination as president of the American Mathematical Society saying "he will try to work strongly and positively for his own race" (p. 101), and Harvard's president A. Lawrence Lowell's policies resulted in halving the percentage of Jewish students admitted to Harvard (p. 101). When Zariski came to Harvard in 1947, he was the first Jew to receive tenure in the Harvard Mathematics Department.

It seems, however, that Zariski's sense of himself as a Jew, and his concern for the new state of Israel was more ethnic than religious, for he "had grown into a man who regarded religious orthodoxy with the same disdain with which he viewed psychoanalysis: both seemed to him irrational dependencies" (p. 133). One wonders, then, what formed the spiritual component of his life. Perhaps as the climber feels awe in the presence of lofty mountains, Zariski could feel the presence of the divine in the subtle interplay of algebra and geometry and in his wondrous

landscape of algebraic varieties. Is God perhaps to be found among the singularities in characteristic p which the best efforts of Zariski and his students could not resolve?

One of the fascinations of algebraic geometry for me has always been the subtle, shifting perspective: if you don't see it geometrically, try to phrase it algebraically; if you get lost in the algebra, ask what is the geometry behind it. In the same way, I believe one can gain great insight by learning to speak another language and seeing life through the eyes of another culture. So, perhaps, some of Zariski's genius in algebraic geometry is linked to his ability to leave Poland and his mother tongue, Yiddish, for high school and Russian, then university in Italian and mature life in English. While his later papers are all written in the algebraic language, he recalled "I was always interested in the algebra which throws light on geometry, but I never did develop the sense for pure algebra... I have too much contact with real life, and that's geometry" (p. 76). Surely it is no accident that his first encounter with the "real life" of geometry dates from his university days in Rome, in a country he romantically thought of as "the land of song, the land of poets, the land of Galileo Galilei" (p. 14), the time of first love and marriage. The algebra needed for his great life work he learned later in America in his mid-thirties. He became so scrupulous in using this algebraic language for the sake of rigor that when I came along, I wished he would explain more of the geometry hidden behind the algebra.

I first met Oscar Zariski when I came to Harvard College as a freshman in the fall of 1955. I enrolled in his course on projective geometry in which he lectured so clearly that my class notes read like a textbook. I recall going to see him in his office at 2 Divinity Avenue one day, bringing with me some papers from my previous year at high school in Germany. I had been a nuisance in the math class there, so to keep me quiet the teacher gave me a textbook of synthetic projective geometry, from the school of von Staudt's "Geometrie der Lage." I read about conics and then invented an analogous theory of plane cubic curves on my own, with many drawings, imitating the style of the book I had read. I showed this to Zariski, hoping perhaps that he would give me some praise and recognition for the clever work I had done. While he said nothing against it, neither did he lavish any praise on my efforts. I came away from that interview with a vision of a broad mathematical world in which I had so far taken only a few steps.

Perhaps a year later, on Christmas eve, I went caroling to Beacon Hill in Boston. I played the piccolo while my friends sang. By chance I met Zariski in the streets, and was surprised that he remembered me. I don't recall what he said, but I felt that he cared for me, and have never forgotten the warmth of that chance encounter.

It was only some years later as a graduate student that I came to realize Zariski's stature in the field of algebraic geometry, and came to work in that area myself. I had heard Chevalley's and Serre's lectures in Paris, and was attracted to the flashy new theory of schemes and cohomology which Grothendieck was then expounding at Harvard. I probably sensed also that to work with Zariski I would have to deal with those awesome singularities with which I saw his students Abhyankar and Hironaka wrestling. Thus I did not work directly with Zariski, though he continued to be a strong influence in my development.

Another incident sticks in my mind. I was interested in a problem about set-theoretic complete intersections. Every time a famous visitor would pass through the doors of the mathematics department, I would eagerly ask my question to see what advice or help I could get. Talking with Zariski one day, I sensed that

he did not find the problem very interesting, but his advice was, well, if you care about it, then settle down and work hard on it yourself. I still have not solved the original problem, but its spinoffs have stimulated much of my subsequent research.

This biography strikes many familiar chords in me. Reading of Zariski's insistence on making proofs which are also valid in characteristic p, for example, I realize I have been telling my students the same thing. Is it a common attitude which accounts for my affinity with Zariski? Or is it more likely something I learned from him and have internalized so that it feels like my own? Specific results, such as "Zariski's Main Theorem" (affectionately known as "ZMT") we can attribute to him consciously. In other cases, the presence of his name, as in "Zariski topology" or "Zariski ring," reminds us of his contribution. But I suspect that often his insights and perspective have been so absorbed into the current attitude and language that we are no longer aware of the extent of his contributions.

We owe a debt of gratitude to Carol Parikh for giving us such a lively view of the life and work of this remarkable man.

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Geometric Etudes in Combinatorial Mathematics. By Vladimir Boltyanski and Alexander Soifer. Center for Excellence in Mathematical Education, Colorado Springs, Colorado, 1991, xii + 236 pp.

Don Chakerian

"Understanding of mathematics cannot be transmitted by painless entertainment any more than education in music can be brought by the most brilliant journalism to those who have never listened intensively. Actual contact with the *content* of living mathematics is necessary." So speaks Richard Courant in his preface to the first edition of *What is Mathematics*? [**2**], authored with Herbert Robbins. Meaningful learning of mathematics takes place only with intensive involvement of the student in the subject matter at a substantial level. It is widely understood that this is best accomplished through methodical work in problem solving, with thoughtfully selected problems that require inventiveness and independence on the part of the student. I remind the reader, incidentally, that the appendix to Courant and Robbins offers, as an adjunct to the exercises in the text itself, a nice assortment of problems "designed not so much to develop routine technique as to stimulate inventive ability" [**2**, p. 487].

In his introduction to *Geometric Etudes in Combinatorial Mathematics*, Branko Grünbaum asks "How do young people develop skills of any kind—from driving cars, to playing basketball or a musical instrument? In all cases the sequence of events is the same: a little instruction, more or less formal, is followed by ample practice. The person wishing to acquire better skills must invest, on his or her own, considerable efforts aimed at gaining better mastery of various aspects of the activity."

Boltyanski and Soifer have titled their monograph aptly, inviting talented students to develop their technique and understanding by grappling with a challenging array of elegant combinatorial problems having a distinct geometric tone. The etudes presented here are not simply those of Czerny, but are better compared to the etudes of Chopin, not only technically demanding and addressed to a variety of specific skills, but at the same time possessing an exceptional beauty that characterizes the best of art.

The preface quotes Hermann Weyl: "The soul of every mathematician is wrestled for by the Devil of Abstract Algebra and the Angel of Topology." Evidently the angels have won the authors' souls. Their selection of subject matter ranges over geometrical aspects of graph theory, combinatorial geometry, convexity, and some of the elementary ideas that have given birth to combinatorial topology.

A large initial portion of the book is taken up with variations on the theme of tiling rectangles, cylinders, tori, and various other surfaces with polyominoes. An n-omino, or polyomino, generalizing the notion of a domino, is a connected and simply connected union of n congruent coplanar squares glued edge to edge. The authors introduce this subject with Gomory's lovely proof that an $m \times n$ checkerboard (with m and n both greater than 1 and mn even) having two squares removed can be tiled with dominoes if and only if the deleted squares are of different colors. Indicative of how quickly one can reach the frontiers of knowledge in these matters, the authors later offer \$50 (in the spirit of Paul Erdös) for the first solution of the following problem contributed by Branko Grünbaum: *Is it true that a tiling of the plane by copies of a given polyomino contains a bounded part that will tile a torus*? (The eager reader should note that "the authors will be judges of what constitutes a solution.") The question is motivated by the fact (given as an exercise) that any tiling of an infinite strip by copies of a given polyomino contains a bounded part that will tile a cylinder. The proof of this can be based on an elegant application of the pigeonhole principle (or Dirichlet box principle), the next major theme taken up in the book.

As a prototypical example of a geometric application of the pigeonhole principle, the authors give one of my favorites, which appeared as the second problem in the morning session of the 1954 William Lowell Putnam Mathematical Competition (consult [**4**, p. 41]): *Prove that among any five points inside a unit square there are two points not more than distance* $\sqrt{2}/2$ *apart*. Solution: Partition the square into four congruent subsquares in the natural way; by the pigeonhole principle some pair of the five given points belong to the same subsquare, and we are done. Their next example applies the pigeonhole principle to six points inside a 3×4 rectangle to prove that some pair are at distance no more than $\sqrt{5}$ (the reviewer would like to ask if $\sqrt{5}$ could be replaced by $\sqrt{145}/6$ in this problem). One readily sees how problems of this type can proliferate (although Boltyanski and Soifer do not pursue this particular variation very far). Let $d(n)$ denote the largest minimum distance that can be achieved among n points in a closed unit square. Since it is clear how to arrange 5 points in a closed unit square so that the distance between

any two is at least $\sqrt{2}/2$, the above shows that in fact $d(5) = \sqrt{2}/2$. What about other values of n? Up to date results on this general problem may be found in the recent book of Croft, Falconer and Guy [**3**, D1]. As noted there, it is easy to see that $d(n)$ is asymptotic to $c/\sqrt{n}$, where c is a universal constant, but exact values for $d(n)$ are known for only relatively few values of n. The second problem in the morning session of the 1960 Putnam Competition was based on the fact that $d(3) = \sqrt{6} - \sqrt{2}$ (see [**4**, p. 58]).

Rather than dealing with distances between pairs of points, we might examine the areas of triangles formed by triples of points. In a similar vein to the preceding, for each set of n points we consider the minimum area that can be produced using triples of points in the set, and let $t(n)$ be the largest of these minimal triangle areas among all sets of n points in a unit square. Heilbronn, some 40 years ago, conjectured that $t(n) < c/n^2$ for some universal constant c (actually, according to Erdös, Heilbronn claimed only to have transmitted the conjecture. But, in the words of Erdös, "since he is unfortunately cured of our incurable disease we cannot find out.") About ten years ago Komlós, Pintz, and Szemerédi [**7**] disproved the conjecture, showing there exists a universal positive constant c such that $t(n) > c(\log n)/n^2$. They also proved that $t(n) < c/n^{\alpha}$ for any exponent α less that $8/7$, for some constant c (see [**6**]). Again, few exact values of $t(n)$ are known. Some specific calculations can be found in [**5**]. Exercises 7.3 and 7.4 of Boltyanski and Soifer provide a starting point for investigations of this nature.

Extending the pigeonhole principle to infinite sets, the Bolzano–Weierstrass theorem is proved, followed by the introduction of the Hausdorff distance between compact sets and a lucid elementary presentation—again with the pigeonhole principle at center stage—of the proof that a bounded sequence of compact sets admits a convergent subsequence. With this the authors are able to give a complete synthetic demonstration of the isoperimetric theorem, including a proof of the crucial point that eluded Jacob Steiner, namely that an extremal figure actually exists. This reviewer suggests that it might not be a bad idea for instructors in advanced calculus courses to take a good look at this treatment. I think it could only benefit a class to emphasize the role of the pigeonhole principle in the usual proof of the Bolzano–Weierstrass theorem, to peek back at the finite analogues, and then look at the powerful generalization to sequences of compact sets. One could then toss out whatever else might be in the course syllabus to leave room for a proof of the isoperimetric theorem.

The chapter dealing with graph theory begins with acquaintanceship problems of a familiar type, the sort of thing that goes through any mathematician's mind when he looks around to see who he knows at a party. How many people do I know? Is it possible that everybody at this party is acquainted with exactly seven others? Are there either three mutually acquainted or three mutually nonacquainted people here? In fact the latter would be true if there were at least 6 people at the party. This is a prototype of Ramsey's theorem and, apparently at Frank Harary's suggestion, found its way into the 1953 Putnam Competition as the second problem (naturally) of the morning session (see [**4**, p. 38]). There it was phrased in an equivalent but more colorful fashion, roughly as follows: *If the line segments joining pairs of six given points in general position in space are colored either red or blue, then a triangle having all edges the same color will be formed*. Ramsey's theorem implies that there is a smallest positive integer $r(m, n)$ such that when the edges of the complete graph with $r(m, n)$ vertices are colored red or blue, a complete subgraph with m vertices will have all its edges colored red or one with n vertices will have all its edges colored blue. The above Putnam problem shows that

$r(3,3) \leqslant 6$. The reader will have no difficulty in two-coloring the edges of the complete graph with 5 vertices in such a way that there are no monochromatic triangles. It then follows that in fact $r(3,3) = 6$. Boltyanski and Soifer lead the reader on a nice excursion through some elementary aspects of Ramsey theory. They then present several other topics in graph theory, including Euler's relation for planar graphs, a discussion of the Kuratowski embedding theorem, and even a proof of the Jordan curve theorem for polygons.

The final third of the book is given over to ideas in combinatorial geometry related to convex figures. This includes the better known, such as Helly's theorem, its variants and its applications, sets of constant width and Borsuk's conjecture, and the lesser known, such as the illumination of convex figures and the Hadwiger covering problem. Much, but not all, of this can be found in the classic introduction to convex figures by Yaglom and Boltyanski **[8]** and another very valuable work on combinatorial geometry by Boltyanski and Gohberg **[1]**. But Boltyanski and Soifer do not give stale replays of this material. Their treatment is fresh and stimulating throughout.

Keep this book at hand as you plan your next problem solving seminar. And place the book of Croft, Falconer, and Guy **[3]** next to it on your desk, for that budding young genius in your seminar who has an insatiable appetite for unsolved problems in geometry.

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General, S, P**, L***.** *More Mathematical People: Contemporary Conversations.* Eds: Donald J. Albers, Gerald L. Alexanderson, Constance Reid. Harcourt Brace Jovanovich, 1990, xviii + 375 pp, $29.95. [ISBN: 0-15-158175-4] Eighteen biographical interviews with famous mathematicians reveal the human face of mathematics—curiosity, persistence, inventiveness, and, most notably, profound respect for the beauty of deep mathematics. Introduced by Martin Gardner, profiles move alphabetically from Lipman Bers to Robin Wilson. Sequel to the pioneering *Mathematical People* (TR, May 1985; Extended Review, November 1986). LAS

Reference, S, P. *MS-DOS 5.0 Command Summary.* Specialized Systems Consultants (POB 55549, Seattle, WA 98155), 1991, 20 pp, $4.50 (P) [ISBN: 0-916151-51-4]; *Korn Shell Reference,* 20 pp, $4.50 (P) [ISBN: 0-916151-50-6]; *C++ Card,* 16 pp, $4.50 (P). [ISBN: 0-916151-49-2] Accordian-fold reference cards listing commands with brief description, syntax, and options. *Korn Shell Reference* includes `vi` and `emacs` modes; *C++ Card* uses examples extensively; *MS-DOS* includes Doskey and DeBug commands. Handy size; useful content; helpful layout. LAS

Reference, L. *Mathematics through History: A Resource Guide.* John Fauvel. QED Books (1 Straylands Grove, York, England), 1990, 47 pp, (P). [ISBN: 0-946544-71-9] A brief annotated bibliography of approximately 250 books (with a few videos) to support the study of history at all levels, primary to tertiary. Due to its source, it includes many British titles not well-known in the United States. LAS

Mathematics Appreciation, T*(13-14: 1). *Excursions in Modern Mathematics.* Peter Tannenbaum, Robert Arnold. Prentice Hall, 1992, xv + 559 pp. [ISBN: 0-13-298233-1] A refreshing text for general education courses stressing applicability, accessibility, age (recent mathematics), and aesthetics. Covers voting, fair division, apportionment; networks applied to management issues; growth, symmetry, fractals; and data analysis (surveys, probability, normal curve). Special innovation: a subscription option providing students with regular reprints of articles from *The New York Times* that are related to themes covered in the text. LAS

Elementary, S. *Passing the City University of New York Mathematics Skills Assessment Test.* Martin M. Zuckerman. Ardsley House, 1983, vi + 362 pp, (P). [ISBN: 0-912675-00-4] A review book with hundreds of worked-out examples, diagnostic tests, and numerous multiple-choice exercises similar to those on the CUNY assessment test. Reviews K-9 mathematics, from whole-number arithmetic through lin-

ear equations. LAS

Education, P. *Psychological Abilities of Primary School Children in Learning Mathematics.* Ed: V.V. Davydov. Soviet Stud. in Math. Educ., V. 6. Transl: Joan Teller. NCTM, 1991, xxi + 376 pp, $25 (P). Six individually-authored studies analyzing the psychological processes by which elementary school children assimilate ideas of multiplication, fraction, number, variables, problem solving, and algebraic methods. Each is based on empirical investigations intended to demonstrate that young children are more capable than traditional curricula would suggest. Instructional methods are didactic (teacher presents, students master), not exploratory. Translation of a Russian monograph published in 1969. LAS

Education, P, L. *Mathematics Assessment: Myths, Models, Good Questions, and Practical Suggestions.* Ed: Jean Kerr Stenmark. NCTM, 1991, iv + 67 pp, $8.50 (P). [ISBN: 0-87353-339-9] A handbook for changing and evaluating assessment: myths about teaching and testing (e.g., "First we teach, then we test"); performance tasks; observations and interviews; portfolios; implementation strategies. Many good examples at various school levels. LAS

Education, P, L. *Survey of Mathematics and Statistics Departments at Higher Education Institutions.* Bradford Chaney, Elizabeth Farris, Patricia White. Higher Educ. Surveys Report, No. 5. NSF, 1990, vi + 79 pp, (P). Report of a 1987 survey of departments of mathematics and statistics: degrees, courses (enrollments, section size), faculty (degrees, recruitment, professional expectations), and problems of departments (teaching load, faculty travel, facilities, computers, etc.). Sample findings: 80% of teaching time is devoted to non-majors; 20% of teachers and students are in departments offering no major in mathematics or statistics; 60% of teachers teach at least one course below the calculus level. LAS

Education, P. *Developing Number Sense in the Middle Grades.* Barbara J. Reys, *et al.* NCTM, 1991, viii + 56 pp, $10.50 (P). [ISBN: 0-87353-322-4] Advice for middle school teachers, with examples about whole numbers, fractions, decimals, percents, and graphs to help children develop an intuitive feeling for numbers and an appreciation for various levels of accuracy—for "common sense" about numbers. LAS

Education, P. *Program Review and Educational Quality in the Major: A Faculty Handbook.* Liberal Learning & Arts & Sci. Major, V. 3. Association of American Colleges, 1992, viii + 32 pp, $12 (P). [ISBN: 0-911696-53-9] A brief guide giving a process and framework for review of undergraduate programs, focused on the quality of educational experience for students, derived primarily from principles of connected learning articulated in the prior AAC report on the undergraduate major (*Volumes 1* and *2*, TR, March 1991). Addressed to all disciplines; written by mathematician John Thorpe; very relevant to current discussion of program review in departments of mathematics. LAS

Education, P. *Graduate Education in Transition.* CBMS, 1992, 16 pp, (P). Report of a May 1991 CBMS workshop in which presidents and senior officers of mathematical societies examined graduate education in mathematics. Recommendations include a call for standards, for greater cooperation with industry, for professional Master's degrees focused on specific market needs, and for more supportive learning environments. LAS

Education, S(17-18). *Develop Your Teaching.* Barbara Jaworski, *et al.* Mathematical Assoc, 1991, 151 pp, (P). [ISBN: 0-7487-0530-9] Practical guide to promoting and structuring professional discussions among teachers via sharing of classroom anecdotes. This "grass-roots" introduction to case-study methods provides details and examples of the anecdoting process and presents the theory linking the process to teacher-instigated instructional improvements. Valuable resource for active teacher networks. MW

Foundations, P*, L.** *The Philosophy of Mathematics Education.* Paul Ernest. Falmer Pr (US Distr: Taylor & Francis), 1991, xiv + 329 pp, $31 (P); $66. [ISBN: 1-85000-667-9; 1-85000-666-0] A sharp, opinionated critique of prevailing philosophies of mathematics ("the absolutist view is an idealization, a myth ...nothing but a fool's paradise") and of mathematics education [e.g., of "industrial trainers" (new right moralists), "technological pragmatists" (economic utilitarians), "old humanists" (conservative classicists), "progressive educators" (child-centered), and "public educators" (social construction)]. Advocates a subjective social constructivism

view of mathematics with associated pedagogical, ethical, and epistemological implications for mathematics education. LAS

Linear Algebra, T(14-15: 1, 2), L. *Linear Algebra with Applications.* John W. Auer. Prentice Hall, 1991, xv + 548 pp. [ISBN: 0-13-538349-8] After an introductory chapter on analytic geometry development follows the pattern of recent years: systems, determinants, vector spaces, diagonalization, inner products. Optional topics include computational considerations, complex spaces, linear functionals, quadratic forms. Appendices on complex numbers, polynomials, linear programming. Exercises, answers, index. JS

Real Analysis, P. *Hausdorff Approximations.* Bl. Sendov. Math. & Its Applic., V. 50. Kluwer Academic, 1990, xix + 364 pp, $134. [ISBN: 0-7923-0901-4] The Hausdorff distance between two real-valued functions is defined (roughly) as the Hausdorff distance between the graphs of the functions. This text "gives an account of the main results in the theory of approximation of functions with respect to Hausdorff distance." Translation from the original Russian text published in Bulgaria in 1979. Clearly written and well-motivated. Extensive bibliography; note price! BH

Complex Analysis, T*(17-18: 1-3), P*. *Complex Variables: An Introduction.* Carlos A. Berenstein, Roger Gay. Grad. Texts in Math., V. 125. Springer-Verlag, 1991, xii + 650 pp, $59. [ISBN: 0-387-97349-4] Introduces theory of functions of one complex variable, stressing the inhomogeneous Cauchy–Riemann equation, and hence connections with function theory in several variables. Exposition is at the graduate level—thorough mastery of undergraduate material and some mathematical maturity are assumed. Further machinery is developed, or reviewed, as needed: e.g., includes overviews of distribution theory, differential forms, rudiments of sheaf theory. In all, an attractive, high-level, and up-to-date treatment. PZ

Differential Equations, T(18), P. *The Riccati Equation.* Eds: Sergio Bittanti, Alan J. Laub, Jan C. Willems. Communic. & Control Engin. Ser. Springer-Verlag, 1991, x + 338 pp, $98. [ISBN: 0-387-53099-1] Growing out of the 1989 workshop on the Riccati equation held in Como, Italy, this volume is a self-contained presentation of the history of the Riccati equation and the state-of-the-art of its solution. Appropriate either as a reference or as a text for a graduate course on the Riccati equation. JO

Dynamical Systems, S(17-18), P. *Fractals and Disordered Systems.* Eds: Armin Bunde, Shlomo Havlin. Springer-Verlag, 1991, xiii + 350 pp, $59. [ISBN: 0-387-54070-9] Intended to bridge the gap between textbooks on fractals and current research in the subject. Ten articles by different authors (but with cross-references and uniform notation) discuss fractals in the context of percolation, random growth, fractures in elasticity, cellular automata, etc. Includes lots of pictures and an introductory chapter outlining the basic concepts used later in the book. JO

Dynamical Systems, T(15-16: 1), S, L. *Chaotic Dynamics: An Introduction.* Gregory L. Baker, Jerry P. Gollub. Cambridge Univ Pr, 1990, x + 182 pp, $49.50; $17.95 (P). [ISBN: 0-521-38258-0] A good introduction to the turbulent subject of chaos, based on dynamics of the driven pendulum. Includes problems and (TrueBASIC) programs. OJ

Dynamical Systems, T(17-18: 2), P, L. *Iteration of Rational Functions: Complex Analytic Dynamical Systems.* Alan F. Beardon. Grad. Texts in Math., V. 132. Springer-Verlag, 1991, xvi + 280 pp, $39.95. [ISBN: 0-387-97589-6] A complete and rigorous introduction to the dynamics of rational functions on the complex plane. Covers material from Fatou and Julia to Sullivan and Shishikura. Assumes a background of one course in each of real and complex analysis. Well-written; lots of examples; many exercises. SP

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the action functional? An interesting book which could be of interest to both mathematicians and physicists. Includes many excellent exercises. Note price. MPR

Operator Theory, S(18), P. *Elements of KK-Theory.* Kjeld Knudsen Jensen, Klaus Thomsen. Math.: Theory & Applic. Birkhäuser, 1991, viii + 202 pp, $49.50. [ISBN: 0-8176-3496-7] Not intended to be a comprehensive treatment but rather a means to introduce the interested reader to the basic approaches to a still developing theory. Chapters on C^*-extensions, Kasparov groups, and Cuntz' approach. Assumes extensive background in operator algebras. Appendices, references, index. JS

Functional Analysis, T(18), S. *Banach Lattices.* Peter Meyer-Nieberg. Universitext. Springer-Verlag, 1991, xv + 395 pp, $49.95 (P). [ISBN: 0-387-54201-9] This nicely presented text contains an introduction to the theory of Banach lattices and the more general class of Riesz spaces, operators on such spaces, properties linked to the underlying topological and lattice structure, and spectral properties of positive and regular operators. Exercises, extensive bibliography. KS

Functional Analysis, T(15-18: 1, 2), L. *A Course on Integral Equations.* Allen C. Pipkin. Texts in Appl. Math., V. 9. Springer-Verlag, 1991, xiii + 268 pp, $39. [ISBN: 0-387-97557-8] Begins with theoretical chapters on Fredholm and Hilbert–Schmidt theory stressing the analogy with linear algebra. Emphasizes problem solving: Volterra equations, convolution equations and Laplace transforms, smoothing operators, Wiener–Hopf method, Cauchy principal value integrals, and analytic continuation method. Complete reading requires understanding of analytic functions. KS

Analysis, P. *Analysis III: Spaces of Differentiable Functions.* Ed: S.M. Nikol'skiĭ. Encyclop. of Math. Sci., V. 26. Springer-Verlag, 1991, 221 pp, $59. [ISBN: 0-387-51866-5] Part I deals with imbedding theorems for Sobolev-type spaces. Well-motivated with extensive (primarily Russian) bibliography. Part II covers the role of capacity in Sobolev-type spaces. Translation from the Russian is a bit choppy in places. BH

Analysis, P. *Spectral Theory of Automorphic Functions and Its Applications.* Alexei B. Venkov. Math. & Its Applic., V. 51. Kluwer Academic, 1990, xiv + 176 pp, $98. [ISBN: 0-7923-0487-X] A preliminary classification of directions and results in "Selberg Theory" or the spectral theory of automorphic functions. Includes an extensive bibliography and two detailed appendices. CEC

Algebraic Geometry, P. *Tata Lectures on Theta III.* David Mumford, Madhav Nori, Peter Norman. Progress in Math., V. 97. Birkhäuser, 1991, vii + 202 pp, $49.50. [ISBN: 0-8176-3440-1] The third and last in the series. The principal goal is to relate three ways of looking at theta function: the analytic, the algebraic, and the representation theoretic. Special emphasis is placed on the algebraic definition of theta functions over any base field. SG

Differential Geometry, P. *Contemporary Geometry: J.-Q. Zhong Memorial Volume.* Ed: Hung-Hsi Wu. Univ. Ser. in Math. Plenum Pr, 1991, xi + 483 pp, $85. [ISBN: 0-306-43742-2] This tribute contains a biography and publications list of Zhong. Three surveys of areas of interest to Zhong: eigenvalue techniques in geometry, the work in several complex variables in China and uniformization in several complex variables, and fourteen papers of Zhong. OJ

Geometry, T(13-15: 1), S*, P*, L*. *Space Structures: Their Harmony and Counterpoint.* Arthur L. Loeb. Birkhäuser, 1991, xx + 188 pp, $34.50. [ISBN: 0-8176-3588-2] Fifth printing of a volume originally published by Addison-Wesley in 1976 (TR, October 1976). Intended to combat "visual illiteracy and mathematics anxiety," it explores subdivisions of space in terms of the Euler–Schlaefli equation, Schlegel diagrams, Dirichlet domains, and numerous types of polyhedra. Includes an extensive up-dated bibliography. Written as a text for design science courses at Harvard. LAS

Optimization, P. *Stability, Duality and Decomposition in General Mathematical Programming.* O.E. Flippo. CWI Tract 76. Centrum voor Wiskunde en Informatica, 1991, vii + 228 pp, Dfl. 59 (P). [ISBN: 90-6196-398-2] Theoretical (conceptual rather than algorithmic) approach to mathematical programming problems. The author argues that stability (roughly, continuity of the objective function value as a function of the right hand sides of the constraints) is essential for convergence of iterative methods based on decomposition; further a good duality theory is neces-

sary for the general decomposition methods used. RM

Mathematical Modelling, P. *Code Recognition and Set Selection with Neural Networks.* Clark Jeffries. Math. Model., No. 7. Birkhäuser, 1991, viii + 166 pp, $49.50. [ISBN: 0-8176-3585-8] Neural networks in this book are dynamical systems, not the more usual layered networks. The systems iterate on an input vector and converge to a "decision" state, the nearest of stored attractors. Each of n neurons is defined by its state x_i and gain function $g_i(x_i)$ and the systems have the form $dx_i/dt = -k_i x_i + p_i(g(x))$, where p_i is a multinomial function. In set selection, answer sets are in correspondence with constant trajectories with each $x_i = \pm 1$. Memory models are applied to error correction for binary codes. RK

Stochastic Processes, P. *Decision Processes in Dynamic Probabilistic Systems.* Adrian V. Gheorghe. Math. & Its Applic., V. 42. Kluwer Academic, 1990, xvii + 354 pp, $112. [ISBN: 0-7923-0544-2] Examines decision making in the context of completely (partially) observable Markov (semi-Markov) processes by risk sensitivity. Applications to stochastic models in engineering, learning, medicine, and planning. RWJ

Stochastic Processes, T(17-18: 1, 2), S, P, L. *Introduction to Multiple Time Series Analysis.* Helmut Lütkepohl. Springer-Verlag, 1991, xxi + 545 pp, $59 (P). [ISBN: 0-387-53194-7] Covers finite- and infinite-order vector autoregressive processes, and systems with exogenous variables and nonstationary processes, but omits spectral methods. Intended for graduate students in business and economics who have a background in matrix algebra, mathematical statistics, and, preferably, univariate time series techniques. RWJ

Computer Literacy, S, L*. *Technobabble.* John A. Barry. MIT Pr, 1992, xv + 268 pp, $22.50. [ISBN: 0-262-02333-4] A witty, informative series of essays on the origins, abuses, and conventions of technical jargon. Thoroughly referenced (with extensive endnotes); well-indexed; glossary and several appendices. Where else can you find three pages on the origin of "kludge" or "nerd," or tables of "TechnoLatin?" LAS

Programming, T(14: 1). *Structuring Data with Turbo Pascal: A Practical Introduction to Abstract Data Types.* William G. McArthur, J. Winston Crawley. Prentice Hall, 1992, xiv + 738 pp, $48 (P). [ISBN: 0-13-853052-1] A standard text on data structures, including such well-known constructs as stacks, queues, linked lists, and trees. Uses the concept of abstract data types to introduce each structure before it proceeds to the more detailed and machine-dependent issue of representation and implementation. The language used in all the example programs is Turbo Pascal, which does support the abstract data type feature. GMS

Computer Systems, S(13-15), L. *Mathematica by Example.* Martha L. Abell, James P. Braselton. Academic Pr, 1992, xv + 654 pp, $32.50 (P). [ISBN: 0-12-041540-2] A thorough collection of increasingly sophisticated examples of *Mathematica* calculations, illustrated with Macintosh notebook windows and annotated with italic boxes that call attention to important features of the syntax or structure. Begins at the very beginning (double-clicking); moves through calculus and linear algebra to ordinary and partial differential equations; samples *Mathematica* packages (numeric and graphical); ends with examples of getting help. Developed primarily for Version 1.2, but contains notes on 2.0 issues. LAS

Computer Graphics, S(16), P. *XView Programming Manual, Third Edition for XView Version 3, Volume 7.* Dan Heller. O'Reilly & Assoc, 1991, xxxvii + 729 pp, $34.95 (P). [ISBN: 0-937175-87-0] XView is a user interface design system which can be used to build graphical applications for the well-known and widely used XWindow system. The package includes a collection of objects for constructing windows, menus, icons, and scrollers. Using this toolkit, users can construct quite sophisticated user interfaces in a straightforward and simple fashion. This text is a reference manual to the XView design system. GMS

Computer Graphics, S(16), P. *XView Reference Manual for XView Version 3.* Ed: Thomas Van Raalte. O'Reilly & Assoc, 1991, xiv + 252 pp, $24.95 (P). [ISBN: 0-937175-88-9] A companion text to the book *XView Programming Manual, Third Edition.* Describes all of the features that are part of XView, including objects, attributes, procedures, macros, and data structures. It is not intended to be used alone, but in conjunction with the associated programming manual. GMS

Computer Science, T?(17: 1), S, P. *De-*

pendencies in Relational Databases. Bernhard Thalheim. Teubner-Texte zur Mathematik, Band 126. BG Teubner Stuttgart, 1991, 220 pp, 38 DM (P). [ISBN: 3-8154-2020-2] Overview of the algebraic and logical foundations of the relational database model, where the semantics of the models are studied through the dependencies (e.g., functional, join, hierarchical decomposition, inclusion) which constitute inherent properties of the models. RM

Applications (Biological Science), P. *Epidemics of Plant Diseases: Mathematical Analysis and Modeling, Second, Completely Revised Edition.* Ed: Jürgen Kranz. Ecological Stud., V. 13. Springer-Verlag, 1990, xv + 268 pp, $98. [ISBN: 0-387-52116-X] Seven chapters written by ten authors present mathematical and statistical methods used for the analysis of plant disease epidemics. Topics treated are multivariate analysis, temporal and spatial aspects of air and soil-borne diseases, competition among subpopulations, assemblage of models, and mathematical simulation. SP

Applications (Biological Science), S (16), P, L. *Randomization and Monte Carlo Methods in Biology.* Bryan F.J. Manly. Chapman & Hall, 1991, xiii + 281 pp, $65. [ISBN: 0-412-36710-6] Examples, references, and explanations for carrying out randomization and Monte Carlo techniques in a variety of settings; one- and two-sample tests, ANOVA, regression, distance matrices and spatial data, time series, multivariate data as well as some other ad hoc methods. All techniques are applied to at least one interesting data set, and often more. Exposition is excellent and references are useful. Some Fortran code is included. MK

Applications (Economics), S(15-16), P, L. *A World Ruled by Number: William Stanley Jevons and the Rise of Mathematical Economics.* Margaret Schabas. Princeton Univ Pr, 1990, xii + 192 pp, $29.95. [ISBN: 0-691-08543-9] A readable historical account of Jevons's pivotal role in the nineteenth-century origins of mathematical economics (it's hard to believe economic theory was ever not mathematical). BC

Applications (Engineering), P. *Proceedings of the Fifth European Conference on Mathematics in Industry.* Ed: Matti Heiliö. ECMI, V. 7. Kluwer Academic, 1991, x + 400 pp, $139. [ISBN: 0-7923-1317-8] Section I consists of seven invited addresses ranging from reflector design to the directional spectra of ocean waves. Section II consists of eight papers relating to optimal supply and distribution of energy (electric); then follow fifty-six contributed papers (eclectic). AWR

Applications (Fluid Dynamics), P. *New Perspectives in Turbulence.* Ed: Lawrence Sirovich. Springer-Verlag, 1991, xv + 367 pp, $49. [ISBN: 0-387-97559-4] Fourteen papers on aspects of turbulence. Includes a nice review by David Ruelle of applications of the theory of differentiable dynamical systems. BC

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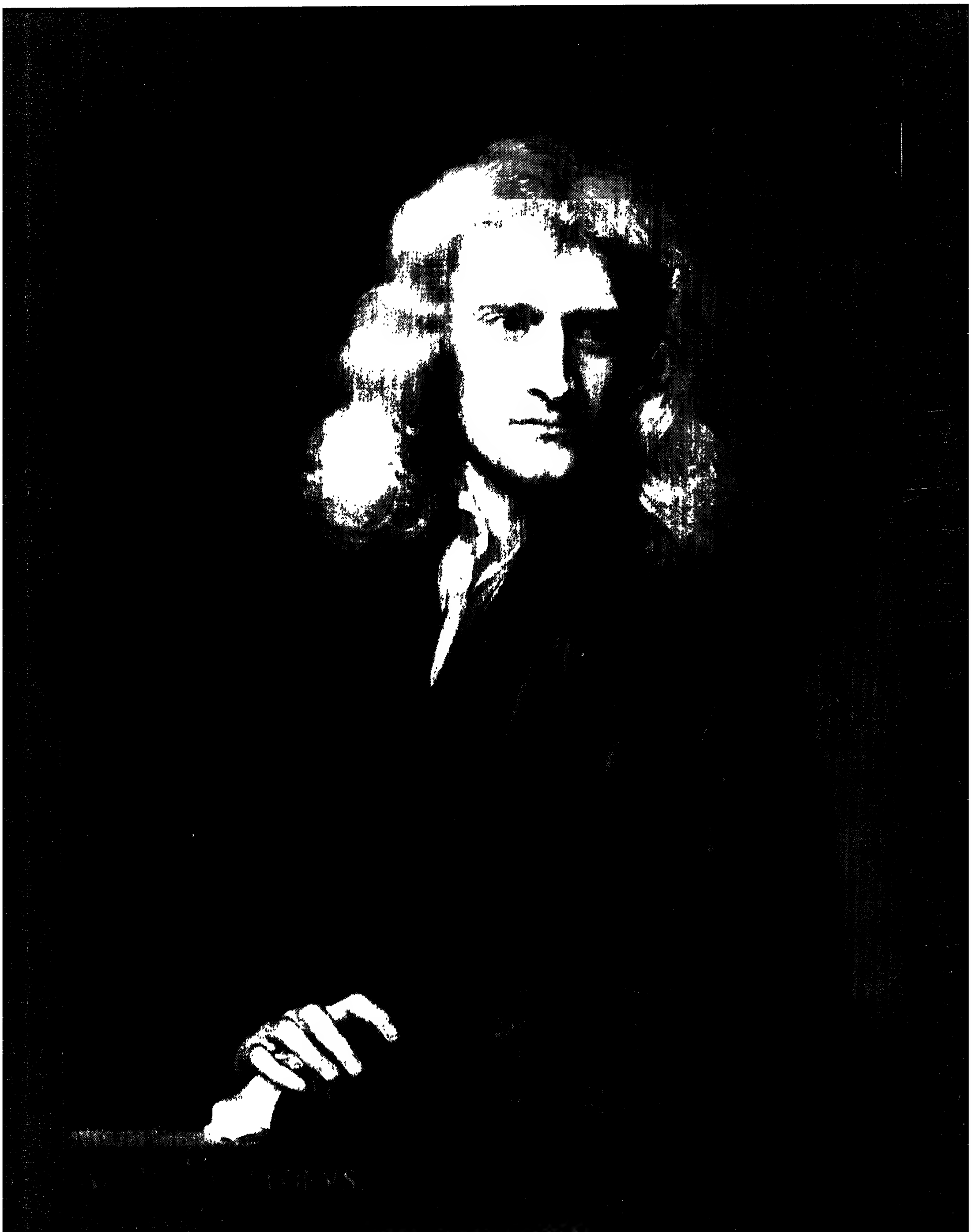

Isaac Newton, 1689

AN OFFICIAL PUBLICATION OF THE MATHEMATICAL ASSOCIATION OF AMERICA

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COVER: Newton's first portrait at age forty six, two years after the publication of *Principia*. The portrait is by Sir Godfrey Kneller; the photograph is by Jeremy Whitaker. Reprinted with permission of the Earl of Portsmouth and Jeremy Whitaker.

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An associate dean (a member of the English Department) recently began an interview with a young job candidate with a short speech. "Everyone knows the teaching of mathematics is a disgrace," he said. "What are *you* going to do about it?" At a party, a member of the Biology Department walked up to me, introduced himself, and began his conversation with a question. "Why is it that no one can teach in the Mathematics Department?" Not long ago, an acquaintance in the Education School called me on the phone: "Why are all our students failing your courses?" he demanded. "Can't you *do* something about teaching in your department?"

All these people are convinced that there is a crisis in Mathematics Education, that mathematicians are a sorry lot in the classroom, and that the scoundrels aren't doing much to fix things. And where did they learn all this? By listening to us. They read that "innovations in undergraduate teaching lag far behind advances in research" and that "both in instructional methodology and in curricular content, undergraduate mathematics is far below what it should be..." They read that "interest in teaching college mathematics has declined significantly at both undergraduate and graduate levels." And they read that the consequence "is a dysfunctional system of undergraduate mathematics beset on all sides by inadequacies and deficiencies..." They read all this in *Moving Beyond Myths*, a recent report from our community.

Enough! I know, I know, some of my colleagues are not always conscientious teachers; but many others are creative and able instructors at every level. These are people who care about students and think about the courses they teach. I know, I know, much can be improved in the curriculum; but there is also much to recommend a curriculum that has some historical roots. Those roots help us to set standards and to compare one generation to the next. And I know, I know, we ought to experiment with new and innovative learning techniques (I think that's what "instructional methodology" means); but many of my colleagues *already* experiment with courses, and indeed *like* to teach new courses in new ways rather than the same stale course year after year.

Should we be satisfied with teaching in mathematics? Of course not. But we ought to realize that the problems of education go far deeper than flawed instructional methodologies and curricula. That dean who complained about mathematics teaching resides in the English Department, where 84% of the grades are A's and B's; the figure in Mathematics is 47%. The Biology professor teaches only students who choose his courses as electives. And the Education School friend? Most of his colleagues in the Education School *want* to use mathematics as a screening device for their students. The Education school awards 87% A's and B's.

Mathematics, like most disciplines, has poor teachers; it also has some great ones, and lots of people in between. We can do better; we *are* doing better. Let's experiment and innovate and be creative teachers. But let's not exaggerate our problems. Mathematicians have a reputation for honesty. When we go about wringing our hands and moaning about the dismal state of mathematics teaching, people begin to believe us.

—John Ewing

From Newton to Einstein

Blake Temple and Craig A. Tracy

1. INTRODUCTION. In 1687 Sir Issac Newton (1642–1727) published *Philosophae Naturalis Principia Mathematica* (known as the *Principia* by those who do not speak Latin), in which he explained the observed motion of the planets in the sky. In particular, he derived Kepler's laws of motion from the assumption that the sun *pulls* on a planet with a *force* that varies inversely with the square of the distance from the sun to the planet. The brilliance of this work lies in the fact that Newton had to invent the meaning of the word *force*, and in so doing he related the change of motion to the force applied through what we now refer to as *Newton's Second Law*:

$$\textit{Force} = \textit{Mass} \times \textit{Acceleration}. \tag{1.1}$$

Newton then postulated that every particle of matter in the universe attracts every other particle with a force whose direction is that of the line joining the two, and whose magnitude varies directly as the product of their masses, and varies inversely as the square of the distance between them. Thus a planet of mass M_p and the sun of mass M_s separated by a distance r each experience an attractive force of magnitude F given by the formula

$$F = \frac{G_0 M_p M_s}{r^2}, \tag{1.2}$$

where G_0 is the universal gravitational constant. From these strikingly simple assumptions, Newton was able to prove mathematically that the planets must obey the celebrated laws of Johannes Kepler (1571–1630); laws that Kepler had earlier formulated on the basis of detailed observational studies of the motions of the heavenly bodies, namely:

(1) *The planets move in elliptical orbits about the sun with the sun fixed at one focus of the ellipse.*
(2) *The velocity of a planet varies in such a way that the line joining the planet to the sun sweeps out equal areas in equal times.*
(3) *The square of the time required by a planet for one revolution around the sun is proportional to the cube of its mean distance from the sun.*

Newton unified all of the planetary laws of motion which were known in his lifetime: laws that were written down by Kepler in the first decade of the seventeenth century and until Newton were understood only as empirical observations. Thus, planetary motion was explained by the assumption that celestial bodies *pull* on each other (across millions of miles of empty space) with a *force* proportional to one over the separation distance squared. This point of view stood as the ultimate explanation of why the stars and planets in the sky move the way they do, and the fundamental starting points, (1.1) and (1.2), were elevated to the

status of *Laws of Nature*. That is, until Albert Einstein (1879–1955) entered the scene in 1916 with his paper *Die Grundlagen der allgemeinen Relativitätstheorie* (*The Foundation of the General Theory of Relativity*). Einstein took the point of view that heavenly bodies don't pull on each other across empty space, but rather the massive objects in the universe cause space itself to be *curved*, and the motions of the planets are explained as bodies moving along *straight lines* in a *curved space*. In fact, it is actually *spacetime* that is curved, and in Einstein's theory the curvature of spacetime evolves dynamically in an elaborate manner determined by the stars and planets in the universe. Einstein made mathematically precise sense of this, and used his constructions to show rigorously that with his assumptions, the planets would almost move in ellipses around the sun, but that there would be a small correction. In 1916 this correction to Newtonian theory was too small to observe in all the planets except Mercury (today this effect has been observed in other planetary orbits, but it is most pronounced in the case of Mercury (see [**7**, **8**])). Einstein showed that if his theory were correct, then the *perihelion* of the orbit of the planet Mercury, the point at which the orbit was closest to the sun, would not be the same in every orbit as Newton's theory predicted, but would precess an angular distance of 43 seconds of an arc per century. This had been observed exactly to be the case in 1859 by Joseph Le Verrier (1811–1877)*, and this gave the first experimental evidence that Newton's theory was only an approximation to Einstein's more general theory. In fact, beyond our solar system the predictions of Einstein's theory diverge dramatically from Newton's predictions. Indeed, Einstein's theory implies the formation of *black holes* in extremely massive stars. These are stars in which everything sufficiently close, including *light*, is sucked into the center of the star. It is no wonder that at the moment of his derivation of the perihelion shift predicted by his theory, Einstein is quoted as saying that his excitement was so great as to give him "palpitations of the heart"! ([**6**], pg. 253).

Both Newton's and Einstein's predictions involve the study of ordinary differential equations. The fundamental ODE is the equation that describes how the radius of the orbit, i.e. the distance from the sun, varies as a function of time along a planet's orbit. In fact, it will be simpler to study the ODE that describes how $1/r$ varies as a function of angle θ (Astronomically, it is angular changes that can be measured most accurately with a telescope). In this paper we will derive this ODE in the case of Newton's assumptions (1.1) and (1.2). We will then write down the corresponding ODE which Einstein gets from his theory. We will observe that this ODE approximates the one Newton gets, but with a small perturbation. We will then use the principle of conservation of energy to determine the qualitative structure of the orbits predicted by these ODE's. The analysis of Einstein's ODE gives an elementary qualitative picture of what happens in a black hole and how black holes arise in the theory of gravitation. Finally, an asymptotic expansion of

*Actually, the observed perihelion advance is 574 arcseconds/century of which 531 arcseconds/century are accounted for due to the perturbing effect of the other planets on the Mercury-Sun system. Le Verrier found that the largest contribution comes from Venus, 278 arcseconds, and next Jupiter at 153 arcseconds. The Earth's effect is third with 90 arcseconds and the remaining planets contribute about 10 arcseconds. Thus the total contribution coming from Newtonian celestial mechanics calculations is about 531 arcseconds per century. The remaining 43 arcseconds/century is called the *anomalous perihelion shift* and it is this that is unaccounted for by Newtonian theory. A compilation of a decade's worth of data (1966–1976) by a group at MIT gave the anomalous part of Mercury's perihelion precession to be 43.11 ± 0.21 arcseconds per century (see [**8**]).

Einstein's ODE will enable us to estimate the difference between the predicted orbits, and we will obtain Einstein's famous result that in the case of Mercury, a precession in the amount of 42.98 arcseconds/century is predicted to occur in the perihelion of the orbit of planet Mercury when Einstein's equation is taken in place of Newton's. To within experimental error this is equal to 43.11 ± 0.21 arcseconds/century which is the observed anomalous precession in Mercury's orbit [**8**].

Once we assume the ODE that comes from Einstein's theory, our treatment is entirely self-contained. The actual derivation of the ODE in Einstein's theory involves an in-depth study of differential geometry and physics which is beyond the scope of this paper. It is remarkable, though, that once the fundamental ODE's are established, both Newton's and Einstein's predictions can be derived by methods taught in an undergraduate course in differential equations.

For an in-depth discussion of the history of this subject, the reader is referred to the book *Subtle is the Lord* by Pais [**6**]. A brief but informative discussion can also be found in the first chapter of *Gravitation and Cosmology* by Weinberg [**7**] (see also [**4**]). An introductory account of the experimental tests of general relativity can be found in *Was Einstein Right?* by Will [**8**]. A comprehensive study of black holes can be found in *The Mathematical Theory of Black Holes* by Chandrasekhar [**5**].

2. THE FUNDAMENTAL ODE'S. We will first derive the fundamental ODE predicted by Newton's theory. So assume that Newton's Laws (1.1) and (1.2) hold. We derive an ODE for the distance r as a function of the angle θ, and the final form of the ODE will be obtained by making the substitution $u = 1/r$. This will give us an ODE that must be satisfied along every trajectory that corresponds to a solution of (1.1) and (1.2). To start, let $\mathbf{r}_p$ and $\mathbf{r}_s$ denote the positions of the planet and sun, respectively, with respect to some (inertial) coordinate system. Then combining (1.1) and (1.2) (and accounting for the direction of the force) we have

$$M_p\ddot{\mathbf{r}}_p = -\frac{G_0 M_p M_s}{|\mathbf{r}_p - \mathbf{r}_s|^3}(\mathbf{r}_p - \mathbf{r}_s), \tag{2.1}$$

$$M_s\ddot{\mathbf{r}}_s = -\frac{G_0 M_p M_s}{|\mathbf{r}_p - \mathbf{r}_s|^3}(\mathbf{r}_s - \mathbf{r}_p). \tag{2.2}$$

The dot here and throughout denotes differentiation with respect to the time t. We introduce $\mathbf{r} = \mathbf{r}_p - \mathbf{r}_s$, the vector that points from the sun to the planet, and the center of mass $\mathbf{r}_0 = (M_p\mathbf{r}_p + M_s\mathbf{r}_s)/(M_p + M_s)$. Adding (2.1) and (2.2) shows that the center of mass $\mathbf{r}_0$ moves freely (that is, its time dependence is $\mathbf{c}_1 t + \mathbf{c}_2$, $\mathbf{c}_1$ and $\mathbf{c}_2$ are vector constants). Subtracting (2.2) from (2.1), expressing $\mathbf{r}_p$ and $\mathbf{r}_s$ in terms of $\mathbf{r}$ and $\mathbf{r}_0$, and using $\ddot{\mathbf{r}}_0 = 0$, we obtain

$$\ddot{\mathbf{r}} = -\frac{G}{|\mathbf{r}|^3}\mathbf{r}, \tag{2.3}$$

where we set

$$G = G_0(M_s + M_p) \approx G_0 M_s.$$

Note that the constant G is essentially independent of the planet considered because for all planets $M_p/M_s \ll 1$. Thus (2.3) is an equation that holds for every planet. Since $M_p/M_s \ll 1$, the center of mass is essentially at the sun; and so, we may think of the sun at the origin and (2.3) describes the motion of the planet

about the fixed sun. Since (2.3) is a second order (nonlinear) ODE, the vector valued function $\mathbf{r}(t)$ that satisfies (2.3) is determined by the initial conditions

$$\mathbf{r}(0) = \mathbf{r}_0 \quad \text{and} \quad \dot{\mathbf{r}}(0) = \dot{\mathbf{r}}_0.$$

As a consequence of (2.3), the orbit $\mathbf{r}(t)$ must lie in a fixed plane containing the sun. To see this, let $\mathbf{r}$ and $\dot{\mathbf{r}}$ be given at time t, and $\mathbf{M} = \mathbf{r} \times \dot{\mathbf{r}}$ denote the cross product of $\mathbf{r}$ and $\dot{\mathbf{r}}$. Since $\mathbf{r} \times \dot{\mathbf{r}}$ is perpendicular to both $\mathbf{r}$ and $\dot{\mathbf{r}}$, it suffices to show that $\mathbf{M}$ is constant in t. Using the Leibniz rule for the cross product, we obtain

$$\dot{\mathbf{M}} = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = 0,$$

because $\ddot{\mathbf{r}}$ is parallel to $\mathbf{r}$ by (2.3), and the cross product of parallel vectors vanishes. Thus the entire trajectory lies in the plane perpendicular to $\mathbf{M}$. Let $\mathbf{r} = (x, y)$ denote Cartesian coordinates in this plane with the sun at the origin, and let r and θ denote the corresponding polar coordinates. Now a given trajectory $\mathbf{r}(t) = (x(t), y(t))$ that satisfies (2.3) determines the functions $r(t)$ and $\theta(t)$ through the relations $x(t) = r(t)\cos\theta(t)$ and $y(t) = r(t)\sin\theta(t)$. We now find the ODE that this trajectory in polar coordinates satisfies. To this end, note that (2.3) reads

$$\ddot{x} = -\frac{G}{r^3}x, \tag{2.4}$$

$$\ddot{y} = -\frac{G}{r^3}y, \tag{2.5}$$

and using the substitution $x = r\cos\theta$, $y = r\sin\theta$, we obtain

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta,$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta.$$

Differentiating again and using (2.4) and (2.5) we have

$$\ddot{x} = \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta - r\ddot{\theta}\sin\theta = -\frac{G}{r^2}\cos\theta, \tag{2.6}$$

$$\ddot{y} = \ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta + r\ddot{\theta}\cos\theta = -\frac{G}{r^2}\sin\theta. \tag{2.7}$$

Now multiplying (2.6) by $\cos\theta$, (2.7) by $\sin\theta$, and adding the result we obtain

$$\ddot{r} - r\dot{\theta}^2 = -\frac{G}{r^2}; \tag{2.8}$$

and multiplying (2.6) by $\sin\theta$, (2.7) by $\cos\theta$, and subtracting we obtain

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0. \tag{2.9}$$

Statements (2.8) and (2.9) hold so long as $r \neq 0$. Assuming this, (2.9) tells us that

$$r^2\dot{\theta} = H, \tag{2.10}$$

where H is a constant determined by the initial conditions. Without loss of generality we assume that H is positive. We can use (2.10) to solve for $\dot{\theta}$ in terms of r, substitute into (2.8), and obtain the following ODE that relates r and t:

$$\ddot{r} = \frac{H^2}{r^3} - \frac{G}{r^2}, \qquad r \neq 0. \tag{2.11}$$

We now use (2.11) to obtain an ODE that is satisfied by r as a function θ. Indeed, (2.10) shows that $\dot{\theta} = H/r^2 \neq 0$ when $H \neq 0$ and $r \neq 0$, so in this case θ is a monotone function of time along trajectories. Let us now assume that $H \neq 0$, $r \neq 0$, let $r = r(\theta)$ give r as a function of θ along a trajectory, and let prime denote differentiation with respect to θ. In this case the chain rule gives

$$\dot{r} = r'\dot{\theta} \quad \text{and} \quad \ddot{r} = r''\dot{\theta}^2 + r'\ddot{\theta}.$$

But $\dot{\theta} = H/r^2$ implies $\ddot{\theta} = -(2H/r^3)r'\dot{\theta} = -(2H^2/r^5)r'$, so we can obtain $\ddot{r} = r''(H^2/r^4) - r'^2(2H^2/r^5)$. Substituting this into (2.11) gives us an ODE for r as a function of θ:

$$\frac{H^2}{r^2}r'' - 2\frac{H^2}{r^3}r'^2 = \frac{H^2}{r} - G. \tag{2.12}$$

We now use one final clever trick to simplify this ODE. We make the definition $u = 1/r$, and substitute u in favor of r in (2.12) using the identities

$$r' = -u'/u^2 \quad \text{and} \quad r'' = -u''/u^2 + 2u'^2/u^3. \tag{2.13}$$

This gives the final remarkably simple linear constant coefficient ODE

$$u'' + u = \frac{G}{H^2}. \tag{2.14}$$

Equation (2.14) is known as *Binet's equation*, and it tells how $u = 1/r$ varies as a function of θ along the trajectory of a planet (assuming Newton's laws are correct). In (2.14) we have transformed a nonlinear equation into a linear one which we can solve explicitly. To summarize, (2.14) is the fundamental ODE predicted by Newton's theory for the orbit of the sun-planet system.

The predictions of Einstein's theory for a sun-planet system are similar. In Einstein's theory a derivation analogous to the derivation above leads to the conclusion that trajectories also lie in a fixed plane containing the sun and equation (2.10) is still satisfied, but the equation that $u = u(\theta)$ satisfies is no longer (2.14), but is instead the following nonlinear ODE which is a perturbation of (2.14) (cf. [**1**], pg. 207):

$$u'' + u = \frac{G}{H^2} + \frac{3G}{c^2}u^2. \tag{2.15}$$

Here c is the speed of light expressed in the units of time and length that G is expressed in. This equation is the same as the equation (2.14) except for the term $(3G/c^2)u^2$, which we might expect is small because the constant c^2 is in the denominator, and the speed of light is very large. In the case of a star in which M_s is large enough so that $G = G_0 M_s$ is on the order of c^2, this term will not be small; and consequently, we expect the orbits of planets to be significantly different from those predicted from Newton's theory. Indeed, Einstein's theory predicts the existence of *black holes* when the density of the star is sufficiently large.

Our analysis of Einstein's ODE in the next section will show that all planets near enough to the star (with low enough energy) will ultimately be sucked into the center of the star as they follow trajectories of (2.15). This contrasts strikingly with the conclusions of Newton's ODE, which predicts that the corresponding planets would enter stable elliptical orbits which would rotate around the star forever. In Einstein's full theory, one can show that when the density of a star is sufficiently large there is a distance, called the *Schwarzschild radius*; and that objects of *all* energies, including *light*, will be drawn into the star when the distance to the star

falls within this radius (the Schwarzchild radius for the sun lies well inside the surface of the sun). Thus radiation emitted from such a star cannot be seen, and hence the name *black hole*. This general result cannot be obtained from the ODE (2.15) alone. In fact the xt-coordinates in terms of which (2.15) is expressed do not separate space and time uniformly, curvature effects become dominant, and (2.15) is not a good approximation to Einstein's theory for distances near the Schwarzschild radius. In fact, the fundamental ODE (2.15) was obtained as an approximation to the *Schwarzschild solution*, an exact solution to the Einstein field equations, under the condition that G/Hc is small*. Even though our analysis of (2.15) is not strictly valid close to the center of very massive stars, the next section gives a nice qualitative indication of how black holes arise in the theory of gravitation.

In Section 3 we determine the qualitative properties of solutions of (2.14) and (2.15) using the principle of conservation of energy, and in the final section we will show that the extra term $(3G/c^2)u^2$ in Einstein's equation (2.15) gives rise to the observed anomalous precession in the perihelion of the orbit of the planet Mercury.

3. STRUCTURE OF SOLUTIONS. First we discuss the solutions of the ODE (2.14). We rewrite (2.14) as

$$u'' = -u + \frac{G}{H^2}. \tag{3.1}$$

This ODE is linear and has the general solution

$$u = \frac{G}{H^2} + D\cos(\theta + K), \tag{3.2}$$

where D and K are arbitrary real constants. It is easily verified that (3.2) defines an ellipse, a hyperbola, or a parabola depending upon whether $|D| < G/H^2$, $|D| > G/H^2$, or $|D| = G/H^2$, respectively. We now verify the qualitative properties of the solutions of (3.1) using the principle of conservation of energy. We could get this information directly from (3.2), but we wish to use a method which applies also to the study of Einstein's ODE which is nonlinear.

Writing $F(u) = -u + G/H^2$, equation (3.1) becomes

$$u'' = F(u). \tag{3.3}$$

For equations of this type, the *energy* $E(u,u') = u'^2/2 + P(u)$ is constant along solutions $u = u(\theta)$. Here $u'^2/2$ is called the *kinetic energy* associated with (3.3); and $P(u)$, the *potential energy*, satisfies $P'(u) = -F(u)$. To check that $E = E(u(\theta), u'(\theta))$ is constant along solutions, we simply differentiate with respect to θ:

$$E'(\theta) = u'u'' + P'(u)u' = u'u'' - F(u)u' = 0.$$

Thus, if our initial conditions for (3.1) are

$$u(0) = u_0 \quad \text{and} \quad u'(0) = u_0'$$

for some constants u_0 and u_0', then $E(u(\theta), u'(\theta)) = E(u_0, u_0') = E$ for all θ.

*As a historical note, this exact solution was derived by Karl Schwarzschild (1873–1916) in December 1915 while serving in the German army on the eastern front. This work was communicated to the Berlin Academy by Einstein on January 13, 1916, shortly before Schwarzschild's untimely death [5, p. 136].

The positivity of the kinetic energy implies,

$$E \geq P(u(\theta)) \quad \text{for all } \theta, \tag{3.4}$$

and so the solution cannot take on values of u where $P(u) > E$. Thus the energy controls "ahead of time" the possible values of u that a solution $u(\theta)$ of (3.1) can assume. In technical terms, we say that (3.4) is an *apriori estimate* for (3.1). A graph of P will thus indicate to us the types of solutions that are possible for a given initial value of the energy E. Since P is any antiderivative of F with respect to u, we can take P to be

$$P(u) = \frac{1}{2}u^2 - \frac{G}{H^2}u. \tag{3.5}$$

P, a quadratic function, is sketched in Figure 1.

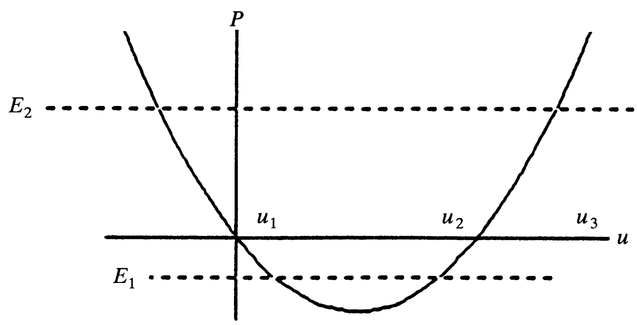

Figure 1

We see that P takes a minimum value of $-G^2/2H^4$ at $u = G/H^2$, so $E_0 = -G^2/2H^4$ is the smallest possible value that the energy E of an orbit can have because $E \geq P$ all along the orbit. For trajectories having $E = E_0$, $u = 1/r = G/H^2$ is constant, so the orbit must be a circle of radius $r = H^2/G$. For trajectories that have energy $E = E_1$, where $-G^2/2H^4 < E_1 < 0$ (see Figure 1), $E \geq P$ implies that the possible values of u taken on in the trajectory lie between the two values u_1 and u_2 which satisfy $P(u_1) = P(u_2) = E_1$. These trajectories correspond to the elliptical orbits in the plane that move between $r = 1/u_1$ and $r = 1/u_2$, the major axis of this ellipse occurring at the value $\theta = \theta_1$ which satisfies $u(\theta_1) = u_1$, and the minor axis occurring at $\theta = \theta_2$ satisfying $u(\theta_2) = u_2$.

The trajectories with energies $E = E_2$, $0 < E_2 < \infty$, are restricted to taking on values of u between 0 and u_3 in Figure 1 (recall that $u = 1/r$ and hence must be nonnegative) with $P(u_3) = E_2$. Such trajectories correspond to hyperbolic orbits that come closest to the sun at $r = 1/u_3$, and then go off to infinity as u tends to zero and $r = 1/u$ tends to infinity. Similarly, the $E = 0$ orbit is the lowest energy orbit for which r tends to infinity, and the nearest this trajectory comes to the sun is $r = H^2/G$. This trajectory corresponds to a parabolic orbit. Note that in the arguments given above for obtaining qualitative structure of orbits at various energies, we used the important observation that the angular velocity u' can be zero only at values of u where $P(u) = E$. This means that u, and hence r, is a strictly increasing or decreasing function of θ when u is in one of the intervals determined by the values of u where $P(u) = E$; and hence solutions can "turn around" only at these special values.

Note that none of the solutions ever crashes into the sun. Thus there is one solution missing from the above analysis; namely, the trajectory corresponding to an object falling straight into the sun. For such a solution, $\theta = constant$, and thus

we lost this one solution when we made the assumption $H = r^2\dot{\theta} \neq 0$. Note also that the above energy analysis told us that a trajectory in Newton's theory behaves like an ellipse, hyperbola, or parabola, but it did not tell us the exact shape of an orbit. For Newton's equation we can find a simple formula (3.2) for the trajectories; and we can verify directly from the formula that the orbits truly describe conic sections in the xy-plane. In the following analysis of Einstein's equation, we do not have the luxury of an elementary formula for the solutions, and we will use the energy method to understand the behavior of the orbits.

We now discuss Einstein's ODE (2.15) which we write as

$$u'' = -u + \frac{G}{H^2} + \frac{3G}{c^2}u^2 = F(u). \tag{3.6}$$

This is a nonlinear equation, and the energy E associated with (3.6) is given by

$$E(u,u') = \tfrac{1}{2}u'^2 + P(u),$$

where P is a cubic function of u given by

$$P(u) = \frac{1}{2}u^2 - \frac{G}{H^2}u - \frac{G}{c^2}u^3.$$

One can verify that the critical points in the graph of P are u_+ and u_- given by

$$u_{\pm} = \frac{1}{2B}(1 + \sqrt{1 \pm 4\varepsilon}),$$

where $A = G/H^2$, $B = 3G/c^2$, and $\varepsilon = AB$ are positive constants. Note that for $\varepsilon \ll 1$, the case for the sun, a Taylor expansion of $\sqrt{1-4\varepsilon}$ shows that it is approximately $1 - 2\varepsilon = 1 - 2AB$, and substituting this into the formula for u_- gives the value $u_- = A$, the critical point in Newton's potential. However, u_+ does not correspond to a critical point in Newton's theory. A graph of P is sketched in Figure 2.

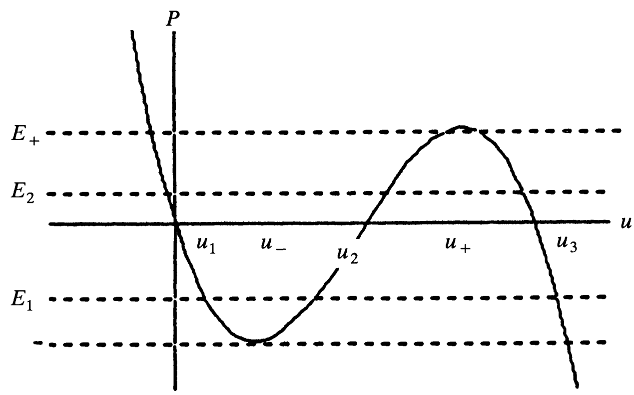

Figure 2

In this figure, $E_- = P(u_-)$, $E_+ = P(u_+)$, and E_i $(i = 1,2)$ are sample values of the energy lying in the intervals determined by E_- and E_+. For fixed E, the states u_i are the values of u where $P(u) = E$. The states u_1, u_2 and u_3 are graphed for energy level $E = E_1$ in Figure 2. The qualitative structure of an orbit depends on which of these intervals the energy of the orbit lies. If $E > E_+$, then $E \geq P$ implies that u' is never zero. Thus, if $u' < 0$, then u tends to zero, r tends to infinity as θ increases, and this corresponds to a trajectory that escapes the sun's gravitational pull in Newtonian theory. Similarly, if $u' > 0$, then $u(\theta)$ will continue

out to infinity as θ increases; and hence, r tends to zero and the planet crashes into the sun. Thus, unlike Newton's equation, Einstein's equation predicts that if an object moves toward the sun with enough energy, it will necessarily crash into the sun.

Consider now the orbits in the case with energy E satisfying $E_- < E < E_+$ say, $E = E_1$. Then $E \geq P$ implies that the values of u taken on by the trajectory must either lie within the interval $[u_1, u_2]$, or else within the interval $[u_3, \infty]$, as the dotted lines at energy level E_1 indicates in Figure 2. The case $u(\theta)$ in $[u_1, u_2]$ corresponds to the orbits of Newton's theory. When $u_1 > 0$ and $u(\theta)$ ranges between u_1 and u_2 (exemplified by $E = E_1$ in Figure 2), we obtain a cyclic trajectory that rotates between $r_1 = 1/u_1$ and $r_2 = 1/u_2$, and these correspond to the elliptical orbits of Newton's theory. When $u_1 < 0$ (exemplified by $E = E_2$ in Figure 2), then $u(\theta)$ in $[u_1, u_2]$ implies that $u(\theta)$ actually ranges between 0 and u_2 because $u(\theta) > 0$. Such solutions can move out to the maximum value u where $P(u) = E_2$ (a distance of closest approach), and then they turn around and move monotonically to $u = 0$ (equivalently $r = \infty$). These solutions correspond to the hyperbolic orbits of Newton's theory. Note that nothing we have said implies that the minimum value r_1 for one of the cyclic orbits will be taken on at the same value of θ in every cycle. Indeed, it is the precession of this angle that we will calculate in the next section for the orbit of Mercury.

For the case $E_- < E < E_+$ (again, say, $E = E_1$), $u(\theta) > u_3$, $u' < 0, u(\theta)$ decreases to $u = u_3$ where $u' = 0$, and then "turns around" and $u(\theta)$ increases to infinity. If $u' > 0$ then $u(\theta)$ increases monotonically to infinity. In either case this corresponds to an orbit crashing into the sun. The same is true for trajectories for which $E < E_-$ (see Figure 2). We can conclude that objects close enough to the sun or with low enough energy will necessarily crash into the origin. There are no corresponding trajectories in Newton's theory. At this point it is important to note, however, that our analysis above assumes throughout that the sun is a point mass located at the origin. In fact, the radius of the sun actually occurs at $u < u_+$, so the solutions with $u > u_+$ that crash into the origin are not really observed in our solar system because $u > u_+$ lies inside the surface of the sun*. In contrast, for very massive stars the radius of the star can lie at a value of u well outside of u_+, and one can show that there is in fact a critical value of r, the Schwarzschild radius, inside of which everything, including light, falls into the star, in analogy with the orbits in the last case above. Although the above analysis is a nice indication of the behavior of orbits near a black hole, a complete analysis requires a deeper understanding of general relativity and cannot be obtained from the ODE (3.6) alone (see [5]). As a final comment, note that the solution corresponding to $E = E_-$ is a stable circular orbit at radius $r = 1/u_-$; and solutions sitting at u_+ with energy E_+ are unstable circular orbits, and can just as well fall into the sun as drift away to infinity. Also there is an omitted solution corresponding to an object falling straight into the sun with $\dot{\theta} = 0$, and as in Newton's theory, this solution is not accounted for in (3.6).

*More precisely,

$$r_+ = \frac{1}{u_+} = \frac{2B}{1 + \sqrt{1 + 4\varepsilon}} \leq 2B = \frac{6G}{c^2}$$

has the dimension of length (cf. §4) and corresponds to a radius much smaller than the radius of the sun.

4. THE PRECESSION IN THE PERIHELION OF THE ORBIT OF MERCURY. In this section we study the precession that occurs in the cyclical trajectories of Einstein's equation (3.6). Now the solutions of (3.6) should approximate the solutions of Newton's equation (3.1) when the term $(3G/c^2)u^2$ is "small". We then need a way to measure how small this term really is. It is tempting to take G/c^2 as a measure of how small the term is, but a closer look shows that this makes no sense. Indeed, the absolute magnitude of G/c^2 depends on the choice of units in terms of which we decide to measure mass, length and time. To make sense of the size of term $(3G/c^2)u^2$, we must construct a constant which has a value independent of units we choose. Then we can write $(3G/c^2)u^2$ in terms of this constant. Such a constant is called a *dimensionless parameter*. To obtain our dimensionless parameter, we must first determine the dimensions of the constants G, H, and c which appear in our equation. To this end, let L denote the dimension of length, T the dimension of time, and M the dimension of mass. Now let square brackets around a quantity denote the dimensions of that quantity. For example,

$$[c] = L/T$$

since c is a velocity. Letting X and Y denote two quantities, $[\cdot]$ has the property that

$$[X^nY^m] = [X]^n[Y]^m$$

for any two integers m and n. Thus, for example,

$$[c^2] = L^2/T^2.$$

We now use the following principle to obtain the dimensions of the quantities G and H: *Every term in the same physical equation must have the same dimensions.* We call such an equation dimensionally correct. Indeed, this principle is really expressing the fact that if we have a function which satisfies a given physical equation expressed in one set of units, then the equation expressed in a new set of units should have as its solution the function obtained from the original one by rescaling it according to the dimensions of the solution variable. We now obtain the dimensions of G and H.

Using that an acceleration has units L/T^2, from (2.3) we obtain

$$L/T^2 = [\ddot{\mathbf{r}}] = \left([G]/[|\mathbf{r}|^3]\right)[\mathbf{r}] = [G]/L^2,$$

so solving for $[G]$ yields

$$[G] = L^3/T^2. \tag{4.1}$$

Equation (2.10) implies

$$[H] = L^2/T \tag{4.2}$$

since the unit of $\dot{\theta}$, a frequency, is $1/T$. Using (4.1) and (4.2) we can verify that G/Hc is the simplest dimensionless parameter constructible from G, H, and c.
Equation (2.10) implies

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since the unit of $\dot{\theta}$, a frequency, is $1/T$. Using (4.1) and (4.2) we can verify that G/Hc is the simplest dimensionless parameter constructible from G, H, and c.

As an aside, statement (4.1) asserts that within Newtonian theory there is a universal constant G, independent of the planet considered, which has the dimension L^3/T^2. This might well lead you to guess there is a quantity of dimension L^3/T^2 associated with each planetary orbit that is independent of the planet

chosen. Kepler's third law verifies that this intuition is correct, and that the simplest guess for such a quantity (mean distance to the sun cubed divided by the period of the orbit squared) is correct! In short, by dimensional analysis, one could guess Kepler's third law without making any headway whatsoever in rigorously solving Newton's ODE. When one is presented with a complicated equation, this type of intuition can be crucial. It can also be incorrect!

We are now ready to study the perihelic motion which occurs in the cyclical trajectories in Einstein's theory when $G/Hc \ll 1$. But there is a problem. Since G/Hc is dimensionless, it will be the same when evaluated under any choice of units, and thus it is tempting to say that G/Hc is a true measure of how small this last term is. However, the rate at which a solution of Einstein's ODE (2.14) diverges from a solution of Newton's equation (2.15) also depends on the size of the initial conditions, and G/Hc is not a measure of the perturbation which is independent of the starting conditions. To obtain such a dimensionless parameter that accounts for the initial conditions as well, we "nondimensionalize" the ODE's (2.14) and (2.15). To begin, let us fix on the underlying elliptical solution u_0 of (2.14) which corresponds to the orbit of Mercury in the Newtonean theory. The solution of (2.14) is given in (3.2) as

$$u_0 = A + D\cos(\theta + K), \tag{4.3}$$

where $A = G/H^2$, and we assume $|D| < A$, so that (4.3) describes an ellipse in $r\theta$-coordinates. Since rotating the coordinate axes by an angle $-K$ would eliminate the constant K in this formula, we can assume with no loss of generality that $K = 0$, in which case the initial conditions are

$$u(0) = A + D, \qquad u'(0) = 0. \tag{4.4}$$

(To specify the orbit of Mercury, we must obtain the values for D and H from astronomical tables, but we will see that only the value of H affects the perihelion shift.) Since $|D| < A$, we can take A as a dimensional measure of the size of the initial conditions. Now back to our problem: we wish to find a dimensionless measure of the perturbation of solutions of the Einstein ODE (2.15) from the solution of (2.14) that accounts for the size of the initial condition. The idea is to obtain the equations for the dimensionless variable

$$\bar{u} = u/A. \tag{4.5}$$

First, for the Newton equation, substituting $\bar{u}$ into (2.14) gives

$$\bar{u}_0'' = -\bar{u}_0 + 1, \tag{4.6}$$

with initial conditions

$$\bar{u}_0(0) = \frac{A+D}{A}, \qquad \bar{u}_0'(0) = 0. \tag{4.7}$$

Similarly, for the Einstein ODE, substituting $\bar{u}$ into (2.15) and assuming the same initial conditions, gives

$$\bar{u}'' = -\bar{u} + 1 + \varepsilon\bar{u}^2, \tag{4.8}$$

$$\bar{u}(0) = \frac{A+D}{A}, \qquad \bar{u}'(0) = 0, \tag{4.9}$$

as the Einstein prediction for the same planetary orbit, where $\varepsilon = 3G^2/H^2c^2$. Since $[\bar{u}] = [\varepsilon] = 1$, the parameter ε is a dimensionless parameter that reasonably gives an absolute measure of the perturbation of the Einstein solution $\bar{u}$ from the

Newtonian solution $\bar{u}_0$. Conclude that by writing the non-dimensional equations (4.6) and (4.8) for the dimensionless variables $\bar{u}_0$ and $\bar{u}$, we have located a dimensionless perturbation parameter ε that incorporates the size of the initial conditions. Thus, let

$$u_0(\theta) = 1 + d\cos(\theta), \tag{4.10}$$

denote the fixed solution of Newton's ODE (4.6) corresponding to the Mercury solution (4.3), $d = D/A$. When $\varepsilon \ll 1$, the solution to Einstein's ODE (4.8) with the same initial data will remain close to this trajectory at least over changes of angle that are not too great. Thus we write the corresponding solution $\bar{u}$ to Einstein's ODE as $\bar{u} = \bar{u}_0 + \varepsilon v$ so that εv is the perturbation from the Newtonian trajectory. We wish to estimate this perturbation. Thus we plug $\bar{u}_0 + \varepsilon v$ into Einstein's ODE (4.8) and collect like powers of ε. If ε is small, and the trajectory ranges over angles that are not too great, we can ignore all terms with powers of ε smaller than or equal to ε^2. The term corresponding to the first power of ε will provide an equation whose solution is a good estimate for the perturbation of the solution u from the underlying orbit u_0 of Newton's theory. Plugging in we obtain:

$$\bar{u}_0'' + \bar{u}_0 - 1 + \varepsilon\left[v'' + v - u_0^2\right] + (\text{higher order terms in } \varepsilon) = 0. \tag{4.11}$$

Equation (4.11) is called an *asymptotic expansion* of (4.8). Now $\bar{u}_0'' + \bar{u}_0 - 1 = 0$ because this is Newton's equation, and $\bar{u}_0$ was assumed at the outset to be some given solution of this equation. Neglecting the higher order terms in ε, we obtain an ODE that approximately describes the function v when ε is small:

$$v'' + v - \bar{u}_0^2 = 0. \tag{4.12}$$

Thus for $\bar{u}_0$ known, the ODE (4.12) is a linear, constant coefficient, inhomogeneous equation in v, and we can solve it directly. We conclude that, given a Newtonian trajectory $\bar{u}_0$, we can approximate the Einsteinian perturbation εv from this orbit by solving (4.12). Plugging (4.10) into (4.12) yields the ODE

$$v'' + v = 1 + \frac{d^2}{2} + 2d\cos\theta + \frac{d^2}{2}\cos 2\theta, \tag{4.13}$$

where we have applied the trigonometric identity $\cos^2\theta = (1 + \cos 2\theta)/2$. Now (4.13) is an inhomogeneous linear ODE with constant coefficients, and $v_0 = d'\cos(\theta + K')$ solves the underlying homogeneous equation for arbitrary constants d' and K'. To obtain a particular solution of the inhomogeneous problem, we can apply superposition and write $v = v_1 + v_2 + v_3$ where v_i solve the separate equations:

$$v_1'' + v_1 = 1 + \frac{d^2}{2},$$

$$v_2'' + v_2 = 2d\cos\theta,$$

$$v_3'' + v_3 = \frac{d^2}{2}\cos 2\theta.$$

One can easily verify that the three solutions are

$$v_1 = 1 + \frac{d^2}{2}, \qquad v_2 = d\theta\sin\theta, \qquad v_3 = -\frac{d^2}{6}\cos 2\theta.$$

Thus the general solution of (4.8) is $v = v_0 + v_1 + v_2 + v_3$. Now in order that $\bar{u}_0$

and $\bar{u}$ satisfy the same initial conditions (4.7) and (4.9), $v(\theta)$ must satisfy

$$v(\theta) = 0, \quad v'(\theta) = 0,$$

and thus it is easy to calculate that

$$K' = 0,$$

and

$$d' = -1 - d^2/3.$$

Our approximation for $\bar{u}$ can now be written down:

$$\bar{u} = \bar{u}_0 + \varepsilon v \approx 1 + d\cos\theta + \varepsilon d' \cos(\theta) + \varepsilon + \varepsilon\frac{d^2}{2} + \varepsilon d\theta \sin\theta - \varepsilon\frac{d^2}{6}\cos 2\theta. \tag{4.14}$$

Now (4.14) is a messy formula, but we are only interested in the perihelion shift (the rotation in the angle at which the maximum value of either u or $\bar{u}$ is taken on in successive orbits) for the cyclical trajectory of u. But it is only the nonperiodic terms in (4.14) that can contribute to such a shift, and the only nonperiodic term in (4.14) is the term $\varepsilon d\theta \sin(\theta)$. To see the effect of this term on successive perihelia, rewrite (4.9) as,

$$\bar{u} = 1 + d(\cos\theta + \varepsilon\theta\sin\theta) + \text{periodic terms of order } \varepsilon. \tag{4.15}$$

The periodic terms can change the angle at which the perihelia are taken on, but being periodic, they cannot significantly affect the shift in the position of the perihelia that occur in successive revolutions. To see this more clearly, we can write

$$\cos\theta + \varepsilon\theta\sin\theta \approx \cos\theta\cos(\varepsilon\theta) + \sin\theta\sin(\varepsilon\theta) = \cos(\theta - \varepsilon\theta) \tag{4.16}$$

because, since we are neglecting higher order terms in ε, Taylor's theorem implies that $\cos(\varepsilon\theta) \approx 1$ and $\sin(\varepsilon\theta) \approx \varepsilon\theta$. Using this in (4.15) we obtain

$$\bar{u} = 1 + d\cos(\theta - \varepsilon\theta) + \text{periodic terms of order } \varepsilon. \tag{4.17}$$

We now claim that the shift in the perihelion during one cycle is affected by the periodic terms of order ε only in an amount that is second order in ε; and so neglecting higher order terms, the shift observed in (4.17) after one revolution will be the same as the shift observed in the function $\bar{u} = 1 + d\cos(\theta - \varepsilon\theta)$ after one revolution. We postpone the proof of this claim until the end of this section, where we show that the claim is a special case of a general principle. The function $1 + d\cos(\theta - \varepsilon\theta)$ takes on successive maxima when $\theta = 2n\pi/(1-\varepsilon) \approx 2n\pi(1+\varepsilon)$. Therefore the shift in the angle at which the perihelia occur after one revolution (ignoring terms quadratic in ε) is estimated as

$$\Delta\theta = 2\pi\varepsilon = 6\pi\frac{G^2}{H^2c^2}, \tag{4.18}$$

since $\varepsilon = 3G^2/H^2c^2$. To apply this formula to the Mercury-Sun system, we must have numerical values for G, c, and H, the latter applying to Mercury's orbit. From **[3]** we obtain current experimental values for G and c:

$$G = 1.32712497 \times 10^{26}\ \text{cm}^3/\text{sec}^2,$$

$$c = 2.99792458 \times 10^{10}\ \text{cm/sec}.$$

The quantity H is somewhat more difficult to find because it is difficult to measure directly by astronomical observations. In constrast, the lengths of the major and

minor axes of the almost elliptical orbit of Mercury are readily observable since these are obtained from measurements of the closest and farthest distances that the planet comes to the sun. We claim that

$$G/H^2 \approx \frac{1}{L}, \tag{4.19}$$

where $L = a(1 - e)$, a is the length of the semi-major axis (the average of the major and minor axes of the ellipse) and e is the eccentricity for the elliptical orbit of Mercury (cf. [7]). We leave the verification of this claim until the end. Assuming (4.19), (4.18) becomes

$$\Delta\theta \approx 6\pi \frac{G}{c^2 L}. \tag{4.20}$$

From [2] we find that $a = 5.7909 \times 10^{12}$ cm and $e = 0.205628$, which implies that $L = 5.5460 \times 10^{12}$ cm. Thus equation (4.20) gives

$$\begin{aligned}\Delta\theta &= 5.0187 \times 10^{-7} \text{ radians per revolution}\\ &= 2.8755 \times 10^{-5} \text{ degrees per revolution}\\ &= 0.103518 \text{ seconds of arc per revolution.}\end{aligned}$$

Since there are 415.2 revolutions in a century, we obtain that the precession in the perihelion of the orbit of the planet Mercury in one century is predicted by Einstein's theory to be approximately $415.2\Delta\theta$, which is approximately 42.98 seconds of an arc per century.

All that remains is to verify (4.19), and to prove our claim that the periodic terms of order ε contribute order ε^2 in the perihelion shift. For (4.19), note that in Newtonian theory the orbit of Mercury is an ellipse with major and minor axes given by

$$r_{\pm} = (1 \pm e)a.$$

At $u = u_{\pm} = 1/r_{\pm}$, $u' = 0$ since $u_{\pm}$ are critical points of $u = u(\theta)$. Thus evaluating the energy integral at these values gives two equations:

$$E = \frac{1}{2}u_{\pm}^2 - \frac{G}{H^2}u_{\pm} - \frac{G}{c^2}u_{\pm}^3.$$

Subtracting these two equations and canceling a common factor of $(u_{+} - u_{-})$ we find

$$\frac{G}{H^2} = \frac{1}{2}(u_{+} - u_{-}) - \frac{G}{c^2}\left(u_{+}^2 + u_{+}u_{-} + u_{-}^2\right).$$

In terms of a and e,

$$u_{+} + u_{-} = \frac{2}{L},$$

$$u_{+}^2 + u_{+}u_{-} + u_{+}^2 = \frac{3 + e}{L^2},$$

so that

$$\frac{G}{H^2} = \frac{1}{L}\left(1 - \frac{G}{c^2 L}(3 + e^2)\right). \tag{4.21}$$

For Mercury, $G/c^2 L = 2.7 \times 10^{-8}$, so we may neglect this term in (4.21) to obtain equation (4.20).

Finally, our claim that the periodic terms of order ε in (4.17) contribute errors in the perihelion shift of order ε^2 follows directly from the following:

Theorem. *Let F and f be smooth, real valued, 2π-periodic functions of θ and set*

$$G(\theta) = F(\theta - \varepsilon\theta) + \varepsilon f(\theta).$$

Assume that $|\varepsilon| \ll 1$, that θ_0 satisfies $G(\theta_0) = 0$ and that $F'(\theta_0 - \varepsilon\theta_0) = a \neq 0$. Then

$$G(\theta_0 + 2\pi + \Delta\theta) = 0,$$

where

$$\Delta\theta = 2\pi\varepsilon + \textit{terms of order } \varepsilon^2.$$

Our claim follows when we let $\theta_0 = 0$, $F(\theta) = \cos\theta$, and $f(\theta) =$ the periodic terms of order ε.

Proof: Let $\theta = \theta_0 + 2\pi + \Delta\theta$ for $|\Delta\theta| \ll 1$, and set $f'(\theta_0) = b$, then

$$F(\theta - \varepsilon\theta) = F(\theta_0 + 2\pi + \Delta\theta - \varepsilon(\theta_0 + 2\pi + \Delta\theta)).$$

Using Taylor's theorem to expand $F(\theta - \varepsilon\theta)$ about the point $\theta_0 - \varepsilon\theta_0 + 2\pi$, we obtain

$$F(\theta - \varepsilon\theta) = F(\theta_0 - \varepsilon\theta_0 + 2\pi) + a(\Delta\theta - 2\pi\varepsilon) + \text{Error}_1$$

where $|\text{Error}_1| \leq \text{const}(|\Delta\theta| + |\varepsilon|)^2$. Similarly,

$$\begin{aligned} \varepsilon f(\theta) &= \varepsilon f(\theta_0 + 2\pi) + \varepsilon b\Delta\theta + \cdots \\ &= \varepsilon f(\theta_0 + 2\pi) + \text{Error}_2 \end{aligned}$$

where $|\text{Error}_2| \leq \text{const}(|\Delta\theta| + |\varepsilon|)^2$. Thus,

$$\begin{aligned} G(\theta) &= F(\theta - \varepsilon\theta) + \varepsilon f(\theta) \\ &= F(\theta_0 - \varepsilon\theta_0 + 2\pi) + \varepsilon f(\theta_0 + 2\pi) + a(\Delta\theta - 2\pi\varepsilon) + \text{Error}_3 \\ &= a(\Delta\theta - 2\pi\varepsilon) + \text{Error}_3 \end{aligned}$$

where $|\text{Error}_3| \leq \text{const}(|\Delta\theta| + |\varepsilon|)^2$ and we have used the fact that G is 2π-periodic and $G(\theta_0) = 0$. Therefore, $G(\theta)$ will be zero when

$$\Delta\theta = 2\pi\varepsilon + \text{Error}_3.$$

But this implies $|\Delta\theta| \leq \text{const}\,\varepsilon$, so $|\text{Error}_3| \leq \text{const}\,\varepsilon^2$, and so we conclude that

$$\Delta\theta = 2\pi\varepsilon + \text{terms of order } \varepsilon^2,$$

which verifies the claim.

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Billiards and Rational Periodic Directions in Polygons

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1. INTRODUCTION. Let $\Gamma \subset \mathbb{R}^2 = \mathbb{C}$ be a plane, non-selfintersecting polygon. The billiard trajectory on Γ is completely determined by the initial data (x, θ) which includes a point $x \in \Gamma$ and a direction specified by the choice of a point $\theta \in S^1 = \{z \in \mathbb{C} | \, |z| = 1\}$ on a unit circle. The point (particle) moves inside Γ along a straight line in the direction θ, with the unit speed, until it reaches the boundary $\partial\Gamma$. Then the direction of motion changes instantaneously according to the rule "the angle of incidence is equal to the angle of reflection", and the point resumes rectilinear motion inside Γ, in the new direction, until the next collision with $\partial\Gamma$, and so on. Thus, the phase space of this billiard dynamical system can be identified with $\Gamma \times S^1$, with the obvious identifications on the boundary $\partial\Gamma \times S^1$ imposed by the reflection rule. A billiard trajectory is called regular if it continues indefinitely without hitting a vertex of Γ. Otherwise the trajectory is called singular, and it terminates upon hitting a vertex. (For a formal definition of a billiard dynamical system see e.g. [**BKM**], [**ZK**] or [**CFS**, Chapter 6].)

A direction $\theta = \exp(2\pi i\phi) \in S^1$ on a billiard table Γ is called rational if $\phi \in \mathbb{R}/\mathbb{Z}$ is rational. A direction θ is called periodic if there is a periodic orbit in this direction (that is, the orbit of (x, θ) is periodic for some $x \in \Gamma$). By a generalized diagonal we mean a billiard trajectory which connects two vertices of Γ. (By definition, the sides of Γ are considered to be generalized diagonals.) A direction θ is said to be exceptional if there exists a generalized diagonal (with one of its segments lying) in this direction. Thus, θ is exceptional if, for some $x \in \Gamma$, the forward trajectories of both (x, θ) and $(x, -\theta)$ are singular (hit a vertex of Γ). Denote by $E(\Gamma), P(\Gamma), Q$ the sets of exceptional, periodic and rational directions, respectively. The three sets are subsets of S^1. Sometimes (when there is no doubt as to what table Γ is considered), the dependence on Γ is suppressed, and $E(\Gamma)$, $P(\Gamma)$ are abbreviated to just E, P.

Note that, for an arbitrary polygon, E is always a countable dense subset of S^1, and $P \subset E$. One way to see it is to view a regular billiard trajectory x_t as an infinite ray on a plane by reflecting the original polygon Γ around each of its sides in the succession they are hit by x_t. (We refer to [**BKM**], or [**CFS**, Chapter 6, §3, Lemma 2] for a more detailed description of this approach and for the proofs of the above stated facts.)

Thus both E and Q are dense subsets of S^1. How much do they have in common? Theorem 1.1 below answers this question. (It turns out that there is only a finite number of exceptional rational directions.) For rational triangles, we describe explicitly (Corollary 1.7) a finite set F containing $E \cap Q$. Some partial results and computer computations indicate that for many–but not all–triangles the equality $E = F$ in fact holds (§2).

Theorem 1.1. *For any billiard table (polygon) Γ, the set of rational exceptional directions $E \cap Q$ is finite.*

The proof is in §3. In the case when Γ is a rational triangle, more can be said on the size of the set $E \cap Q$ (Theorem 1.6 and Corollary 1.7 below). A periodic direction $\theta \in P(\Gamma)$ is said to be purely periodic (notation: $\theta \in P_0(\Gamma)$) if every regular (i.e., avoiding the vertices) orbit in this direction is periodic. Thus, for an arbitrary polygon, E is a countable dense set of S^1, and $P_0 \subset P \subset E$.

We point out that the cardinality of $E \cap Q$, and even of $P_0 \cap Q$, though always finite, can be arbitrarily large as one can see from the following proposition.

Proposition 1.2. *Assume that Γ_n, $n \geq 3$, is either a regular n-gon G_n, or an isosceles triangle T_n with the angles $(2/n)\pi$, $(n-2/2n)\pi$ and $(n-2/2n)\pi$, and assume that one of the sides of T_n is either horizontal or vertical. Then*

$$E(\Gamma_n) \cap Q = P_0(\Gamma_n) \cap Q = S^1(n'),$$

where

$$S^1(n) = \{z \in S^1 \subset \mathbb{C} | z^n = 1\}, \qquad n \geq 1, \tag{1.3}$$

and n' is the least common multiple of 2 and n.

The (easy) proof is in the end of §5. Note that H. Masur has recently proved the following.

Theorem 1.4 [M]. *For any rational billiard table Γ, $P(\Gamma)$ (the set of periodic directions) is dense in S^1.*

A polygon Γ is said to be rational if all its angles are rational multiples of π. Otherwise Γ is called irrational.

In the case when the billiard table Γ is a rational triangle, one can be more specific regarding the set $E(\Gamma) \cap Q$ (see Theorem 1.6 and Corollary 1.7 below, cf. Theorem 1.1).

Notation 1.5. In what follows, $\Delta = (\alpha, \beta, \chi)$ denotes a rational triangle with the angles $\alpha\pi$, $\beta\pi$ and $\chi\pi$. (Thus α, β and χ are positive rationals whose sum is 1.) Clearly, α, β and χ define the triangle $\Delta = (\alpha, \beta, \chi)$ completely, up to scaling. By $d = d(\alpha, \beta, \chi) = d(\Delta)$ we denote the least common denominator of the rationals α, β and χ. Define $d' = d'(\alpha, \beta, \chi) = d'(\Delta)$ to be the least common multiple of 2 and d: thus $d' = d$ if d is even, and $d' = 2d$ if d is odd.

Theorem 1.6. *Let $\Delta = (\alpha, \beta, \chi) \subset \mathbb{R}^2$ be a rational triangle. If the angle ϕ between two exceptional directions is a rational multiple of π, then ϕ must be a multiple of $\pi/d'(\Delta)$. In particular, the cardinality of the set $E(\Delta) \cap Q$ is at most $2d'(\Delta)$.*

The proof is in §5. Since the directions parallel to the sides of Δ lie in $E(\Delta)$ (by definition), the following follows.

Corollary 1.7. *Let $\Delta = (\alpha, \beta, \chi) \subset \mathbb{R}^2$ be a rational triangle such that one of its sides v is either horizontal or vertical (or, more generally, the angle formed by v with the horizontal direction is a multiple of $\pi/d'(\Delta)$). Then $E(\Delta) \cap Q \subset S^1(2d'(\Delta))$ (see (1.2) for notation).*

We point out that for some triangles the equality $E \cap Q = P_0 \cap Q = S^1(2d')$, in fact, holds (see Proposition 1.2 and Conjectures 2.2(a), (c) and (e)).

Remark. Note that the problem of existence of a periodic orbit in an arbitrary (irrational) polygon—even in an arbitrary right triangle—is open. (The author believes that all triangles have periodic orbits, and that, however, the claim fails for some polygons). On the other hand, the existence of periodic orbits in a rational polygon is easily seen from Proposition 1.8 below. An orbit in a polygon is called symmetric if it hits perpendicularly λ where λ is either a side of Γ, or an axis of symmetry of Γ. Note that the set of singular (hitting a vertex) symmetric orbits is at most countable, so that most symmetric orbits are regular (avoid vertices).

Proposition 1.8. *Let Γ be a rational polygon. Then every regular symmetric orbit must be periodic.*

Proposition 1.8 is easily proved by considering the interval exchange transformation induced on $(\lambda \cup \partial\Gamma) \times F$ (where F is a finite set of directions which lie in the orbit of the direction which is perpendicular to λ.) One uses the fact that when an interval exchange transformation is restricted to a subinterval, the resulting induced transformation is itself an interval exchange map (see e.g. **[K]** or **[CFS**, Chapter 5, §3, Lemma 2]). The fact allows us to conclude that if a regular orbit hits λ perpendicularly once, it hits λ perpendicularly once again, and therefore must be periodic. (See e.g. **[BKM]** or **[CFS**, Chapter 6, §2, p. 148] for the description of the way rational billiards induce interval exchange transformations).

Note that Theorem 1.4 is much deeper than Proposition 1.8, and its proof relies heavily on Teichmüller theory.

2. A CONJECTURE AND SOME RESULTS OF COMPUTER COMPUTATIONS. By Corollary 1.7, the set of rational exceptional directions in a rational triangle is contained in the set $S^1(2d')$: $E \cap Q \subset S^1(2d')$. On the other hand, some results and computer computations indicate that, in fact, for many (but not all) triangles, the equality

$$S^1(2d') = P_0 \cap Q = P \cap Q = E \cap Q \tag{2.1}$$

holds (see Proposition 1.2 and Conjectures 2.2 (a), (c) and (e)).

Recall that for an arbitrary polygon Γ, the inclusions $P_0 \subset P \subset E$ takes place where P_0, P, E stand for the sets of periodic, purely periodic and exceptional directions, respectively.

Conjecture 2.2. *Assume that a rational triangle $\Delta = (\alpha, \beta, \chi)$ satisfies the conditions of Corollary* 1.7. *Then*:

(a) *If d is even (equivalently, $d' = d$), we have $S^1(2d') \subset P_0$. (This would imply that* (2.1) *holds, see Corollary* 1.7.)

(b) *If d is odd (equivalently, $d' = 2d$), we have $S^1(2d') \setminus S^1(d') \subset P_0 \cap Q$.*

(c) *If $d \leq 12$, and if $\Delta \neq (2/11, 3/11, 6/11)$, we have $S^1(2d') \subset P_0$. (This would imply that* (2.1) *holds, see Corollary* 1.7).

(d) *If $d \leq 12$, and if $\Delta \neq (2/11, 3/11, 6/11)$ or $(4/11, 4/11, 3/11)$, we have $P_0 = E$.*

(e) *If Δ is either isosceles, or right triangle, we have $S^1(2d') \subset P_0$ (and hence* (2.1) *holds in view of Corollary* 1.7).

Note that (a) $\Rightarrow$ (e). Both (a), (b) have been verified* for all rational triangles Δ with $d = d(\Delta) \leq 16$ (and for many others), while (c) has been confirmed* for all $d \leq 12$. The conjecture (d) has been tested for all triangles Δ, with $d \leq 12$, in many directions in E (not only in $E \cap Q$).

There is a good evidence, for the triangle $\Delta = (2/11, 3/11, 6/11)$, with the largest side being horizontal, that the rational direction $\theta = \exp((5/11)\pi i)$ lies in $P \setminus P_0$, that is, is periodic, but not purely periodic (cf. Conjecture 2.2, (d)). (In this direction there are periodic orbits involving 34 and 286 reflections, but there are also orbits which do not close even after $2 \cdot 10^8$ reflections.)

On the other hand, the isosceles triangle $(4/11, 4/11, 3/11)$ seems to have an irrational direction in $P \setminus P_0$ (containing a periodic trajectory involving 18 reflections and a trajectory which does not close even after $2 \cdot 10^8$ reflections; cf. Conjecture 2.2, (e)).

W. Veech has recently proved [V] that for the triangles

$$\Delta_n = \left(\frac{1}{n}, \frac{1}{n}, \frac{n-2}{n}\right), \qquad n \geq 3, \tag{2.3}$$

(and some other closely related triangles and polygons, in particular, regular n-gons) we have $P_0 = E$. For $n = 3$, 4 and 6 the result easily follows from the fact that the triangle Δ_n and its reflections pack the plane; however, for all other n, even $n = 5$, the result does not seem to be simple. Note that the claim of Conjecture 2.2 (e) certainly holds for the triangles T_n and Δ_n (Proposition 1.2).

It would be interesting to find a general procedure to decide, for a given rational triangle Δ, whether or not $P_0 = E$. The simplest triangle for which the question is open is $\Delta = (2/7, 2/7, 3/7)$ (although, for this triangle, computer computations support the equality $P_0 = E$).

Some other computer computations suggest that the approach in [V] is not suitable for proving that $P_0 = E$ for $\Delta = (2/7, 2/7, 3/7)$. (This triangle does not seem to lead to a lattice in the sense of [V] because it failed the test described in the concluding remark in [V].)

3. PROOF OF THEOREM 1.1. Let $\mathbb{C} \supset \mathbb{R} \supset \mathbb{Q}$ denote the fields of complex, real and rational numbers, respectively. For $z \in \mathbb{C}$, denote $|z| = (z \cdot \bar{z})^{1/2} \geq 0$ and, if $z \neq 0$, $\mathrm{Dir}(z) = z/|z| \in S^1$ (where S^1 denotes the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$). For any field $\Phi \subset \mathbb{C}$, the set

$$\mathrm{Dir}(\Phi) = \{\mathrm{Dir}(z) | z \in \Phi\}$$

will be called the set of directions in Φ. $\mathrm{Dir}(\Phi)$ is clearly a subgroup of S^1.

For any polygon $\Gamma \subset \mathbb{R}^2 = \mathbb{C}$, with n sides v_k, $1 \leq k \leq n$, a field $\Phi = \Phi(\Gamma) \subset \mathbb{C}$ is associated in the following way. Each side v_k of Γ is identified with the complex number (vector in $\mathbb{R}^2$) corresponding to this side and defined up to the sign. $\Phi(\Gamma)$ is defined as the field generated by the numbers v_k and $(\mathrm{Dir}(v_k))^2 = v_k/\overline{v_k} \in S^1$:

$$\Phi(\Gamma) = \mathbb{Q}\left(\left\{v_k, (\mathrm{Dir}(v_k))^2\right\}_{1 \leq k \leq n}\right) = \mathbb{Q}(\{v_k, \bar{v}_k\}_{1 \leq k \leq n}). \tag{3.1}$$

Note that (3.1) implies that $\Phi(\Gamma)$ is a field closed under the complex conjugations: $z \to \bar{z}$. Without loss of generality, we shall always assume that $0 \in \mathbb{C}$ is one of the vertices of Γ. Then all the vertices of Γ lie in $\Phi(\Gamma)$.

*Supported by computer computations (which have been carried out with more than 30 decimal digits accuracy). No symbolic computations have been used. The conclusions derived can be considered as quite reliable (though not certain).

There is a standard procedure for straightening a billiard trajectory in an arbitrary polygon (see e.g. **[BKM]** or **[Gu]**). In order to view a billiard trajectory x_t on Γ as a straight line in $\mathbb{R}^2 = \mathbb{C}$, one reflects Γ in succession around each of the sides of Γ, in the order they are hit by x_t.

We claim that, for every side v' of every reflected copy of Γ, we have both v' and $(\mathrm{Dir}(v'))^2$ lie in $\Phi(\Gamma)$. Indeed, let Γ' be the polygon obtained as a result of reflection of Γ relative to the side w of Γ. Then every side v' of Γ' is an image of some side v of Γ, and we obtain

$$v' = v(\mathrm{Dir}(w))^2 (\mathrm{Dir}(v))^{-2} \in \Phi(\Gamma)$$

whence

$$(\mathrm{Dir}(v'))^2 \in \Phi(\Gamma).$$

The induction on the number of reflections completes the proof of the claim in the preceding paragraph. The claim implies (under the assumption that $0 \in \mathbb{C}$ is one of the vertices of Γ) that the vertices of all reflected copies of Γ—as well as the vertices of Γ itself—lie in $\Phi(\Gamma)$. This allows us to conclude the following (for notation see the paragraphs which precede and follow Theorem 1.1).

Proposition 3.2. *For every billiard table (polygon) Γ,*

$$P_0 \subset P \subset E \subset \mathrm{Dir}(\Phi(\Gamma)).$$

Recall that Q stands for the set $\bigcup_{k \geq 1} S^1(k)$ of all rational directions (see (1.3)). In the next section we shall prove the following proposition.

Proposition 3.3. *For every finitely generated (over $\mathbb{Q}$) field $\Phi \subset \mathbb{C}$, we have* $\mathrm{Dir}(\Phi) \cap Q = S^1(k)$ *for some (even) integer k. In particular,* $\mathrm{Dir}(\Phi) \cap Q$ *must be finite.*

Theorem 1.1 now follows easily from the above proposition.

Proof of Theorem 1.1. In view of Proposition 3.2, $E(\Gamma) \cap Q$ is a subset of a set $\mathrm{Dir}(\Phi(\Gamma)) \cap Q$ which is finite by Proposition 3.3 since $\Phi(\Gamma)$ is finitely generated. □

4. PROOF OF PROPOSITION 3.3. For integers $q \geq 1$, denote

$$\mathbb{C}_q = \mathbb{Q}(S^1(q)) = \mathbb{Q}\left(\exp\left(\frac{2}{q}\pi i\right)\right) \tag{4.1}$$

to be the splitting field (over $\mathbb{Q}$) of $z^q = 1$ (see (1.3)). In the first three lemmas we recall some well known facts in algebra and number theory.

Lemma 4.2. *For any integer $q \geq 1$, $[\mathbb{C}_q : \mathbb{Q}] = \phi(q)$ where $\phi(q)$ denoted the number of positive integers $\leq q$ which are relatively prime with q. ($[\mathbb{C}_q : \mathbb{Q}]$ stands for the degree of $\mathbb{C}_q$ over $\mathbb{Q}$.)*

Lemma 4.3. *Let $q = \prod_{k=1}^n p_k^{\alpha_k}$ be a prime factorization of an integer $q \geq 2$. Then*

$$\phi(q) = \prod_{k=1}^{n} (p_k - 1) p_k^{\alpha_k - 1}.$$

Lemma 4.4. $\mathrm{Lim}_{q \to \infty} \phi(q) = \infty$.

Lemma 4.5. *Let $\Phi \subset \mathbb{C}$ be a field closed under the conjugation $(z \to \bar{z})$ and not contained in $\mathbb{R}$ (i.e., $\Phi \setminus \mathbb{R} \neq \varnothing$). Let $\theta \in S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Then the following two conditions are equivalent*:

(1) $\theta \in \mathrm{Dir}(\Phi)$ (*see* §3 *for notation*);

(2) $\theta^2 \in \Phi$.

Proof: (1) $\Rightarrow$ (2). $\theta \in \mathrm{Dir}(\Phi)$ means that there is $r \in \mathbb{R}$, $r \neq 0$, such that $r\theta \in \Phi$. Since Φ is closed under conjugation, we have $r\bar{\theta} = r\theta^{-1} \in \Phi$. Thus the quotient $\theta^2 \in \Phi$.

(2) $\Rightarrow$ (1). First assume that $\theta^2 \neq -1$. Then $\theta + \theta^{-1} = r \in \mathbb{R}$, $r \neq 0$. Thus $r\theta = \theta^2 + 1 \in \Phi$, and therefore $\theta \in \mathrm{Dir}(\Phi)$.

In the remaining case $\theta^2 = -1$, we take any $z \in \Phi \setminus \mathbb{R}$ and consider $w = z - \bar{z}$ (where $\bar{z}$ denotes the complex conjugate of z). Then $\theta = \pm\mathrm{Dir}(w) \in \mathrm{Dir}(\Phi)$. □

Lemma 4.6. *Let $\Phi \subset \mathbb{C}$ be a finitely generated subfield, $\Phi = \mathbb{Q}(\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_k \in \mathbb{C}$. Then Φ is algebraically finite.*

A field $\Phi \subset \mathbb{C}$ is said to be algebraically finite if $\Phi \cap \mathbb{A}$ is a finite extension of $\mathbb{Q}$: $[(\Phi \cap \mathbb{A}):\mathbb{Q}] < \infty$. $\mathbb{A}$ denotes the field of algebraic numbers (in $\mathbb{C}$).

The proof of Lemma 4.6 is obtained by induction on n (the number of generators of Φ). The inductional step reduces to the following:

Sublemma. *Let $\Phi \subset \mathbb{C}$ be an algebraically finite field. Then, for any $c \in \mathbb{C}$, the field $\Phi_1 = \Phi(c)$ is also algebraically finite.*

Proof of Sublemma. If c is transcendental over Φ, then $\Phi \cap \mathbb{A} = \Phi_1 \cap \mathbb{A}$. Thus we may assume that c is algebraic over Φ: $[\Phi_1:\Phi] = n < \infty$. Let $[(\Phi \cap \mathbb{A}):\mathbb{Q}] = m$. Let $\alpha \in \Phi_1 \cap \mathbb{A}$ be arbitrary. Denote $p_1(x) \in \mathbb{Q}[x]$, $p_2(x) \in \Phi[x]$ to be the minimal monic polynomials for α, over $\mathbb{Q}$ and Φ respectively. Then $p_2(x)$ divides $p_1(x) \in \mathbb{Q}[x]$ and hence $p_2(x) \in (\Phi \cap \mathbb{A})[x]$. It follows that

$$[(\Phi \cap \mathbb{A})(\alpha):(\Phi \cap \mathbb{A})] = \deg(p_2(x)) = [\Phi(\alpha):\Phi] \leq [\Phi_1:\Phi] = n$$

and hence $[(\Phi \cap \mathbb{A})(\alpha):\mathbb{Q}] \leq m \cdot n$. Since $\alpha \in \Phi_1 \cap \mathbb{A}$ is arbitrary, it follows that $[(\Phi_1 \cap \mathbb{A}):\mathbb{Q}] \leq m \cdot n$, that is, $\Phi_1 \cap \mathbb{A}$ is algebraically finite. □

Now we are in the position to prove Proposition 3.3.

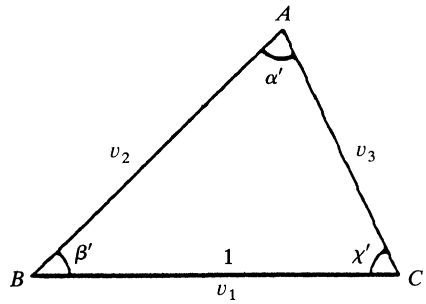
Proof of Proposition 3.3. Denote $\Phi' = \mathrm{Dir}(\Phi) \cap \mathbb{Q}$. Let $\theta \in \Phi'$, then $\theta = \exp[(2p/q)\pi i]$ for some relatively prime integers p, q. Then, by Lemma 4.5, $\theta^2 \in \Phi$. Since Φ is finitely generated, we have $[(\Phi \cap \mathbb{A}):\mathbb{Q}] = m < \infty$ (by Lemma 4.6). Therefore $[\mathbb{Q}(\theta^2):\mathbb{Q}] \leq m$. Let q' denote q if q is odd and $q/2$ if q is even. Then $\mathbb{Q}(\theta^2) = \mathbb{C}_{q'}$.

Therefore (Lemma 4.2) $\phi(q') = [\mathbb{C}_{q'}:\mathbb{Q}] \leq m$, and, by Lemma 4.4, all q', and hence q, must have a uniform bound (depending only on m), for all choices of $\theta = \exp[(2p/q)\pi i] \in \Phi'$. Therefore Φ' is finite. Since Φ' is a finite subgroup of the unit circle S^1 closed under the π-rotation: $\theta \to -\theta$, the claim of the proposition follows. □

5. PROOFS OF COROLLARY 1.7 AND PROPOSITION 1.2.

Proposition 5.1. *Let $\Delta = (\alpha, \beta, \chi) \in \mathbb{R}^2$ be a rational triangle. Assume that one of its sides has length 1 and is horizontal. Then $\Phi(\Delta) \subset \mathbb{C}_d = \mathbb{C}_{d'}$ where $d = d(\Delta)$ and $d' = d'(\Delta)$ (Notations 1.5 and 4.1)).*

Proof: Recall (see (3.1)) that $\Phi(\Delta) = \mathbb{Q}(\{v_k, (\mathrm{Dir}\, v_k))^2\}_{1 \le k \le 3})$ where $v_k \in \mathbb{C}$ correspond to the sides of the triangle $\Delta = ABC$: $v_1 = BC = 1$, $v_2 = CA$ and $v_3 = BA$. Denote $\alpha' = \pi\alpha = \angle A$, $\beta' = \pi\beta = \angle B$ and $\chi' = \pi\chi = \angle C$.



Since $\mathrm{Dir}(v_k)$ are powers of $\exp(\pi i/d)$, it follows that $(\mathrm{Dir}(v_k))^2$ are powers of $\tau = \exp(2\pi i/d)$, and hence belong to the field $\mathbb{C}_d = \mathbb{Q}(\tau)$.

It remains to show that $v_3 \in \mathbb{C}_d$ (because $v_1 = 1$ and $v_2 = v_3 - 1$). It is easy to verify that v_3 is equal to

$$v_3 = \frac{\sin(\chi') \cdot \exp(i\beta')}{\sin(\alpha')} = \frac{L_1}{L_2},$$

where

$$L_1 = i \sin(\chi') \cdot \exp(-i\chi')$$

and

$$L_2 = i \cdot \sin(\alpha') \cdot \exp(-i\alpha').$$

Each angle δ in the triangle $\Delta = ABC$ is a multiple of π/d. Therefore the numbers $\exp(\pm 2\delta i)$, $\cos(2\delta)$, $i\sin(2\delta)$ and hence

$$2i\sin(\delta) \cdot \exp(\pm\delta i) = i\sin(2\delta) \pm (\cos(2\delta) - 1)$$

lie in the field $\mathbb{C}_d = \mathbb{Q}(\tau)$, $\tau = \exp(2\pi i/d)$. We conclude that both $L_1, L_2 \in \mathbb{C}_d$, whence $v_3 = L_1/L_2 \in \mathbb{C}_d$, and the proof is complete. □

Recall that Q stands for the set of rational directions (§1).

Proposition 5.2. *For even integers $q \geqslant 2$, we have $\mathbb{C}_q \cap Q = S^1(q)$ (see* (1.3) *and* (5.1)).

The proof of Proposition 5.2 will be given in the end of the section. Now we are in position to prove Theorem 1.6.

Proof of Theorem 1.6. Without loss of generality, we may assume that the rational triangle Δ satisfies the conditions of Proposition 5.1. Thus $\Phi(\Delta) \subset \mathbb{C}_{d'}$ where $d' = d'(\Delta)$ (Notation 1.5). Let $\theta_1, \theta_2 \in S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ be two exceptional directions in Δ such that the angle ϕ between them is a rational multiple of π.

By Proposition 3.2, $\theta_1, \theta_2 \in \mathrm{Dir}(\Phi(\Gamma))$. Therefore the ratio $\theta = \theta_1/\theta_2 = \exp(i\phi)$ lies in $\mathrm{Dir}(\Phi(\Gamma)) \cap Q$. Applying in succession Lemma 4.5, Proposition 5.1 and Proposition 5.2, we get $\theta^2 \in \Phi(\Gamma) \cap Q \subset \mathbb{C}_{d'} \cap Q) = S^1(d')$. (Note that d' is always even, see Notation 1.5). This implies that $\theta = \exp(i\phi) \in S^1(2d')$. Therefore, ϕ is a rational multiple of π/d'. □

It remains to prove Proposition 5.2.

Proof of Proposition 5.2. Clearly $S^1(q) \subset \mathbb{C}_q \cap Q$. Let $\theta \in \mathbb{C}_q \cap Q$. Then, for some relatively prime integers m and n, we have $\theta = \exp((2m/n)\pi i)$. Denote p be the least common multiple of q and n. We obviously have $\mathbb{C}_q = \mathbb{Q}(S^1(q)) = \mathbb{Q}(\theta, S^1(q)) = \mathbb{Q}(S^1(p)) = \mathbb{C}_p$. By Lemma 4.2, $\phi(p) = \phi(q)$. Since q is even and p is a multiple of q, $\phi(p) = \phi(q)$ is possible only if $p = q$. Thus n divides q, and hence $\theta = \exp((2m/n)\pi i) \in S^1(q)$. □

Proof of Proposition 1.2. Since a regular n-gon can be obtained as a union of n reflected copies of T_n, it is easily seen that $P_0(G_n) \subset P_0(T_n)$ and $E(G_n) \subset E(T_n)$. Since $d'(T_n) = n'$, it follows from Corollary 1.7 that $E(T_n) \cap Q \subset S^1(n')$. On the other hand, the orbits in G_n belonging to the directions $\theta \in S^1(n')$ must be symmetric, and therefore $S^1(n') \subset P_0(G_n)$ (by Proposition 1.8). We obtain

$$S^1(n') \subset P_0(G_n) \cap Q \subset P_0(T_n) \cap Q \subset E(T_n) \cap Q \subset S^1(n')$$

and also

$$S^1(n') \subset P_0(G_n) \cap Q \subset E(T_n) \cap Q \subset E(T_n) \cap Q \subset S^1(n')$$

whence the claim of the proposition follows. □

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Some Elementary Properties of Infinite Products

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1. INTRODUCTION. Infinite products provide an important analytical tool in different branches of mathematics such as number theory, classical complex analysis, or the theory of function spaces (examples: Jacobi's triple product identity, Weierstraß products, Blaschke products, H^p spaces; see, e.g., Hardy and Wright [4], p. 282 ff., Rudin [10], chap. 15 and 17).

It is therefore desirable and also of interest in its own to have a convergence theory for products which is as elaborated as that for infinite series. In fact, one tries to relate both theories. To this end the following definition of convergence is used:

Definition. *An infinite product* $\prod_{n=1}^{\infty}(1+x_n)$ *is said to be* convergent, *if there is a number* $n_0 \in \mathbf{N}$ *such that* $\lim_{N\to\infty}\prod_{n=n_0}^{N}(1+x_n)$ *exists and is different from zero; otherwise the product is called divergent* (*see, e.g.*, **[1]**, **[5]**, **[6]**, **[9]**, **[11]**).

Thus a product $\prod(1+x_n)$ is convergent if and only if, $-\pi < \mathscr{I} \log z \leqslant \pi$ being supposed, $\sum_{n=n_0}^{\infty}\log(1+x_n)$ is convergent for some $n_0 \in \mathbf{N}$. As a result, there is a very simple correspondence between the unconditional convergence of series and that of products:

A product $\prod_{n=1}^{\infty}(1+x_n)$ converges for arbitrary arrangements of its factors if and only if the series $\sum_{n=1}^{\infty}|x_n|$ is convergent; in this case the product is called *absolutely* (or *unconditionally*) *convergent* (**[1]**, **[5]**, **[6]**, **[9]**, **[11]**).

There is no equally simple correspondence between the *conditional* convergence of the product $\prod(1+x_n)$ and that of the series $\sum x_n$, as can be seen by simple examples:

$$\left(1-\frac{1}{\sqrt{2}}\right)\left(1+\frac{1}{\sqrt{2}}\right)\left(1-\frac{1}{\sqrt{3}}\right)\left(1+\frac{1}{\sqrt{3}}\right)\cdots$$

diverges to zero ($\prod_{n=2}^{N}(1-1/n) = 1/N$), while

$$-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}-+\cdots$$

is convergent;

$$\left(1-\frac{1}{\sqrt[4]{2}}\right)\left(1+\frac{1}{\sqrt[4]{2}}\right)\left(1+\frac{1}{\sqrt{2}}\right)\left(1+\frac{1}{2}\right)$$
$$\times\left(1-\frac{1}{\sqrt[4]{3}}\right)\left(1+\frac{1}{\sqrt[4]{3}}\right)\left(1+\frac{1}{\sqrt{3}}\right)\left(1+\frac{1}{3}\right)\cdots$$

is convergent ($\prod_{n=2}^{N}(1 - 1/n^2) = (N + 1)/2N$), whereas

$$\sum x_n = -\frac{1}{\sqrt[4]{2}} + \frac{1}{\sqrt[4]{2}} + \frac{1}{\sqrt{2}} + \frac{1}{2} - \frac{1}{\sqrt[4]{3}} + \cdots$$

and even $\Sigma(x_n - (1/2)x_n^2)$ (observe $\log(1 + x_n) = x_n - (1/2)x_n^2 + O(x_n^3)$) diverges.

A useful general criterion for the conditional convergence of an infinite product was formulated by Cauchy is his famous *Analyse algébrique* [2], the first book containing a systematic treatment of infinite series ([2], p. 563):

Let $x_n > -1$ *for all n. If* $\lim_{N\to\infty}\Sigma_{n=1}^{N} x_n$ *exists then so does* $\lim_{N\to\infty}\Pi_{n=1}^{N}(1 + x_n)$; *the limit is zero if and only if* $\Sigma x_n^2 = \infty$.

Cauchy ([2], p. vii) explicitly attributes this theorem to Gaspard-Gustave Coriolis who is well-known for his work in mechanics.† I propose to name the corresponding general convergence test for complex products after Coriolis (although I suspect this test to be less impressive than the *Coriolis force* which can be watched via satellite).

Coriolis test. *If* $(z_n)_{n\in N}$ *is a sequence of complex numbers such that* Σz_n *and* $\Sigma|z_n|^2$ *are convergent then* $\Pi(1 + z_n)$ *converges.*

As can be seen from the second one of the examples mentioned above, the Coriolis conditions are not necessary for convergence. In fact, it seems impossible to find a simple *necessary* and sufficient condition for the convergence of the product $\Pi(1 + x_n)$ in terms of that of series like Σx_n: If any such convergent product is given, we can choose an arbitrary sequence (y_n) converging to zero and conclude that

$$\prod (1 + x_n)(1 + y_n)\left(1 - \frac{y_n}{1 + y_n}\right)$$

is convergent, too, while

$$\sum\left(x_n + y_n - \frac{y_n}{1 + y_n}\right) = \sum\left(x_n + \frac{y_n^2}{1 + y_n}\right)$$

may be strongly divergent.

Nevertheless, the Coriolis test can be complemented by certain converse results which, together with some noteworthy examples and an open problem, will be presented in this article.

2. REAL PRODUCTS. If a real sequence $(x_n)_{n\in N}$ fulfils the Coriolis conditions then necessarily not only $\Pi(1 + x_n)$, but also $\Pi(1 + cx_n)$ for every $c \in \mathbf{R}$ is convergent. It turns out that this or even the convergence of $\Pi(1 + c_1x_n)$, $\Pi(1 + c_2x_n)$ for any two different nonzero real numbers c_1, c_2 is *equivalent* to the Coriolis conditions.

There remains a "pathological" special case of convergence of a real product which is characterized by the following properties: $\Pi(1 + x_n)$ converges, $\Sigma x_n = \Sigma x_n^2 = \infty$; in this case the balance of factors is destroyed by any scaling of the

†Hence Hardy [3] is not completely right in calling the theorem "Cauchy's test."

deviations from unity: $\prod(1 + cx_n)$ diverges for $c \in \mathbf{R} \setminus \{0, 1\}$. Obviously, every convergent product can be transformed into such a pathological one by means of the recipe mentioned at the end of the introduction.

We now formulate these assertions as a theorem.

Theorem 1. *Let $(x_n)_{n \in N}$ be a sequence of real numbers.*

a) *If any two of the four expressions*

$$\prod_{n=1}^{\infty}(1 + x_n), \qquad \prod_{n=1}^{\infty}(1 - x_n),$$
$$\sum_{n=1}^{\infty} x_n, \quad \sum_{n=1}^{\infty} x_n^2$$

are convergent, then this holds also for the remaining two.

b) *If $\sum_{n=1}^{\infty} x_n$ is convergent and $\sum_{n=1}^{\infty} x_n^2$ is not, then $\prod_{n=1}^{\infty}(1 + x_n)$ diverges to zero.*

c) *If $\sum_{n=1}^{\infty} x_n^2$ is convergent and $\sum_{n=1}^{\infty} x_n$ is not, then $\prod_{n=1}^{N}(1 + x_n)/\exp(\sum_{n=1}^{N} x_n)$ tends to a finite limit for $N \to \infty$.*

d) *If $\prod_{n=1}^{\infty}(1 + x_n)$ is convergent and $\sum_{n=1}^{\infty} x_n^2$ is not, then $\sum_{n=1}^{\infty} x_n = \infty$.*

e) *If $\prod_{n=1}^{\infty}(1 + cx_n)$ is convergent for two different values $c \in \mathbf{R} \setminus \{0\}$, then the product is convergent for every $c \in \mathbf{R}$.*

(The premises of b), c) can be slightly weakened to "$\limsup \sum x_n < \infty$ and $\sum x_n^2 = \infty$" and "$\sum x_n^2 < \infty$", respectively, without changing the proofs.)

Proof: Ad a): If any two of the four expressions are convergent then there is an $n_0 \in \mathbf{N}$ such that $|x_n| \leqslant 1/2$ for $n \geqslant n_0$. Hence for $n_0 \leqslant n_1 \leqslant n_2$

$$\sum_{n=n_1}^{n_2} \log(1 + x_n) = \sum_{n=n_1}^{n_2}\left(x_n - \frac{\vartheta_n}{2}x_n^2\right) = \sum_{n=n_1}^{n_2} x_n - \frac{\tilde{\vartheta}}{2}\sum_{n=n_1}^{n_2} x_n^2, \tag{1}$$

where $\vartheta_n, \tilde{\vartheta} \in (\frac{4}{9}, 4)$ by Taylor's theorem.

Thus, by Cauchy's criterion for series, if any two of the three expressions $\prod(1 + x_n)$, $\sum x_n$, $\sum x_n^2$ are convergent, then so is the third one.

Since the convergence behaviour of $\sum x_n$ and $\sum x_n^2$ is not affected by changing the sign of each x_n, there only remains to be shown that the convergence of $\prod(1 + x_n)$ and $\prod(1 - x_n)$ implies that of $\sum x_n^2$.

But if $\prod(1 + x_n)$ and $\prod(1 - x_n)$ are convergent then so is $\prod(1 + x_n)(1 - x_n) = \prod(1 - x_n^2)$, and hence $\sum x_n^2$ converges (see, e.g., Titchmarsh **[11]**, p. 14).

The assertions b), c), and d) follow almost immediately from (1).

Ad e): Without loss of generality we assume that $\prod(1 + x_n)$ and $\prod(1 + c_0 x_n)$ are convergent, where $c_0 \in \mathbf{R} \setminus \{0, 1\}$. Then with $|x_n|, |c_0 x_n| < 1$ for $n \geqslant n_0$

$$\left(\prod_{n=n_0}^{\infty}(1 + x_n)\right)^{c_0} = \prod_{n=n_0}^{\infty}(1 + x_n)^{c_0},$$

and thus also

$$\prod_{n=n_0}^{\infty}\frac{(1 + x_n)^{c_0}}{1 + c_0 x_n}$$

converges. Since

$$\frac{(1+x_n)^{c_0}}{1+c_0x_n} = 1 + x_n^2 \cdot \frac{c_0(c_0-1)}{2}(1+\epsilon_n) \qquad (n \geqslant n_0)$$

with $\epsilon_n \to 0$ for $n \to \infty$, the convergence of Σx_n^2 and hence by a) the conclusion follows. ■

One may give a more elementary (but also slightly more complicated) proof of the theorem which avoids Taylor approximations of logarithms and powers and only uses the series $e^a = 1 + a + a^2/2 + \cdots$ and the Cauchy product $e^a e^b = e^{a+b}$, by means of the inequalities

$$1 + x + \tfrac{1}{3}x^2 \leqslant e^x \leqslant 1 + x + \tfrac{3}{4}x^2 \qquad (|x| \leqslant 1), \tag{2}$$

$$e^{x - x^2/(1+x)} \leqslant 1 + x \leqslant e^{x - x^2/4} \qquad (|x| < 1), \tag{3}$$

and

$$1 + \frac{\alpha x}{1 + (1-\alpha)x} \leqslant (1+x)^\alpha \leqslant 1 + \alpha x \qquad (0 \leqslant \alpha \leqslant 1, x > -1), \tag{4}$$

of which the first two follow directly from the e^x series (geometric series estimates), while the third one can be deduced from the arithmetic-geometric mean inequality.

Instructive examples are furnished by the products

$$\prod_{n=0}^{\infty} \left(1 + c \cdot \binom{\alpha}{n}\right) \quad \text{for } \alpha, c \in \mathbf{R}, \tag{5}$$

and

$$\prod_{n=0}^{\infty} \left(1 + c \cdot \frac{\alpha^{2n}}{\sqrt{n} - 1/2}\right)\left(1 - c \cdot \frac{\alpha^{2n+1}}{\sqrt{n} + 1/2}\right) \quad \text{for } \alpha, c \in \mathbf{R}. \tag{6}$$

In order to discuss (5), the convergence properties of $\Sigma\binom{\alpha}{n}$ and $\Sigma\binom{\alpha}{n}^2$ need to be investigated. To do this it is most convenient to use the Gauß test (see, e.g., Hyslop **[5]**) which, in a slightly generalized form, may be stated as follows:

Gauß test. *Let $a_n > 0$ for almost all $n \in \mathbf{N}$.*

If there is a bounded sequence $(b_n)_{n \in N}$ and a constant $a > 1$ such that

$$\frac{a_{n+1}}{a_n} \leqslant 1 - \frac{a}{n + b_n} \quad \textit{for almost all } n,$$

then the series Σa_n is convergent;

if there is a bounded sequence $(b_n)_{n \in N}$ such that

$$\frac{a_{n+1}}{a_n} \geqslant 1 - \frac{1}{n + b_n} \quad \textit{for almost all } n,$$

then the series Σa_n is divergent.

Using this criterion and the fact that $|\binom{\alpha}{n}| \searrow 0 (n \to \infty)$ for $-1 < \alpha < 0$, whereas $|\binom{\alpha}{n}| \to \infty (n \to \infty)$ for $\alpha < -1$, one can deduce (we omit the details):

- If $\alpha \geqslant 0$, (5) is absolutely convergent for every $c \in \mathbf{R}$;
- if $-1/2 < \alpha < 0$, (5) is conditionally convergent for $c \in \mathbf{R} \setminus \{0\}$;
- if $-1 < \alpha \leqslant -1/2$, (5) diverges to zero for $c \in \mathbf{R} \setminus \{0\}$;

- if $\alpha = -1$, (5) diverges to zero for $0 < |c| < \sqrt{2}$, and is indefinitely divergent for $|c| \geqslant \sqrt{2}$;
- if $\alpha < -1$, (5) is indefinitely divergent for $c \in \mathbf{R} \setminus \{0\}$.

Now we discuss the products (6).

- If $|\alpha| < 1$, (6) obviously is absolutely convergent;
- if $|\alpha| > 1$, (6) is divergent for $c \in \mathbf{R} \setminus \{0\}$;
- if $\alpha = 1$, (6) is convergent for $c = 1$, since

$$\left(1 + \frac{1}{\sqrt{n} - 1/2}\right)\left(1 - \frac{1}{\sqrt{n} + 1/2}\right) = 1$$

  for all $n \in \mathbf{N}$; thus from $\Sigma x_n^2 = \infty$ and Theorem 1 we conclude that for every $c \in \mathbf{R} \setminus \{0, 1\}$ the product is divergent.
- Moreover, one can prove (see Wermuth [12])

$$\lim_{\alpha \nearrow 1} \prod_{n=1}^{\infty} \left(1 + \frac{\alpha^{2n}}{\sqrt{n} - 1/2}\right)\left(1 - \frac{\alpha^{2n+1}}{\sqrt{n} + 1/2}\right) = 2, \tag{7}$$

  whereas

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{\sqrt{n} - 1/2}\right)\left(1 - \frac{1}{\sqrt{n} + 1/2}\right) = 1. \tag{8}$$

Hardy [3] gave a slightly more complicated example of this "non-Abelian" behaviour. On the other hand, Hardy proved an analogue of Abel's theorem for products fulfilling the Coriolis conditions:

Theorem 2. *Let the series Σa_n and $\Sigma |a_n|^2$ be convergent. Then*

$$\lim_{z \to 1} \prod_{n=1}^{\infty} (1 + a_n z^n) = \prod_{n=1}^{\infty} (1 + a_n),$$

if z approaches unity inside a Stolz angle.

3. COMPLEX PRODUCTS. We now consider the question whether there is a converse to the Coriolis test for complex products.

At first glance, the complex case seems to be easily reducible to the real one: Writing $1 + z_n = (1 + r_n)\exp(i\varphi_n)$, where $r_n \geqslant -1$, $-\pi < \varphi_n \leqslant \pi$, it is not difficult to show that

$\Pi(1 + z_n) = \Pi(1 + r_n)e^{i\varphi_n}$ *is convergent if and only if both* $\Pi(1 + r_n)$ *and* $\Sigma\varphi_n$ *are convergent.*

Nevertheless, there is no obvious converse to the Coriolis test. A simple counterexample: The polynomial $z^4 - c^4/k^2 = 0$ with $c \in \mathbf{C}$ and $k \in \mathbf{N}$ has the roots

$$\frac{c}{\sqrt{k}} e^{l\pi i/2} \qquad (l = 0, 1, 2, 3).$$

Thus the product $\Pi(1 - cz_n)$ is convergent for every $c \in \mathbf{C}$, whereas $\Sigma |z_n|^2 = \infty$, if z_n runs through all values $e^{l\pi i/2}/\sqrt{k}\,(k \in \mathbf{N}, l \in \{0, 1, 2, 3\})$.

But there is a partial converse:

Theorem 3. *Assume $\Sigma |z_n|^{k+1} < \infty$ for some $k \in \mathbf{N}$. Then $\Pi(1 + cz_n)$ is convergent for every $c \in \mathbf{C}$ if and only if $\Sigma z_n, \Sigma z_n^2, \ldots, \Sigma z_n^k$ converge.*

Proof: For all sufficiently large n we have

$$\log(1 + cz_n) = cz_n - \frac{1}{2}c^2z_n^2 + \cdots + \frac{(-1)^{k-1}}{k}c^kz_n^k + \vartheta c^{k+1}z_n^{k+1}$$

with $|\vartheta| \leqslant 1$. Thus $\prod(1 + cz_n)$ is convergent for every $c \in \mathbf{C}$ if and only if

$$\sum \left\{cz_n - \frac{1}{2}c^2z_n^2 + \cdots + \frac{(-1)^{k+1}}{k}c^kz_n^k\right\} \tag{9}$$

is convergent for every $c \in \mathbf{C}$.

Hence one direction of the proof is trivial (and, in fact, already appears in Pringsheim [**8**]). Suppose now that (9) is convergent for every $c \in \mathbf{C}$. Replace c by $c \cdot e^{\pi i/k}$ and add both expressions; in the resulting expression, replace c by $c \cdot e^{\pi i/(k-1)}$ and add both expressions, etc., to get finally:
$\sum c(1 + e^{\pi i/2})(1 + e^{\pi i/3}) \cdots (1 + e^{\pi i/k})z_n$ is convergent. Hence $\sum z_n$ is convergent. Now if $\prod(1 + cz_n)$ is convergent for every $c \in \mathbf{C}$ then so is $\prod(1 + cz_n^k)$ for every $k \in \mathbf{N}$, since with

$$1 + z^k = (c_1 + z) \cdots (c_k + z) \quad \text{for all } z \in \mathbf{C} \quad \text{and} \quad c_0^k = c$$

we have

$$\prod \left(1 + c_0c_1^{-1}z_n\right) \cdots \left(1 + c_0c_k^{-1}z_n\right) = \prod \left(1 + cz_n^k\right).$$

Hence the conclusion follows. ∎

This theorem includes no *complete* converse of the Coriolis test, since the counterexample may be modified to get an example such that $\prod(1 + cz_n)$ converges for all $c \in \mathbf{C}$, whereas $\sum |z_n|^k = \infty$ for every $k \in \mathbf{N}$. To this end, let (z_n) be the sequence of all numbers

$$\hat{z}_{klm} = \frac{1}{m\sqrt[m]{l^2}}e^{2k\pi i/m} \qquad \left(0 \leqslant k \leqslant m - 1, m^{2m} \leqslant l < (m + 1)^{2m+2}, m \geqslant 2\right), \tag{10}$$

arranged in such order that k changes most quickly and m changes most slowly. Then because of

$$\left(1 - c\hat{z}_{0lm}\right) \cdots \left(1 - c\hat{z}_{m-1,l,m}\right) = 1 - \frac{c^m}{l^2},$$

$$\left(1 + |c\hat{z}_{0lm}|\right) \cdots \left(1 + |c\hat{z}_{m-1,l,m}|\right) = \left(1 + \frac{|c|}{m\sqrt[m]{l^2}}\right)^m \leqslant \left(1 + \frac{|c|}{m^4}\right)^m,$$

a similar lower estimate, and

$$\sum_{m \geqslant 2} \sum_{m^{2m} \leqslant l < (m+1)^{2m+2}} \left|\frac{c^m}{l^2}\right| \leqslant \sum_{m \geqslant 2} \left(\frac{|c|}{m}\right)^m \sum_{m^{2m} \leqslant l < (m+1)^{2m+2}} \frac{1}{l^{3/2}} < \infty,$$

the product $\prod(1 + cz_n)$ converges for every $c \in \mathbf{C}$. Moreover

$$\sum |z_n|^j = \sum |\hat{z}_{klm}|^j \geqslant \sum_{l \geqslant (2j)^{4j}} \frac{1}{l} = \infty$$

for every j. But nevertheless $\sum z_n^j$ is convergent for every j.

Hardy [3] raised the question whether there are convergent products $\prod(1+z_n)$ such that $\sum|z_n|^k=\infty$ for every k while $\sum z_n^k$ is convergent for every k. Littlewood [7] gave an affirmative answer by proving that

$\prod(1+x_n e^{n\varphi i})$ is convergent, if φ/π is a non-rational algebraic number and the sequence (x_n) tends to zero monotonously.

Our counterexample is a more elementary example of this type.

Hardy [3] also asked for an example of a divergent product $\prod(1+z_n)$ such that $\sum z_n^j$ is convergent for every j. This question seems to have remained unanswered. If we modify our example by putting

$$\hat{z}_{klm}=\frac{1}{\sqrt{m}}e^{2k\pi i/m}\qquad (0\leqslant k\leqslant m-1, 1\leqslant l<m^m, m\geqslant 1), \tag{11}$$

then $\sum z_n^j$ is convergent for every $j\in\mathbf{N}$, whereas $\prod(1+cz_n)$ is divergent for every $c\in\mathbf{C}\setminus\{0\}$. To prove divergence, we first observe

$$(1-c\hat{z}_{0lm})\cdots(1-c\hat{z}_{m-1,l,m})=1-\left(\frac{c}{\sqrt{m}}\right)^m;$$

hence, with $N_m=1^2+2^3+3^4+\cdots+m^{m+1}$ for $m\in\mathbf{N}$,

$$\prod_{n=N_{m-1}+1}^{N_m}(1-cz_n)=\left(1-\left(\frac{c}{\sqrt{m}}\right)^m\right)^{m^m}. \tag{12}$$

Writing $c=re^{i\varphi}$ with $r>0$, $-\pi<\varphi\leqslant\pi$ for $c\in\mathbf{C}\setminus\{0\}$, we get:
If $\varphi=0$,

$$\left(1-\left(\frac{c}{\sqrt{m}}\right)^m\right)^{m^m}=\left(1-\frac{(r\sqrt{m})^m}{m^m}\right)^{m^m},$$

if $\varphi\neq 0$,

$$\left|1-\left(\frac{c}{\sqrt{m}}\right)^m\right|^{m^m}=\left|1-e^{i\varphi m}\frac{(r\sqrt{m})^m}{m^m}\right|^{m^m}\geqslant\left(1+\frac{\frac{1}{2}(r\sqrt{m})^m}{m^m}\right)^{m^m}$$

for infinitely many m, since in this case there are infinitely many $m\in\mathbf{N}$ such that $e^{i\varphi m}$ belongs to the sector $\frac{2}{3}\pi\leqslant\arg z\leqslant\frac{4}{3}\pi$. Thus

$$\lim_{m\to\infty}\prod_{n=N_{m-1}+1}^{N_m}(1-cz_n)=1$$

doesn't hold for any $c\in\mathbf{C}\setminus\{0\}$, i.e., the product $\prod(1-cz_n)$ is divergent for every $c\in\mathbf{C}\setminus\{0\}$.

The last example shows that the convergence of $\sum z_n^k$ for every $k\in\mathbf{N}$ is not sufficient to guarantee the convergence of $\prod(1+cz_n)$ for any nonzero c; on the other hand, by the previous example we saw that the convergence of $\prod(1+cz_n)$ for every complex c does not imply the convergence of $\sum|z_n|^k$ for any integer k.

So it seems difficult to sharpen Theorem 3, but a certain refinement would be the solution of the following

Open Problem. *To show that $\sum z_n$ (and thus $\sum z_n^k$ for every $k\in\mathbf{N}$) is convergent if $\prod(1+cz_n)$ converges for every $c\in\mathbf{C}$, or to give a counterexample.*

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In most sciences one generation tears down what another has built and what one has established another undoes. In Mathematics alone each generation builds a new story to the old structure.

—*Hermann Hankel*

Pascal's Triangle and the Tower of Hanoi

Andreas M. Hinz

INTRODUCTION. The most genuine examples for the principle of complete induction are the arithmetic triangle (AT) and the Tower of Hanoi (TH). They also reveal an unexpected mathematical relation which will be developed here.

The AT has been studied by Blaise Pascal in a treatise published posthumously in 1665 and is therefore often called Pascal's triangle. However, it was known before in Europe (Peter Apian, 1527), China (Jia Xian, 11th century), the Islamic world (al-Karajī, ca. 1000), and possibly India (Piṅgala, ca. −200). The TH is an invention of the French mathematician Édouard Lucas, who first published the puzzle in 1883. An account of its history and basic mathematical properties can be found in [**4**].

Recently, connections between the AT and the Sierpiński gasket (SG) have been observed. The SG is obtained from a closed equilateral triangle by deleting the open middle triangle and iterating this step for the remaining subtriangles ad infinitum. It turns out that its fractal geometry is the same as that of the AT modulo 2 (see [**2**, p. 10f]). On the other hand the SG can be viewed, in a certain sense, as the limit of the graph of the TH for an increasing number of discs (see Hinz and Schief [**5**]). So, by transitivity, there must be a link between the TH and the AT. Since the TH is closely related to binary structures, it is not surprising that this connection is again with AT mod 2.

The TH with $n \in \mathbb{N}_0$ discs will be identified with the graph TH_n, whose vertices are the distributions of n discs among three pegs which are regular (i.e. no disc lies on a smaller one), and whose edges are legal moves of a single top disc, leading from one such distribution to another. This graph is simple, undirected, planar, and connected. Figure 1 shows the example $n = 3$. (The discs being numbered from 1 to n, the pegs named 0, 1, and 2, the state with disc 1 on peg 0, disc 2 on peg 2, and disc 3 on peg 1 is abbreviated 021, for instance.)

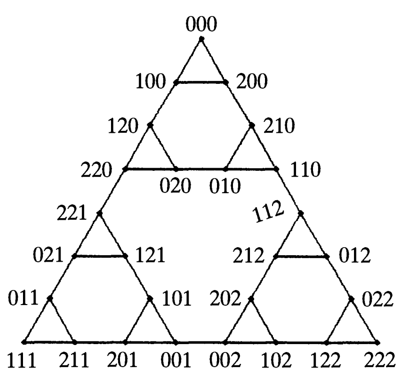

Figure 1

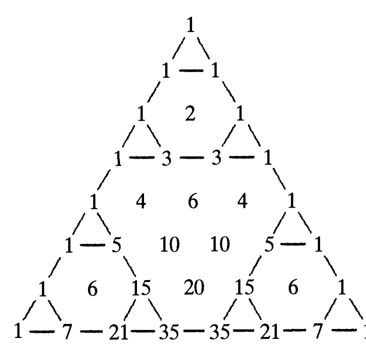

Figure 2

AT_m will denote the AT with $m \in \mathbb{N}$ rows, counted as usual from the 0th row at the apex to the $(m-1)$-th row at the base. Assume further that the geometry of AT_m is as symmetric as possible, i.e. nearest neighbors are a unit apart. Then the basic observation is the following: The graph $AT_{2^n} \bmod 2$, consisting of the odd numbers in AT_{2^n} joined by an edge if one unit apart (see Figure 2), is isomorphic to TH_n.

1. THE PARITY OF BINOMIAL COEFFICIENTS.

The parity of binomial coefficients $\binom{\mu}{k}$ has recently played an important role in a paper of Jones and Matijasevič [6] in connection with Hilbert's tenth problem, Gödel's undecidability proposition, and computational complexity. They base their Lemma on the following theorem of Lucas [9, Section XXI].

Theorem 0. *Let p be a prime. Then*

$$\binom{\mu}{k} \equiv \prod_{i=0}^{n-1} \binom{\mu_i}{k_i} \bmod p,$$

where μ_i and k_i are the p-ary digits (or pits) of μ and k, respectively.

Since Jones and Matijasevič only need the case $p = 2$, they could have relied on an older result of Kummer [7, p. 115f], namely, that the highest power of p dividing $\binom{k+\nu}{k}$ is equal to the number of carries in the p-ary addition of k and ν, which for $p = 2$ means that $\binom{\mu}{k}$ is odd if and only if $k_i \leq \mu_i$ for all i.

Lucas states in his famous book *Théorie des nombres* of 1891 (p. 420) that all binomial coefficients in a row of the AT are odd only if the row number is one less than a power of two:

$$\left(\forall\, 0 \leq k \leq \mu: \binom{\mu}{k} \text{ odd}\right) \Leftrightarrow (\exists\, n \in \mathbb{N}_0: \mu = 2^n - 1), \tag{1}$$

while the complementary statement, namely, they are all even (except the outer ones) if the row number is a power of two:

$$\left(\forall\, 0 < k < \mu: \binom{\mu}{k} \text{ even}\right) \Leftrightarrow (\exists\, n \in \mathbb{N}_0: \mu = 2^n), \tag{2}$$

is due to Fine. Fine also proved that odd binomial coefficients are sparse:

$$\#\left\{\text{odd } \binom{\mu}{k} \in \mathrm{AT}_m\right\} / \#(\mathrm{AT}_m) \to 0, \qquad \text{as } m \to \infty. \tag{3}$$

All these results arise from Theorem 0 and can be extended to a general prime p. As another consequence of Theorem 0, Glaisher represented the number of odd binomial coefficients in the μth row as:

$$\forall\, \mu \in \mathbb{N}_0: \#\left\{\text{odd } \binom{\mu}{k}\right\} = 2^{\beta(\mu)}, \tag{4}$$

where $\beta(\mu)$ is the number of non-zero bits of μ.

The references to Fine and Glaisher, as well as to many other works on parity of binomial coefficients, can be found in Stolarsky [13], who was interested primarily

in the asymptotic behavior, as $m \to \infty$, of

$$\mathrm{B}(m) := \#\left\{\text{odd}\begin{pmatrix}\mu\\k\end{pmatrix} \in \mathrm{AT}_m\right\},$$

the number of odd binomial coefficients in the first m rows. It turns out that $\mathrm{B}(m)$ behaves essentially like m^s, where $s = \ln 3/\ln 2$ is the Hausdorff dimension of the SG (see [**2**, pp. 157–159]; roughly speaking, doubling the linear extension of the SG means tripling its measure, whence $2^s = 3$).

An explicit formula for $\mathrm{B}(m)$ can be obtained from the special case $p = 2$ of Corollary 4 in Roberts [**10**]:

$$\forall\, m \in \mathbb{N}: \mathrm{B}(m) = \sum_{i=0}^{n-1} m_i 2^{\sum_{j=i+1}^{n-1} m_j} \cdot 3^i, \tag{5}$$

for $m = \sum_{i=0}^{n-1} m_i \cdot 2^i$, $m_i \in \{0, 1\}$.

Because of the isomorphy of $\mathrm{AT}_{2^n} \bmod 2$ and TH_n all these results can be reinterpreted in terms of the TH puzzle. For instance, $\mathrm{B}(m)$ is the number of states accessible from the perfect initial distribution (i.e., all discs are on the same peg) in less than m moves. However, most of these statements are easier derived from the properties of the TH. This and some additional results will be achieved in Section 3. Before doing that, it seems adequate to honor the person who stands for these things.

2. ÉDOUARD LUCAS (1842–1891). François Édouard Anatole Lucas was born on April 4th, 1842 in Amiens (France). Son of a worker, his talents earned him a scholarship for higher education. In 1861 he was accepted by the most prestigious French institutions of the time, the *École polytechnique* and the *École normale*. Lucas attended the latter and left it in 1864 as *Agrégé des sciences mathématiques*.

Édouard Lucas (1842–1891)

The employment at the Paris Observatory as an assistant of Leverrier was interrupted by his active participation in the Franco-Prussian war of 1870/71. His last twenty years Lucas held positions as a teacher of higher mathematics at the high schools of Moulins (72–76), Paris Charlemagne (76–79, 90–91), and Paris St. Louis (79–90). Being a mathematician out of line who was described as young, ardent, and energetic till the end of his life, this professional situation was not adequate since "his character of a noble independence, his spontaneous mind were not able to bend into the narrow mould of university or even high school

teaching, not more than his high intelligence, we may say his genius, could stay a prisoner of programmes. (A. Béligne)" His research interests being centered in number theory, Lucas himself felt that he was living "at a time and in a country where higher arithmetic is forsaken by mathematicians and public education." So his main activities were focussed to learned societies of France and other countries and, of course, to his written works, which unfortunately are not accessible in collected form, but a catalogue of which has been compiled by Harkin [**3**].

Besides some papers on geometry, most of his articles and books concentrate on number theory, recurrent series, and recreational mathematics. He was the last "largest prime number record" holder in pre-computer age, has a series of numbers, namely $2, 1, 3, 4, 7, 11, \ldots,$ called after him, and published, in addition to the famous TH of N. Claus de Siam (= Lucas d'Amiens), a collection of scientific puzzles, now apparently lost, which won a gold medal at the world's fair of 1889. He left a couple of books unfinished, in particular the planned sequel of the *Théorie des nombres*. So large was the interest in his unpublished papers, that, as E. T. Bell once remarked, "the fantastic price of thirty thousand dollars was being asked for Lucas's manuscripts. In all his life Lucas never had that much money." One may doubt at least the last sentence, since it is known that Lucas donated a collection of calculating machines, among which those of Chebyshev and Roth, to a museum in Paris. This was in connection with his efforts to make mathematics popular, and it is said that he was an entertaining teacher in lectures for a general audience. Here and in his papers he took an interest in the history of mathematical problems, definitions, and theorems—not a very common attitude at his time. Lucas was actively involved in the publication of Fermat's collected works and mentioned, an interesting detail in connection with the present note, that he lived for a while "No. 56 rue Monge in Paris, in the house built on the site of the one where Pascal died on August 19th, 1662."

Édouard Lucas himself died in Paris, aged only 49, on October 3rd, 1891.

3. THE TOWER OF HANOI. The TH graphs as defined in the introduction can be obtained recursively in the following way: TH_0 has only one vertex (three empty pegs) and no edges (there are no discs to be moved); TH_{n+1} is composed of three triangular TH_n graphs (movements in the n smaller discs are not affected by the largest disc which is on one of three possible pegs) joined at their base corners (disc $n + 1$ can move only if the other discs are on the peg not involved in that move).

Similarly, $AT_{2^0} \bmod 2$ consists of just one 1, and $AT_{2^{n+1}} \bmod 2$ is constructed recursively in the same manner as TH_{n+1} as can be seen from the following lemma.

Lemma 1. $\forall\, n \in \mathbb{N}_0\ \forall\, 0 \leq \nu,\ k < 2^n$: $\binom{2^n+\nu}{k} \equiv \binom{\nu}{k} \bmod 2.$

This lemma is an immediate consequence of Theorem 0 (or of Kummer's theorem) or can easily be proved by induction.

Thus the basic theorem of this paper is established, namely.

Theorem 1. *For any* $n \in \mathbb{N}_0$, $AT_{2^n} \bmod 2$ *and* TH_n *are isomorphic.*

With the help of this observation, properties of TH_n, as developed e.g. in [**4**], will now be turned into statements about odd binomial coefficients. For instance,

from the trivial fact that there are precisely 3^n regular distributions of n discs among three pegs it follows that for any $n \in \mathbb{N}$ (#(graph) means the number of vertices):

$$\forall\, 2^{n-1} \le m \le 2^n \colon \frac{\#(AT_m \bmod 2)}{\#(AT_m)} \le \frac{\#(AT_{2^n} \bmod 2)}{\#(AT_{2^{n-1}})}$$

$$= \frac{\#(TH_n)}{\#(AT_{2^{n-1}})} = \frac{3^n}{2^{n-2}(2^{n-1}+1)},$$

which yields Fine's result (3).

As mentioned in the introduction, $\mathrm{B}(m)$, the number of odd binomial coefficients in the AT with $m \in \mathbb{N}$ rows, is, by Theorem 1, equal to the number of states of the TH which are accessible from a perfect starting configuration in at most $m-1$ moves. Since it is obvious from the TH graph that $\mathrm{B}(1) = 1$ and for any $n \in \mathbb{N}$

$$\mathrm{B}\left(\sum_{i=0}^{n} m_i \cdot 2^i\right) = m_n \cdot 3^n + 2^{m_n}\mathrm{B}\left(\sum_{i=0}^{n-1} m_i \cdot 2^i\right),$$

an induction on n, the length of the binary representation of m, yields Roberts' formula (5).

Another look at the TH graph shows that

$$\forall\, n \in \mathbb{N} \colon 2^{n-1} \le m < 2^n \Rightarrow 3^{n-1} \le \mathrm{B}(m) < 3^n,$$

from which follows the rough estimate of Stolarsky [**13**, Th. 1]:

$$\forall\, m \in \mathbb{N} \colon \frac{1}{3} < \frac{\mathrm{B}(m)}{m^s} < 3.$$

Glaisher's formula (4) for the number of odd binomial coefficients in row μ is a direct consequence of Proposition 5 in [**4**] which says that the number of states of the TH that are precisely μ steps away from a specific perfect state (here the top apex of TH_n) is $2^{\beta(\mu)}$. (1) and (2) are, of course, just special cases of this. Applying the same proposition to the lower left apex, however, one learns that the number of odd binomial coefficients at a (graph) distance $\bar{\nu}$ from this corner is $2^{\beta(\bar{\nu})}$. But these are just the odd numbers in the $(2^n - 1 - \bar{\nu})$-th diagonal of AT_{2^n}, consisting of binomial coefficients of the form $\binom{k+\nu}{k}$ with $\nu = 2^n - 1 - \bar{\nu}$, i.e. figurate numbers of order ν (so called as generalizations of triangular numbers ($\nu = 2$) and tetrahedral numbers ($\nu = 3$)). This can be summarized as follows:

Proposition 1. *Among the first $2^n - \nu$ figurate numbers of order ν $(0 \le \nu < 2^n)$, $2^{\bar{\beta}(\nu)}$ are odd, where $\bar{\beta}(\nu)$ is the number of zero bits in the n-bit representation of ν.*

Although this result may not be too surprising, it is amazing how it came about from the TH. But there are some deeper insights which stem from considering yet another counting on the graph TH_n, namely the function z_n that gives for an integer μ the number of states in TH_n for which the difference of the distances to two distinct corners is exactly μ. (By symmetry, this does not depend on the pair considered.) Before discussing the functions z_n more detailed in the next section, their appearance in the AT should be pointed out.

Odd binomial coefficients in AT_{2^n} for which the difference of the distances to the base corners is $\nu \in \mathbb{N}_0$ are those of the form $\binom{2k+\nu}{k}$ with $0 \le k < (2^n - \nu)/2$.

Hence Theorem 1 gives for any $n \in \mathbb{N}_0$:

Proposition 2. *Among the first* $\lceil (2^n - \nu)/2 \rceil$ *numbers in the* ν*-th column of the AT* $(0 \le \nu < 2^n)$, $z_n(\nu)$ *are odd; here the columns are counted from the* 0*th at the center.*

Note that the ν-th column consists of the coefficients of the Chebyshev polynomial y_ν in the development of $\frac{1}{2}(2x)^{\nu+2k}$ (here $y_0 := \frac{1}{2}$).

Adding up the entries of the νth subdiagonal of the AT, one gets the Fibonacci number F_ν:

$$F_\nu = \sum_{k=0}^{\lfloor \nu/2 \rfloor} \binom{\nu - k}{k}. \tag{6}$$

This representation can be found in Siebeck [**12**, p. 71] (the last term in his first formula should be $1 \cdot c^{(r-1)/2}$). The geometrical interpretation of the AT has been given by, of course, Lucas [**8**, p. 138f], who also introduced in that article the name *Fibonacci series*. Since for odd binomial coefficients of the ν-th subdiagonal $(0 \le \nu < 2^n)$ the difference between the distances to the bottom right and top corner of AT_{2^n}, respectively, is $\bar{\nu} = 2^n - 1 - \nu$, one has for any $n \in \mathbb{N}_0$:

Proposition 3. $z_n(\bar{\nu})$ *of the binomial coefficients in the* ν*-th subdiagonal* $(0 \le \nu < 2^n)$ *are odd.*

4. THE FUNCTIONS z_n. The following has been proved in [**4**, L.2].

Lemma 2. o) $z_0(0) = 1$, $\forall\, \mu \in \mathbb{Z} \setminus \{0\}$: $z_0(\mu) = 0$;

$$\forall\, n \in \mathbb{N}_0\ \forall\, \mu \in \mathbb{Z}\colon z_{n+1}(\mu) = z_n(\mu - 2^n) + z_n(\mu) + z_n(\mu + 2^n);$$

$$\text{i) } \forall\, n \in \mathbb{N}_0\ \forall\, \mu \in \mathbb{Z}\colon z_n(-\mu) = z_n(\mu),\ |\mu| \geqslant 2^n \Rightarrow z_n(\mu) = 0;$$

$$z_n(0) = 1,\ z_n(1) = n,\ z_n(2^n - 1) = 1.$$

Note that, since

$$z_n(\mu) = \sum_{\xi \in \{-1,0,1\}^n} z_0\left(\mu + \sum_{i=0}^{n-1} \xi_{i+1} \cdot 2^i\right)$$

by induction from Lemma 2o, $z_n(\mu)$ is just the number of ways μ can be written as $\sum_{i=0}^{n-1} \xi_i \cdot 2^i$ with $\xi_i \in \{-1, 0, 1\}$. This shows that the z_n are not very easily accessible functions (cf. the discussion at the end of [**5**]). However, some special relations are feasible.

Let $2^a \le \mu < 2^{a+1}$ for an $a \in \mathbb{N}_0$. Then (by Lemma 2i)

$$\forall\, n \le a\colon z_n(\mu) = 0$$

and (by Lemma 2o)

$$\forall\, n > a\colon z_{n+1}(2^{n+1} - \mu) = z_n(2^n - \mu),$$

whence (by induction)

$$\forall\, k \in \mathbb{N}\colon z_{a+k}(\mu) = z_{a+1}(\mu) + (k-1) z_{a+1}(2^{a+1} - \mu). \tag{7}$$

That is to say, for fixed μ, $z_n(\mu)$ is eventually in arithmetic progression, while e.g. the lengths of the columns in Proposition 2 are essentially in geometric progression.

By Lemma 2, $z_n(0) = 1$, such that $\forall\ k \neq 0$: $2 | \binom{2k}{k}$ by Proposition 2.

For the special cases $\mu = 2^a,\ 2^a + 1,\ 2^{a+1} - 1\ (a \in \mathbb{N}_0)$, (7), and Lemma 2 yield

$$\forall\ k \in \mathbb{N}: z_{a+k}(\mu) = \begin{cases} k \\ a + (k-1)(a+1) \\ 1 + (k-1)(a+1), \text{ respectively.} \end{cases}$$

As an example, the sum for F_{22} in (6) is made up of $z_5(9) = 7$ odd and 5 even numbers by Proposition 3. (Though parity has been associated with gender, these values should not be taken as the numbers of male and female rabbits, since by definition of F_ν they always appear in pairs!)

5. PASCAL'S TRIANGLE AND THE TOWER OF HANOI WITH MORE THAN THREE PEGS. It should be noted that the AT has been used in algorithms for a solution of the TH with more than three pegs; see e.g. Rohl and Gedeon **[11]**. Although in this paper, as in many others on the subject, minimality of the solution is claimed, there is no proof for that. As it stands, *Monthly* Problem 3918 [1939, p. 363] is still unsolved (cf. **[1]**), namely: What is the minimum number of moves required to transfer n discs from one of $k > 3$ pegs to another?

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On the Uniqueness of the Cyclic Group of Order n

Dieter Jungnickel

When is there a unique group of order n? (Such a group, of course, must be cyclic.) When teaching a beginning course in group theory, we point out there is a unique group when n is a prime. Usually, we go on to discuss the Sylow theorems and apply them to groups of order pq ($p < q$ primes). Such a group is unique, we show, if and only if p does not divide $q - 1$. It is natural, therefore, to ask when the group of order n is unique. The answer is "well known", but not widely known, and seldom mentioned in such classes. Here is a simple proof that is suitable for even an elementary class in group theory.

Theorem. *Let n be a positive integer. Then the cyclic group $C(n)$ of order n is the only group of order n if and only if one has $(n, \phi(n)) = 1$, where ϕ denotes the Euler phi function.*

Proof: We first note that both conditions imply that n is square-free. For assume that $n = mp^a$, where p is a prime not dividing m and where $a \geqslant 2$. Then both n and $\phi(n) = p^{a-1}(p-1)\phi(m)$ are divisible by p. Also, the group $C(m) \times C(p)^a$ is clearly not isomorphic to $C(n)$. From now on, let n be square-free. Then

$$(*)\ n = p_1 \cdots p_k \text{ is a product of distinct primes and}$$

$$\phi(n) = (p_1 - 1) \cdots (p_k - 1).$$

Thus $(n, \phi(n)) \neq 1$ implies the existence of primes p and q dividing $n = pqm$, say, for which p divides $q - 1$. Then there exists a non-abelian group H of order pq (a semidirect product), and so $H \times C(m)$ is a non-abelian group of order n.

It thus remains to assume $(n, \phi(n)) = 1$ and to show that there is only one group of order n in this case. Assume the contrary, and let n be the least positive integer for which a counter-example G exists. We shall now reach a contradiction in the following steps.

Step 1. One has $(m, \phi(m)) = 1$ for every divisor m of n.
This follows immediately from $(*)$ above.

Step 2. Every proper subgroup and every non-trivial factor group of G are cyclic.
This is clear from Step 1 and the minimality of n.

Step 3. The center $Z(G)$ is trivial.

Otherwise $G/Z(G)$ would be cyclic by Step 2, and therefore G would be abelian and hence cyclic.

Step 4. Let $x \neq 1$ be an element of a maximal subgroup U of G. Then U is the centralizer $C_G(x)$ of x in G.

For $C_G(x)$ is a proper subgroup of G by Step 3, and U is cyclic and therefore contained in $C_G(x)$ by Step 2; thus the maximality of U shows $U = C_G(x)$.

Step 5. Any two distinct maximal subgroups U and V of G have trivial intersection.

For assume that $x \neq 1$ is in $U \cap V$. Then Step 4 would give the contradiction $U = C_G(x) = V$.

Step 6. Any maximal subgroup U equals its own normalizer $N_G(U)$.

To see this, let $x \neq 1$ be any element in $N_G(U)$. Then the conjugation with x induces an automorphism α of the cyclic group U. If U has order m, then the automorphism group of U has order $\phi(m)$ which divides $\phi(n)$ because of $(*)$. Since x and hence α have order dividing n, Step 1 shows that α has to have order 1. Thus x centralizes U and by Step 3 belongs to U.

Step 7. Let U be a maximal subgroup of order u of G. Then the conjugates of U contain exactly $n - n/u$ elements $\neq 1$.

Note that the number of conjugates of U is the index of the normalizer of U in G. By Step 6, this index is n/u. By Step 5, any two distinct conjugates of U intersect trivially. Thus the conjugates of U contain altogether $(u-1)n/u$ elements $\neq 1$.

Step 8. Now let U be as in Step 7 and choose an element x not contained in any of the conjugates of U. Let V be a maximal subgroup containing x and therefore not conjugate to U. Then any conjugate of V and any conjugate of U intersect trivially by Step 5. Applying Step 7 also to V, we obtain $n - n/v$ elements $\neq 1$ in the conjugates of V. But there are only $n - 1$ elements $\neq 1$, giving the inequality

$$n - n/u + n - n/v < n$$

which results in the contradiction $uv < u + v$. $\square$

Some historical remarks: The preceding theorem is a special case of a result due to Dickson **[1]** who determined those n for which every group of order n is abelian; his 1905 paper is, as far as the author knows, the earliest reference for our theorem. Simpler proofs were given by Szele **[4]** and Szep **[5]** who seem not to have been aware of Dickson's result. Regarding further reading, the reader might be interested to go on to study related questions, e.g. for which orders n every group is abelian or nilpotent; for these and similar questions, we recommend Pazderski **[3]**. Another problem that is suggested by the proof given above is the determination of those non-abelian groups for which all proper subgroups are abelian; this problem was considered by Miller and Moreno **[2]**.

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THE PARADOX OF FAIRNESS

Let's say the coin is fair
And we toss it in the air.

Heads or Tails?
Who's the first to pick?
Shall we toss another?
To avoid a Diaconis trick.

But then what side
Will decide
The options on which
Our game does ride?

Let a third person toss it in the air!
And we'll call it while it's there.

But who's first to call it—
While it's there?
Again we shout
All is still unfair!

Play until fortunes tie.
Won't that now satisfy?

Might as well play for fun
Or never start the run
Than await boring ties
And even triter lies.

Cooperation is what's fair.
You cut the cake...
I'll pick from the pair.

But Beware!
Let not Tarski make the tear!
Otherwise, and it's okay—
The Game is Solitaire—
With its fun and lonely fare
Free of all competing dare.

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Sequences with Many Primes

Robin Forman

A basic problem is to investigate the number of primes which appear in various sequences. Euclid proved that the sequence

$$a_n = n \qquad n = 0, 1, 2, \ldots$$

contains an infinite number of primes. Dirichlet extended this result to sequences of the form

$$a_n = pn + q \qquad n = 0, 1, 2, \ldots$$

where p and q are relatively prime integers, $p > 0$. No such result is known for any other "simple" function of x.

As a first step towards treating polynomials of higher degree, Sierpinski showed in **[3]** that for any M there is an integer c such that the sequence

$$a_n = n^2 + c \qquad n = 0, 1, 2, \ldots$$

contains at least M primes. It should be noted that there is a long-standing conjecture that the sequence

$$n^2 + 1 \qquad n = 0, 1, 2 \ldots$$

contains infinitely many primes.

In **[1]**, Garrison extended Sierpinski's result to polynomials of degree $k \geq 2$ and proved that for any such k and any M there is an integer $c \geq 0$ such that the sequence

$$n^k + c \qquad n = 0, 1, 2, \ldots$$

contains at least M primes.

In this note, by modifying Garrison's proof we extend the Garrison-Sierpinski theorem to a large class of sequences. In particular, as a corollary to our main theorem we have

Proposition. *Let f be any non-constant polynomial with positive leading coefficient (the coefficients need not be integers). Then for any M there are infinitely many $c \geq 0$ such that the sequence*

$$[f(n)] + c \qquad n = 0, 1, 2, \ldots$$

contains at least M primes (where [ ] denotes the greatest integer function).

Letting $f(x) = x^k$ recovers Garrison's theorem.

Our main theorem is much more general. Roughly speaking we show that the desired property is true for any sequence which grows slower than $e^{\sqrt{n}}$, but still is likely far from the most general possible. In particular, this property is conjectured to be true for every sequence. That is, if

$$a_0, a_1, a_2, \ ..$$

is any sequence of integers tending to $+\infty$, then given any M there are infinitely many $c \geq 0$ such that the sequence

$$a_0 + c,\ a_1 + c,\ a_2 + c, \dots$$

contains at least M distinct primes. This conjecture is not universally accepted, and may very well fail to be true for sequences with extremely rapid growth, such as

$$a_n = (10^{10^n})!$$

In section 2 of this paper we relax the problem slightly and ask whether, given any M, there is a $c \in \mathbf{Z}$ (i.e. c is not necessarily non-negative) such that the sequence

$$a_n + c \qquad n = 0, 1, 2, \dots$$

contains at least M primes. Roughly speaking, we show that this is true as long as the a_n grow slower than e^n.

Before specializing to the case of prime numbers, we consider the relationship between general pairs of increasing sequences of integers A and B, where

$$A = \{a_0 < a_1 < a_2 < \cdots\}$$
$$B = \{b_0 < b_1 < b_2 < \cdots\}.$$

We write

$$A \succeq B$$

if for any M there is a $c \in \mathbf{Z}^{\geq 0}$ such that B and $A + c$ have at least M elements in common. That is

$$\#\{B \cap (A + c)\} \geq M$$

(where $A + c$ denotes the sequence $\{a_0 + c < a_1 + c < a_2 + c < \cdots\}$). We write

$$A \approx B$$

if for any M there is a $c \in \mathbf{Z}$ such that

$$\#\{B \cap (A + c)\} \geq M.$$

Clearly

$$A \succeq B \Rightarrow A \approx B.$$

Moreover, the relation $\approx$ is symmetric. That is

$$A \approx B \Rightarrow B \approx A.$$

It may not be immediately clear that the relation $\approx$ is non-trivial, i.e. that there exist sequences A and B with $A \not\approx B$. However, consider the sequences

$$A = \left\{a_n | a_n = \sum_{i=0}^{n} 10^i\right\}$$

$$B = \left\{b_n | b_n = \sum_{i=0}^{n} 2 \times 10^i\right\}.$$

Then $A \not\approx B$ since for any c

$$\#\{B \cap (A + c)\} \leq 1.$$

[*Proof:* Suppose

$$b_{m_1} = a_{n_1} + c,\ b_{m_2} = a_{n_2} + c$$

with $m_1 \geq m_2$, $n_1 \geq n_2$. Then

$$b_{m_1} - b_{m_2} = a_{n_1} - a_{n_2}.$$

The left hand side is a number of the form

$$\underbrace{222\ldots2}_{m_1 - m_2 \text{ times}}\ 00\ldots0.$$

The right hand side is a number of the form

$$\underbrace{111\ldots1}_{n_1 - n_2 \text{ times}}\ 00\ldots0.$$

These can be equal only if they are both zero, i.e.

$$m_1 = m_2,\ n_1 = n_2.]$$

For future reference, we note that all of our results will be in terms of the counting function π_A of A, where, for $r \in \mathbf{R}$

$$\pi_A(r) = \sup_{k=0,1,2,\ldots} \{k \mid a_k \leq r\}.$$

For example, for any k

$$\pi_A(a_k) = k + 1.$$

1. THE RELATION $\succeq$. Let A and B be increasing sequences of integers. Our main result is the following

Theorem 1.1. *If*

$$\lim_{i\to\infty} \frac{a_i - a_{i-1}}{\pi_B(a_i)} = 0 \tag{1.2}$$

then $A \succeq B$.

Proof: Given M, by (1.2) it is possible to choose N_1 such that for all $n \geq N_1$

$$\frac{a_n - a_{n-1}}{\pi_B(a_n)} < \frac{1}{2M}.$$

Now choose $N \geq N_1$ such that

$$\pi_B(a_N) - \pi_B(a_{N_1}) \geq \tfrac{1}{2}\pi_B(a_N).$$

Then, if we define $\tilde{B} \subset B$ by

$$\tilde{B} = \{b \in B \mid a_{N_1} < b \leq a_N\}$$

we have

$$\#\tilde{B} = \pi_B(a_N) - \pi_B(a_{N_1}) \geq \tfrac{1}{2}\pi_B(a_N).$$

Moreover, for each $b \in \tilde{B}$ there is a k, $N_1 < k \leq N$, such that

$$a_{k-1} < b \leq a_k$$

so that

$$1 \leq b - a_{k-1} \leq a_k - a_{k-1} < \frac{1}{2M}\pi_B(a_k) \leq \frac{1}{2M}\pi_B(a_N).$$

Therefore, for each $b \in \tilde{B}$, there is an integer

$$c_b \in \left[1, \frac{1}{2M}\pi_B(a_N)\right]$$

such that

$$b \in A + c_b.$$

By the pigeon-hole principle (i.e. if r pigeons are distributed among s pigeon-holes, then at least one pigeon-hole has at least r/s pigeons) there is a $c \in [1, (1/2M)\pi_B(a_N)]$ which is equal to c_b for at least

$$\frac{\#\tilde{B}}{\frac{1}{2M}\pi_B(a_N)} \geq \frac{\frac{1}{2}\pi_B(a_N)}{\frac{1}{2M}\pi_B(a_N)} = M$$

distinct values of b. Thus

$$\#\{B \cap (A + c)\} \geq M$$

as desired. ∎

Note that if $k \in \mathbf{Z}$ is any fixed constant

$$\lim_{i \to \infty} \frac{\pi_B(a_i + k)}{\pi_B(a_i)} = 1.$$

Thus, if (1.2) is satisfied, then

$$\lim_{i \to \infty} \frac{(a_i + k) - (a_{i-1} + k)}{\pi_B(a_i + k)} = 0.$$

This implies, by Theorem 1.1,

$$A + k \succeq B.$$

Corollary 1.3. *If*

$$\lim_{i \to \infty} \frac{a_i - a_{i-1}}{\pi_B(a_i)} = 0$$

then for any M there are infinitely many distinct $c \in \mathbf{Z}^{>0}$ such that

$$\#\{B \cap (A + c)\} \geq M.$$

Proof: By Theorem 1.1, there is a $c_1 \in \mathbf{Z}^{>0}$ such that

$$\#\{B \cap (A + c_1)\} \geq M.$$

By the above discussion,

$$A + c_1 \succeq B$$

so there is a $c_2 \in \mathbf{Z}^{>0}$ such that

$$\#\{B \cap (A + c_1 + c_2)\} \geq M.$$

Continuing in this fashion yields the desired infinite sequence of constants. ∎

We now restrict our attention to the case of

$$B = \mathscr{P} = \{\text{prime numbers}\}.$$

The prime number theorem states that

$$\lim_{x\to\infty} \pi_{\mathscr{P}}(x) \cdot \frac{\log x}{x} = 1.$$

Thus, Corollary 1.2 implies

Corollary 1.4. *Let A be an increasing sequence which satisfies*

$$\lim_{i\to\infty} \frac{a_i - a_{i-1}}{a_i} \log a_i = 0. \tag{1.5}$$

Then given any M, there are infinitely many distinct positive integers c such that the sequence

$$A + c$$

contains at least M primes.

The reader can easily check that if $f(x)$ is any non-constant polynomial with integer coefficients and a positive leading coefficient, then

$$a_n = f(n)$$

satisfies condition (1.5) (note that $f(n)$ is increasing for n sufficiently large), so that Corollary 1.4 does, in fact, generalize the theorems of Sierpinski and Garrison. Moreover, the condition (1.5) holds much more generally. If

$$\alpha_1 < \alpha_2 < \alpha_3 \cdots \to +\infty$$

is any sequence of real numbers which satisfies (1.5), then the sequence $\{a_n\}$ where

$$a_n = [\alpha_n]$$

also satisfies (1.5), where [ ] denotes the greatest integer function. Thus, if $f(x)$ is any non-constant real polynomial with positive leading coefficient then

$$a_n = [f(n)]$$

satisfies (1.5).

Condition (1.5) is not satisfied by any sequence of the form

$$\alpha_n = e^{n^k} \text{ for } k \geq \tfrac{1}{2}.$$

However, it is easy to construct sequences which satisfy (1.5) and which grow faster than any polynomial. For example, let

$$a_n = [f(n)]$$

for any function f of the form

$$f(x) = \left[e^{x^{k_1}(\log x)^{k_2}} p(x)\right]$$

where $p(x)$ is any polynomial with positive leading coefficient and $0 < k_1 < \frac{1}{2}$ or $k_1 = 0$ and $k_2 > 1$.

2. THE RELATION $\approx$. The relation $\approx$ is symmetric between A and B. Thus, our sufficient condition should also be symmetric between A and B. This is, in fact, the case.

Theorem 2.1. *If*

$$\limsup_{i\to\infty} \frac{\pi_A(i)\pi_B(i)}{i} = \infty \tag{2.2}$$

(*that is, the sequence* $\pi_A(i)\pi_B(i)/i$, $i = 1, 2, 3, \ldots,$ *is unbounded*) *then* $A \approx B$.

Proof: Let $l \in \mathbf{Z}$ be a lower bound for $A \cup B$. That is, for all $a \in A$, $b \in B$

$$a \geq l, b \geq l.$$

Given M, choose N such that

$$N - l + \tfrac{1}{2} < 2N$$

and (using (2.2))

$$\frac{\pi_A(N)\pi_B(N)}{N} > 4M.$$

Let

$$A_N = \{a \in A | a \leq N\}$$

$$B_N = \{b \in B | b \leq N\}$$

(so that $\#A_N = \pi_A(N)$, $\#B_N = \pi_B(N)$).

For each $b \in B_N$, $a \in A_N$

$$-(N - l) \leq b - a \leq (N - l)$$

and

$$b \in A + (b - a).$$

Thus, for $b \in B_N$,

$$b = A + c_b$$

for $\#A_N$ distinct values of c_b with

$$-(N - l) \leq c_b \leq (N - l).$$

There are $\#B_N$ such values of b. Hence, by the pigeonhole principle there is a value of c such that

$$b \in A + c$$

for at least

$$\frac{\#A_N \#B_N}{2(N - l) + 1} = \frac{\pi_A(N)\pi_B(N)}{2(N - l + \frac{1}{2})} > \frac{\pi_A(N)\pi_B(N)}{4N} > M$$

distinct values of b. That is

$$\#\{B \cap (A + c)\} > M$$

as desired. ∎

Now specializing to the case

$$B = \mathscr{P} = \{\text{prime numbers}\}$$

and using

$$\pi_{\mathscr{P}}(x) \sim \frac{x}{\log x}$$

we learn

Corollary 2.3. *If A satisfies*

$$\limsup_{i \to \infty} \frac{\pi_A(i)}{\log i} = \infty \tag{2.4}$$

then given any M there is a $c \in \mathbf{Z}$ such that the sequence

$$A + c$$

contains at least M primes.

Note that if

$$a_i = e^i$$

then

$$\pi_A(i) \sim \log i$$

so the condition (2.4) requires, essentially, that the a_i grow slower than e^i. For example, (2.4) is satisfied by

$$a_i = [f(i)]$$

where f is any function of the form

$$f(x) = e^{x^{k_1}(\log x)^{k_2}} p(x)$$

where p is a polynomial with positive leading coefficient and $k_1 < 1$.

Before leaving this section, we consider the relationship between Theorems 1.1 and 2.1. Since

$$A \succeq B \Rightarrow A \approx B$$

it would certainly be desirable if our sufficient conditions satisfied the same relation. That is

$$(1.2) \Rightarrow (2.2).$$

We complete this section with a proof that this is true.

Theorem 2.5. *Let A and B be 2 increasing sequences of integers. Then, if A and B satisfy* (1.2) *they satisfy* (2.2).

Proof: From (1.2)

$$\lim_{i \to \infty} \frac{a_i - a_{i-1}}{\pi_B(a_i)} = 0.$$

This implies that

$$\lim_{k \to \infty} \left[\frac{1}{k} \sum_{i=1}^{k} \frac{a_i - a_{i-1}}{\pi_B(a_i)} \right] = 0. \tag{2.5}$$

Since

$$i < k \Rightarrow a_i < a_k \Rightarrow \pi_B(a_i) \le \pi_B(a_k)$$

(2.5) implies

$$0 = \lim_{k\to\infty}\left[\frac{1}{k}\sum_{i=1}^{k}\frac{a_i - a_{i-1}}{\pi_B(a_k)}\right] = \lim_{k\to\infty}\frac{\sum_{i=1}^{k} a_i - a_{i-1}}{k\pi_B(a_k)} = \lim_{k\to\infty}\frac{a_k - a_0}{k\pi_B(a_k)}.$$

Therefore,

$$\lim_{k\to\infty}\frac{a_k}{k\pi_B(a_k)} = 0. \tag{2.6}$$

We note that $\lim_{k\to\infty} k/k + 1 = 1$ and, by definition,

$$k + 1 = \pi_A(a_k).$$

Hence, (2.6) is equivalent to

$$\lim_{k\to\infty}\frac{a_k}{\pi_A(a_k)\pi_B(a_k)} = 0$$

or, equivalently

$$\lim_{k\to\infty}\frac{\pi_A(a_k)\pi_B(a_k)}{a_k} = \infty.$$

Thus, the sequence

$$\frac{\pi_A(i)\pi_B(i)}{i}$$

is unbounded, which is precisely (2.2). ■

3. FINAL COMMENTS. The question remains open for sequences with exponential growth. That is:

Q1. For $r \geq 2$ one can find arbitrarily large numbers of primes in the sequences

$$r^n + c \qquad n = 0, 1, 2, \ldots ?$$

The answer is very likely "yes." We present a heuristic argument below. A fundamental question is whether it is possible to prove such a result using only the density of the prime numbers. In other words

Q2. Fixing $r \in \mathbf{Z}$, $r \geq 2$, is there a sequence B with

$$\pi_B(s) \sim \frac{s}{\log s}$$

such that

$$\{r^n\} \not\approx B.$$

I am not sure what to think about Q2. My first thought was that the answer is clearly "yes," but I have been unable to produce such a sequence B.

Now for the above-mentioned heuristic argument. The reader should be warned that the following discussion involves some large leaps of faith.

We begin with a simple lemma

Lemma. *Suppose that the function*

$$f(T) = \frac{1}{T}\sum_{b\in B}\left(\pi_A(b + T) - \pi_A(b - T)\right)$$

is unbounded (*note that* f *takes values in* $[0, +\infty]$). *Then*

$$A \approx B.$$

Proof: Fixing M, choose T such that

$$f(T) \geq 2M.$$

Then for each $b \in B$ and each of the

$$\pi_A(b + T) - \pi_A(b - T)$$

values of $a \in A$ satisfying

$$b - T < a \leq b + T$$

there is a $c_{a,b} \in [-T, T)$ such that

$$b = a + c_{a,b}$$

so that

$$b \in A + c_{a,b}.$$

By considering all values of b, we arrive at

$$Tf(T) > 2MT$$

such $c_{a,b}$'s. However, there are only $2T$ possible values for $c_{a,b}$, so some value of c occurs at least M times, and we learn

$$\#\{B \cap (A + c)\} \geq M. \qquad \blacksquare$$

The above argument shows that if

$$f(T) = +\infty$$

then there is a $c \in [-T, T)$ such that

$$\#\{B \cap (A + c)\} = \infty.$$

Now we specialize to the case of interest. For any sequence of the form

$$A_\alpha = \{a_n = r^n + \alpha\}$$

where $\alpha \in \mathbf{Z}$, we have

$$\pi_{A_\alpha}(s) \sim \log_r(s - \alpha) = \gamma \log(s - \alpha)$$

where $\gamma = (\log r)^{-1}$. Fix T, then for large b

$$\pi_{A_\alpha}(b + T) - \pi_{A_\alpha}(b - T) \sim \gamma(\log(b + T - \alpha) - \log(b - T - \alpha)) \sim \frac{2\gamma T}{b - \alpha}$$

(by Taylor series). Thus

$$f(T) \sim 2\gamma \sum_{b \in B} \frac{1}{b - \alpha}.$$

Therefore, if $\sum_{b \in B} 1/b$ diverges (so that $\sum_{b \in B} 1/(b - \alpha)$ diverges) then for any T we "should" have $f(T) = +\infty$. This would imply that for any T there is a $c \in [-T, T)$ such that

$$\#\{B \cap (A_\alpha + c)\} = \infty.$$

Taking $T < 1$, we must have $c = 0$. Thus, if

$$\sum \frac{1}{b} = +\infty$$

we expect

$$\#\{B \cap A_\alpha\} = \infty.$$

The flaw in this argument is that for $T < 1$

$$\pi_{A_\alpha}(b + T) - \pi_{A_\alpha}(b - T) = 0 \quad \text{or} \quad 1$$

depending on whether $b \in A_\alpha$. Our approximation

$$\gamma(\log(b + T - \alpha) - \log(b - T - \alpha))$$

is only correct "on average." Hence our conclusion is correct only for suitably "random" sequences.

Now we specialize to the case

$$B = \mathscr{P}.$$

By a theorem of Euler ([2] Theorem 414)

$$\sum_{\text{primes } p} \frac{1}{p} = +\infty.$$

The question is whether A_α and $\mathscr{P}$ are "random" enough. The primes $\mathscr{P}$ appear to be randomly distributed according to the distribution implied by the prime number theorem. (I warned you about the leaps of faith.) Thus, we guess that for "random" α

$$\#\{\mathscr{P} \cap A_\alpha\} = \infty.$$

This, of course, goes far beyond an affirmative response to Q1. An extreme optimist might conjecture, based on the above analysis, that as long as α and r are relatively prime, the sequence

$$r^n + \alpha \qquad n = 0, 1, 2, \ldots$$

contains infinitely many primes. Such questions are very difficult. The simplest case, $r = 2$ and $\alpha = 1$, is a famous long-standing conjecture. These are the so-called Fermat primes. The conjecture is likely false in this case, as $2^n + 1$ is not "random" enough. In particular, $2^n + 1$ has a chance of being prime only if $n = 2^m$ for some m ([2] Theorem 17).

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Parabolic Mirrors, Elliptic and Hyperbolic Lenses

Mohsen Maesumi

The functioning of parabolic mirrors and antennas are based on one of the many wonderful properties of conic sections. It is well known that if a mirror is in the shape of a conic and a beam of light emanates from one of its focuses and is reflected by the mirror, then the reflected beam (or its extension) passes through the second focus. For a parabola one focus is at infinity, hence a beam of light that is parallel to the axis of parabola will be focused at the (finite) focus.

We may wonder about the shape of lenses with a similar focusing property. First we note that a major difference exists between a lens and a mirror. For a lens the path of light depends on the color (i.e. wavelength) of light as well as the properties of the glass used, while for a mirror the path is independent of the color. This complication has made it a challenge to design a simple perfect lens [1]. We can, however, design a lens to focus a beam of light of a given single wavelength. The design is simplest if the beam is deflected by only one surface, that is if the incident side of the lens is a flat plane perpendicular to the beam. In this case we show that the curved side of the lens is a conic section.

The propagation of light can be explained by either of the following principles:

(I) Fermat's Principle: A ray of light travels a path between two given points requiring a minimum time.

(II) Huygens' Principle: Light propagates as a wave front. At any time t each point on the wave front is the center of a semi-circular wavelet of radius cdt, where c is the local speed of light. The envelope of the wavelets forms the wave front at time $t + dt$. The rays of light are perpendicular to the wave fronts. The time of flight for *any* ray of light from the position of the wave front at time t to the one at time T is simply $T - t$. (See Figure 1a.)

Either principle can be used to deduce Snell's Law of refraction. This law states that at the interface between two transparent media $\sin\alpha/\sin\beta = v_o/v_g = n$. (See Figure 1b.) Here v_g is the speed of light inside the glass, v_o is the speed outside of the glass and n is called the index of refraction of the interface. v_o, v_g and n depend on the wavelength of light and the properties of the two media.

To describe the lens we assume it is a solid of revolution about the y-axis and consider its intersection with the $x - y$ plane. We will see that there are two possibilities, as shown in Figure 2a and 2b. To consider both cases simultaneously, we define $\eta = -1$ for Figure 2a and $\eta = 1$ for 2b. We assume the incident side of the lens is flat and is perpendicular to the y-axis. The vertex of the lens is at the origin O. The beam of light travels in the positive y direction and focuses at $F = (0, f)$, $f > 0$.

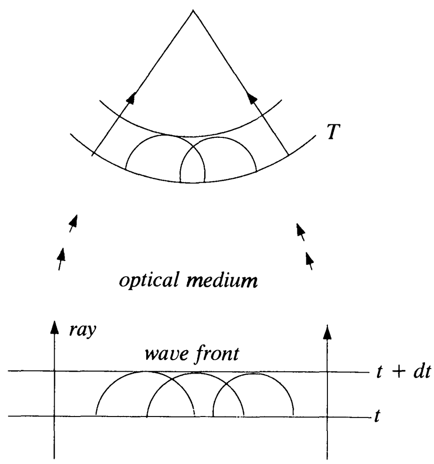

Figure 1a

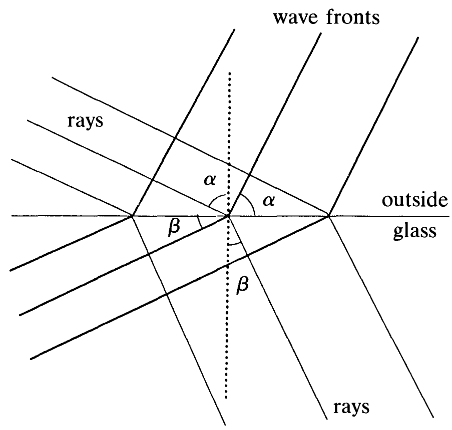

Figure 1b

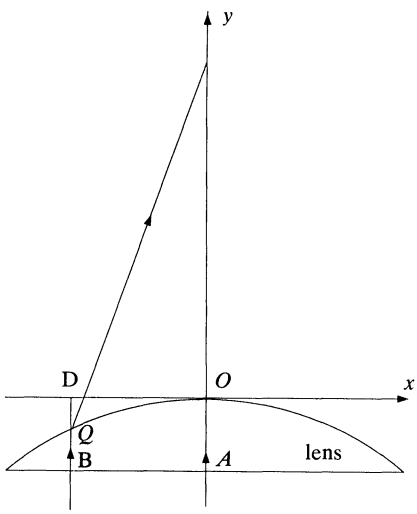

Figure 2a

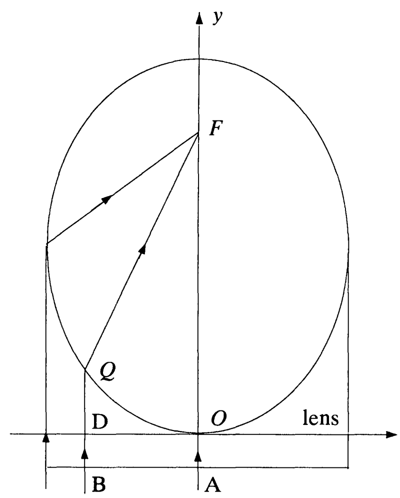

Figure 2b

Proposition. *The curved side of the lens in Figure* $2a$ *is a hyperbola and for Figure* $2b$ *it is an ellipse. In both cases the index of refraction of the lens is equal to the eccentricity of the conic section, and the focus of the lens coincides with a focus of the conic.*

Proof: This solution is based on the directrix definition of conics. The time it takes for a ray of light to go from A to F is $\tau_A = |AO|/v_g + |OF|/v_o$, and from B to F the elapsed time is $\tau_B = |BQ|/v_g + |QF|/v_o$. Since the line AB and the point F, by itself, are wavefronts then $\tau_A = \tau_B$. Noticing that $|BQ| - |AO| = \eta|QD|$ we get $|QF|/v_o + \eta|QD|/v_g = |OF|/v_o$. We write this in Cartesian coordinates. If $Q = (x, y)$ then $\eta|QD| = y$ and we obtain

$$\frac{\left((y-f)^2 + x^2\right)^{1/2}}{v_o} + \frac{y}{v_g} = \frac{f}{v_o}. \tag{1}$$

We assume $v_o \neq v_g$ and define

$$\zeta = \operatorname{sgn}(v_g - v_o), \qquad a = f\left(\frac{|v_o - v_g|}{v_o + v_g}\right)^{1/2}, \qquad b = f\frac{v_g}{v_g + v_o}. \tag{2}$$

Now (1) can be written as

$$\frac{x^2}{\zeta a^2} + \frac{(y-b)^2}{b^2} = 1, \qquad y \leq b. \tag{3}$$

Hence for $v_g < v_o$, $\zeta = -1$ and the lens is hyperbolic (Figure 2a). For $v_g > v_o$, $\zeta = 1$ and the lens is elliptic (Figure 2b). The length of axes are $2a$ and $2b$, and $\zeta = \eta$. The lens is formed from the lower half of each conic. The center of either conic is at $(0, b)$ and the two foci are on the y-axis at $(0, b \pm c)$, where

$$c = \left(b^2 - \zeta a^2\right)^{1/2} = f\frac{v_o}{v_g + v_o} = f - b. \tag{4}$$

Therefore the "upper" focus of the conic is at $(0, b + c) = (0, f)$, i.e. the focus of the lens coincides with a focus of the conic. The eccentricity of the conic section is $e = c/b$, while the index of refraction of the lens is $n = v_o/v_g$. From (2) and (4) it follows that $e = n$. QED.

Since the eccentricity of the conic should match the index of the refraction and the latter depends on the color of light, a single homogeneous lens cannot be used to focus all colors. Moreover the lens that we described is not reversible. The design of a symmetric lens results in an algebraic-differential equation.

ACKNOWLEDGMENT. The author would like to thank the referee for helpful suggestions.

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LETTERS

Conway's Challenge

In volume 98, number 1, of the *American Mathematical Monthly* an article by Colin L. Mallows appears with the title "Conway's Challenge Sequence" (see pages 5–20). I found the article most interesting and enjoyable. The reason I am writing to you is that a step is missing from the sequence of arguments which leads to the final result.

Consider the proof of the theorem at the end of section 5 (page 11). The object is to show that Rules M, $\overline{M}$, and L, generate the same sequence as Rule C. (Rule C was previously shown to produce the differences of Conway's sequence.) However, the proof mentions but never actually shows that $M(d_{k-2})$ and $\overline{M}(d_{k-2})$ are copies of D_{k-1}. What the proof does show is that, assuming $M(d_{k-2}) = D_{k-1}(F_k)$ and $\overline{M}(d_{k-2}) = D_{k-1}(G_k)$, the two copies of D_{k-1} will interleave properly and give us $L(d_{k-2}) = D_k$.

It remains to be shown that the assumption, $M(d_{k-2}) = D_{k-1}(F_k)$ and $\overline{M}(d_{k-2}) = D_{k-1}(G_k)$, is correct. While I believe the assumption is true, I do not have a proof.

I hate to bring up what is essentially an oversight in an otherwise most elegant discussion. Still, I felt it best to bring it to your attention.

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Mallows Replies

Richard E. Stone has pointed out that there is a serious gap in the proof of the main result in my paper [**1**]. The following development, based on an observation of Donald Girod, (personal communication, 10/14/91) fills the gap, and provides some new insights into the structure of the Conway sequence.

Define a triangular array $\begin{bmatrix} n \\ k \end{bmatrix}$, $n \geq 1$, $0 \leq k \leq n$ of strings of 1's and $-$'s by setting

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1, \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \mathrm{c}\left(\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \begin{bmatrix} n-1 \\ k \end{bmatrix}\right)$$

where "c" is the concatenation operator. Thus:

$$
\begin{array}{ll}
\begin{bmatrix} 2 \\ k \end{bmatrix}: & 1 \quad\quad 1\,- \quad\quad - \\
\begin{bmatrix} 3 \\ k \end{bmatrix}: & 1 \quad\quad 1\,1\,- \quad\quad 1\,-\,- \quad\quad - \\
\begin{bmatrix} 4 \\ k \end{bmatrix}: & 1 \quad\quad 1\,1\,1\,- \quad\quad 1\,1\;\;-\,1\,-\,- \quad\quad 1\,-\,-\,- \quad\quad - \\
\begin{bmatrix} 5 \\ k \end{bmatrix}: & 1 \quad 1\,1\,1\,1\,- \quad 1\,1\,1\,-\,1\,1\,- \quad 1\,-\,-\,1\;\;1\;\;-\,1\,-\,-\,1\,-\,-\,- \quad 1\,-\,-\,-\,-\,- \quad -
\end{array}
$$

Clearly $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$, $\begin{bmatrix} n \\ n \end{bmatrix} = -$; for $1 \le k \le n-1$, $\begin{bmatrix} n \\ k \end{bmatrix}$ starts with a 1 and ends with a $-$. Also length $\left(\begin{bmatrix} n \\ k \end{bmatrix}\right) = \binom{n}{k}$, and $\begin{bmatrix} n \\ k \end{bmatrix}$ contains exactly $\begin{bmatrix} n-1 \\ k \end{bmatrix}$ 1's and $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ $-$'s. We observe that $c_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}$ appears to be the sequence D_n of my paper. To establish this, we can prove (by easy induction)

$$c\left(\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \begin{bmatrix} n-1 \\ k \end{bmatrix}\right) = C\left(\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \begin{bmatrix} n-1 \\ k \end{bmatrix}\right)$$

where "C" is derived from Rule C in my paper: cut the first-argument string after every 1, cut the second-argument string after every $-$, and interleave the pieces, starting with the first piece of the first argument.

Thus this "concatenate" version of the Pascal triangle generates the successive parts of (the differences of) the Conway sequence.

The following result is also easily established by induction.

It now follows, by another easy induction, that

$$M'\left(\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\right) = \begin{bmatrix} n \\ k \end{bmatrix}$$

where M' is a more regular version of Rule M in [1];

$$
\begin{aligned}
M'(1) &= 1\,- \\
M'(2) &= 2 \leftarrow 1\,- \\
M'(3) &= 3 - 2 - 1\,-
\end{aligned}
$$

etc., and

$$M'(-n) = -n$$

The claim in [1] that $M(d_{k-1}) = d_k$ now follows.

By the way, a better name would be the "Newman-Conway" sequence, since it appears that David Newman [2] and John Conway invented it independently.

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Nowhere-Differentiable Functions

This is a comment on the paper "Continuous Nowhere-Differentiable Functions—an Application of Concentration Mappings," (May, 1991) by H. Katsuura [1].

The example of the paper is just one of a class of functions called *Kieswetter Curves* [2] or *fractal interpolating functions* [3].

The general construction of these functions is as follows. Given a set of points in $\mathbb{R}^2$ to be interpolated, $(x_0, y_0), (x_1, y_1), \ldots, (x_N, y_N)$, with $(x_0, y_0) = (0,0)$, $(x_N, y_N) = (1,1)$, and $x_0 < x_1 < \cdots < x_N$, define functions w_n by

$$w_n\colon \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x_n - x_{n-1} & 0 \\ 0 & y_n - y_{n-1} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}, \qquad n = 1, 2, \ldots, N.$$

For any initial compact set S_1 in $\mathbb{R}^2$, the sequence of sets $\{S_n\}$ is defined by

$$S_{n+1} = w_1(S_n) \cup w_2(S_n) \cup \cdots \cup w_N(S_n), \qquad n = 1, 2, \ldots .$$

If $\max |y_n - y_{n-1}| < 1$, it is well-known that this is a contraction mapping in the Hausdorff metric [4], and consequently converges to a *unique* limit set S_0, which is *independent* of the choice of the initial set. The set $\{w_1, \ldots, w_N\}$ is an example of what is called an *iterated function* system (IFS) in [3], and the limit set S_0 is called the attractor of the IFS. This is a standard method of constructing fractals. In the present case the attractor is a Kieswetter "curve." If the initial set S_1 is chosen to be the line from $(0,0)$ to $(1,1)$, then the iterates are all curves in the usual sense.

For equally spaced abcissas, $|x_n - x_{n-1}| = 1/N$, $n = 1, \ldots, N$, it can be shown that the *fractal dimension* (or *Hausdorff-Besicovitch dimension*) D is given by

$$\left(\frac{1}{N}\right)^{D-1} \sum_{n=1}^{N} |y_n - y_{n-1}| = 1.$$

[Remark: the more general result for unequally-spaced abcissas claimed in the theorem on pp. 225–226 of [3] does not appear to be correct.]

For the example of Katsuura, the interpolation points are $(0,0)$, $(1/3, 2/3)$, $(2/3, 1/3)$, $(1,1)$, and $D = \log 5/\log 3$, a number strictly between dimensions one (lines) and two (areas). Another interesting example has the interpolation points $(0,0), (1/3, 1/2), (2/3, 1/2), (1,1)$. The Kieswetter curve $f(x)$ generated by these points is the familiar *Cantor function*, or *Devil's staircase*, with dimension $D = 1$. It *is* differentiable almost everywhere, with $f'(x) = 0$ a.e., and is the standard example of a function which is both continuous and monotone (hence of bounded variation), but is not absolutely continuous, because absolutely continuous functions have derivatives which satisfy the fundamental theorem of calculus, and clearly

$$1 = f(1) - f(0) \neq \int_0^1 f'(x)\,dx = 0.$$

Other fractal interpolating functions can be used to give examples of space-filling curves (see [3], p. 240 ff.).

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MARY SOMERVILLE

Clarity and a gift for organizing masses of specialized information into short, lucid accounts are the outstanding features of her scientific writings. She was an expositor of science rather than a popularizer of science. In all her scientific books her chief purposes are alike and clear-cut: (i) to present an account of "the present state" of the science, together with whatever background material, definitions, diagrams and drawings are necessary to render it understandable to any tolerably educated reader and (ii) to show various important connections or dependences between that "present state" and other knowledge. In doing so she uses in almost every instance the vocabulary and terminology of the advanced scientific practitioners of the time. Her style is simple and direct, uncoloured—save for occasional passages in the last two editions of *Physical Sciences*—by a Victorian need to preach or prettify.

From an article by Elizabeth C. Patterson in *The British Journal for the History of Science*, Vol. 4, No. 16, 1969.

UNSOLVED PROBLEMS

Edited by: **Richard Guy**

In this department the MONTHLY presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial or related results. Typescripts should be sent to Richard Guy, Department of Mathematics & Statistics, The University of Calgary, Alberta, Canada T2N 1N4.

The Gordon Game of a Finite Group

John Isbell

Several papers have been published on the problem of *sequencing* a finite group G. Informally, this is hopping around G (x_1 to x_2 to ...), like a knight's tour of a chessboard; but instead of the steps $x_i^{-1}x_{i+1}$ being knight's moves, they are required to be all different. Thus all non-identity elements of G occur as steps.

Basil Gordon invented the problem and settled the abelian case; that is, he showed that (finite) abelian G can be sequenced if and only if $[G: 2G] = 2$. For the cyclic groups Z_{2k}, one sequencing is 0 to 1 to $2k - 1$ to 2 to $2k - 2 \ldots$ to $k - 1$ to $k + 1$ to k. He also found the only known non-sequenceable nonabelian groups, namely S_3, D_4, and Hamilton's quaternion group H (the three smallest nonabelian groups) [3]. Some maximal recent results are that nonabelian groups of order > 8 and ≤ 32 can be sequenced [1] and that dihedral D_n can be sequenced if $n > 3$ and $n \not\equiv 0 \pmod 4$ [4].

There have been only some conversations and letters on competitive sequencing of G by two players moving alternately. Precisely, the *Gordon Game* $\Gamma(G)$ is played as follows. A counter is placed on the identity e, or 0, of the group G. A player, White, moves it to another element x_1 of G. In general, $x_1, x_2, \ldots, x_n$ having been played, for odd (even) n Black (resp. White) moves the counter, if possible, to a group element x_{n+1} such that

(1) $x_{n+1} \notin \{e, x_1, x_2, \ldots, x_n\}$, and

(2) $x_n^{-1}x_{n+1} \notin \{x_1, x_1^{-1}x_2, \ldots, x_{n-1}^{-1}x_n\}$.

The first player unable to move loses.

The theoretical winner of the Gordon Game is known only in finitely many cases, and only by brute force. However, there is a conjecture that seems interesting, especially because it comes with a plausibility argument that offers several faces for criticism. **Conjecture**: Black wins almost all Z_p. ('Almost all' should mean 'all but finitely many'. However, the plausibility argument is probabilistic, and one might want to consider relative density.)

Why should Black, the second player, win Z_p? There is an 'argument from ignorance', like Laplace's well known argument that the probability that the sun

will rise tomorrow is $(n+1)/(n+2)$ where n is the number of known previous sunrises—starting from $1/2$ when nothing is known, because the two cases (rise; not rise) can only be considered equally likely. Of course, arguments from ignorance are treacherous. In a sense, we are not ignorant of anything about Z_p, but as a practical matter, if it is ignorance on which we must depend, we have enough to last for millennia.

It seems worth taking a moment to consider a fallacious argument from ignorance and its refutation. "**White should win in most large groups. For he has the first move, with $n-1$ choices in an n-element group. After the first move (as well as before it) the outcome is determined, assuming best play. For White to lose, all of his $n-1$ different possibilities for first move must be losing moves. For large n, this is very unlikely.**"

Whatever value that argument may have, it is unsound for Z_p. For White certainly does not have $p-1$ really different choices at the first move. The automorphisms of Z_p are transitive on nonzero elements; so all opening moves for White are equivalent.

However, the automorphisms of Z_p are just barely transitive on nonzero elements (a unique automorphism takes 1 to $g \neq 0$). So we come to the argument for Black's winning. "**In the unique game which Black faces after White's first move in $\Gamma(Zp)$, the $p-3$ possible opening moves are all different (i.e. inequivalent by automorphisms). For large p, it is very unlikely that all are losing moves.**"

Actual study of random games is harder work, perhaps worth pursuing. It seems worth noting two easy results. (1) Take a binary tree of length n, and label the 2^n maximal elements with "Win" or "Lose" by standard coin tossing. Then the probability that the first player can force a win is $(1/2)(1-(-1)^n)+o(1)$. (2) If the tree is not binary but, as in $\Gamma(G)$, the number of alternatives decreases as you go $(n+1, n, \ldots, 2, 0)$, the result is the same except that "$o(1)$" is smaller (for $n>2$). —If the 'model' of (2) is further complicated to resemble $\Gamma(Z_p)$ more closely, then its solution will be a better guide to the solution of $\Gamma(Z_p)$. Or (*pace* Hilaire Belloc) "perhaps it will not; I am not quite positive which".

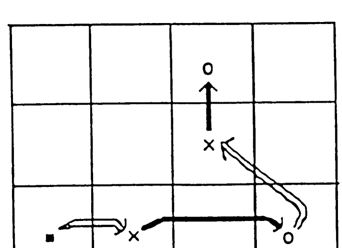
What is the length of $\Gamma(G)$? The **remoteness** [2, p. 258; references p. 278] is the numbers of moves (*half*-moves) the game takes if the theoretical winner aims (rationally) to win as soon as possible while the loser aims to survive as long as possible. Thus its parity tells you the winner, White if odd, Black if even. The lengths of $\Gamma(G)$ for very small G are as follows.

$Z_2, 1$; $Z_3, 1$; $Z_2 \oplus Z_2, 2$; $Z_4, 3$; $Z_5, 3$; $S_3, 3$; $Z_6, 4$; $Z_7, 4$; $Z_2 \oplus Z_4, 5$; $H, 5$; $D_4, 6$;
$Z_8, 6$; $Z_2^3, 6$; $Z_9, 6$; $Z_3^2, 6$; $Z_{10}, 7$; $Z_{11}, 8$; $Z_{13}, 10$.

One could conjecture—something perhaps to be sooner answered—

¿ Black wins Z_p for prime $p>5$?

Here is an illustration in $\Gamma(Z_{12})$. 12 is just past the frontier of knowledge at present. Also, since $Z_{12} \approx Z_4 \oplus Z_3$, it is easy to draw pictures.



If play begins:

1. White, (0, 0)	to (1, 0)	(*by* (1, 0))	Black	to (3, 0)	(*by* (2, 0))
2. White	to (2, 1)	(*by* (3, 1))	Black	to (2, 2)	(*by* (0, 1)),

then White has four possible next moves: to (0, 1), or (1, 1), or (1, 2), or (3, 1). (He cannot move the counter to any of the four places it has been or leave it where it is, or move it to (0, 2) or (2, 0) or (3, 2) since the required increments, respectively (2, 0), (0, 1), (1, 0), have already been used.) Three of these are losing moves. It is time for

3. White to (3, 1)!

Black's legal responses are (0, 2), which loses after White moves to (0, 1); (1, 2), which loses after White moves to (2, 0); and (2, 0), refuted by (0, 1). I do not know, though, if Black has thrown away the game with the opening moves shown.

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Committee—a group of men who individually can do nothing but as a group decide that nothing can be done.

—*Fred Allen*

PROBLEMS AND SOLUTIONS

Edited by:
Richard T. Bumby, Fred Kochman and Douglas B. West

Proposed problems should be sent to the MONTHLY PROBLEMS address given on the inside front cover. Please include solutions, relevant references, etc. Three copies are requested.

Solutions of published problems should arrive before November 30, 1992 at the MONTHLY PROBLEMS address given on the inside front cover. Solutions should be typed with double spacing, including the problem number and the solver's name and mailing address. Two copies suffice. A self-addressed postcard or label should be included if an acknowledgement is desired.

*An asterisk (*) after the number of a problem, or part of a problem, indicates that no solution is currently available. Partial solutions will be useful in such cases. Otherwise, the published solution is likely to be based on a solution which is complete and correct. Of course, an elegant partial solution or a method leading to a more general result is always useful and welcome. In addition, references to other appearances of MONTHLY problems or to solutions of these problems in the literature are also solicited.*

PROBLEMS

10229. *Proposed by Herman Bavinck, Delft University of Technology, Delft, The Netherlands.*

Given that m and p are integers with $m \geq p \geq 1$, evaluate

$$\sum_{j=1}^{p} \binom{1/2}{m-j+1}\binom{1/2}{m+j}.$$

10230. *Proposed by Peter L. Montgomery, University of California, Los Angeles, CA, and J. L. Selfridge, Northern Illinois University, DeKalb, IL.*

Find all perfect numbers of the form $n^n + 1$, where n is a positive integer.

10231. *Proposed by Adrian Riskin, Northern Arizona University, Flagstaff, AZ.*

For positive integers m and n, let

$$f(m,n) = \sum_{k=1}^{\infty} k^n \left(\frac{m}{m+1}\right)^k.$$

(a) Prove that $f(m, n)$ is an integer.

(b) Show that the last digit of the decimal expansion of $f(1, n)$ can only be 0, 2 or 6.

10232. *Proposed by Serge Zakharov, Tumen State University, Tumen, Russia.*

Let M_n be the n by n matrix whose (i, j)-entry is $\operatorname{lcm}(i, j)$. Evaluate $\det(M_n)$.

10233. *Proposed by M. A. Khan, RDSO, Lucknow, India.*

For any odd positive integer $n = 2r - 1$, prove that

$$\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \binom{n}{k} \sum_{j=0}^{r-1} (-1)^j \binom{n}{j} (r-j)^{n-k} = \frac{n!}{2}.$$

10234. *Proposed by Götz Trenkler, University of Dortmund, Dortmund, Germany.*

Let A and B be nonnegative definite Hermitian matrices such that $A - B$ is also nonnegative definite. Show that $\operatorname{tr}(A^2) \geq \operatorname{tr}(B^2)$.

10235. *Proposed by Daniel Goffinet, Saint Étienne, France.*

(a) Determine the set $\mathscr{F}$ of those continuous maps f from $\mathbb{R}^2$ to $\mathbb{R}$ such that, for every rectangle $ABCD$, one has $f(A) + f(C) = f(B) + f(D)$.

(b) Let $KLMN$ be a quadrangle in the plane such that $f(K) + f(M) = f(L) + f(N)$ for every $f \in \mathscr{F}$. Is it true that $KLMN$ must be a rectangle?

10236. *Proposed by M. J. Pelling, University College, London, England.*

(a) Let $f \in L^1(\mathbb{R})$ have period 2π. Suppose that, for a given x and s, the function $\phi(u) = f(x + u) + f(x - u)$ is differentiable in an interval $(0, \delta)$, and that $\lim_{u \to 0} \phi(u) = 2s$ and $\lim_{u \to 0} u\phi'(u) = 0$. Prove that the Fourier series for f converges to s at x.

(b) Give an example for which the test in (a) succeeds while de La Vallée Poussin's test (and *a fortiori* Jordan's and Dini's tests) fails.

(c) Let $f(x) = \Sigma c_n x^n$ be a real power series such that Σc_n converges. By Abel's theorem, it follows that f is continuous on $[0, 1]$. Construct an example where $f(x)$ fails to be of bounded variation on $[0, 1]$.

10237. *Proposed by Paul R. Chernoff, University of California, Berkeley, CA.*

Consider the Laplace transform $\mathcal{L}$ as an operator on $L^2(0, \infty)$. Show that $\mathcal{L}$ is a bounded self-adjoint operator which is unitarily equivalent to the "position operator" X = multiplication by the coordinate x on $L^2(-\sqrt{\pi}, \sqrt{\pi})$.

NOTES

(10230) A "perfect number" n is one like 6 or 28 for which the sum of all divisors of n is $2n$. Both Dickson, *History of the Theory of Numbers*, and Shanks, *Solved and Unsolved Problems in Number Theory*, begin with a study of this definition,

which is traced back to Euclid. **(10232)** A well-known result with a similar flavor is that the n by n matrix D_n whose (i, j) entry is $\gcd(i, j)$ has $\det(D_n) = \prod_{i=1}^{n}\phi(i)$. **(10234)** The point of this problem is that the desired inequality on the trace holds although the matrix $A^2 - B^2$ may not be nonnegative definite (see "Hermitian matrix inequalities and a conjecture" by N. N. Chan and Man Kam Kwong, this MONTHLY, 92(1985), 533–541. **(10235)** Following the usual convention, the quadrilaterals are named by their vertices with (cyclicly) adjacent vertices joined by an edge. Such a quadrilateral is a rectangle if consecutive edges are perpendicular. In particular, if $ABCD$ is a rectangle, then $ACBD$ is not (except in degenerate cases). **(10236)** Further details on the convergence tests for Fourier series referred to here may be found in N. K. Bari, *A Treatise on Trigonometric Series* (Vol. I). Jordan's and Dini's tests are discussed in sections 38 and 39 of Chapter I; de La Vallée Poussin's test can be found in sections 1–3 of Chapter III. **(10237)** The Laplace transform is defined by $(\mathcal{L}f)(x) = \int_0^\infty e^{-xt}f(t)\,dt$. In this problem, x is restricted to satisfy $0 < x < \infty$ and f is restricted to satisfy $\int_0^\infty |f(t)|^2\,dt < \infty$. The problem of finding an explicit representation for such a position operator as a Carleman integral operator is mentioned by Halmos and Sunder, *Bounded Integral Operators on L^2 spaces*, Springer, 1978, p. 99.

SOLUTIONS

Constructing Special Points on a Hyperbola

E 2980 [1983,54]. *Proposed by Jordi Dou, Barcelona, Spain.*

Given the points A_1, A_2, A_3, M and the line s, construct P, Q such that $\overline{PQ}$ is equal and parallel to A_1M and $\overline{P_1Q_1} = \overline{P_2Q_2} = \overline{P_3Q_3}$, where P_i, Q_i are the intersections of PA_i, QA_i with s.

Describe the locus of the point M for which the problem has a solution when A_1, A_2, A_3 and s are known (fixed).

Solution by the proposer. To avoid the consideration of degenerate cases, we suppose that A_1, A_2, A_3 are distinct points not on s and that the lines A_2A_3, A_3A_1, A_1A_2 are distinct lines not parallel to s. In order that the projectivity on s as a section of the projectivity π between the pencils of lines $P(A_1, A_2, A_3)(\pi/\Lambda)Q(A_1, A_2, A_3)$ be a translation it is necessary and sufficient that s be an asymptote of the conic H formed by the homologous lines of π. [For definitions, notation, and constructions see any standard reference on projective geometry such as O. Veblen and J. W. Young, *Projective Geometry*, volumes I and II.]

For equality of the directed segments P_iQ_i it is necessary and sufficient that P and Q be points of the conic H that is uniquely determined by A_1, A_2, A_3 and asymptote s.

We consider the involution J on conic H determined by the pencil of lines parallel to A_1M. The projection of the pairs of points of J onto the line A_1M and

parallel to the asymptote s of H is an involution J_1. The center (or limit point) of J_1 is the intersection point L of lines s and A_1M. A pair of J_1 will be A_1, A_1' where A_1' is the second intersection of line A_1M with conic H.

On A_1M we construct a pair P_1Q_1 of J_1 so that $P_1Q_1 = A_1M$ (that is, $\overline{LQ_1} \cdot \overline{LP_1} = LA_1 \cdot LA_1'$ and $LP_1 - LQ_1 = A_1M$). The intersection points of the lines parallel to s through P_1, Q_1 with conic H give P, Q. The points $P'Q'$ symmetric to P and Q with respect to the center H provides a second solution.

For the locus of points M of which the problem has no solution, it is enough to notice that the involution J_1 is elliptic only when the direction of A_1M falls within the angle of the asymptotes containing the curve H. In this case the minimum distance $\overline{P_1Q_1}$ of a pair P_1, Q_1 of J_1 is precisely the diameter of the conic H parallel to A_1M. Therefore the locus is the part of the plane between the asymptotes and the curve of a hyperbola with center A_1 homothetic with H with ratio 2.

Editorial comment. The proposer also included the details of a straightedge and compass construction of the solution.

No other solutions were received

Expressing n as a Sum of Two Squarefree Positive Integers Relatively Prime to n

6623 [1990,162]. *Proposed by Ernesto Bruno Cossi, Universidade Federal do Rio Grande do Sul, Porto Alegre, Brazil, and the editors.*

Let $R(n)$ denote the number of ways of expressing the positive integer n (greater than 1) as a sum of two squarefree positive integers relatively prime to n. Is it true that $R(n) > c\phi(n)$ for some positive constant c, where ϕ denotes the Euler function?

Composite solution by Joachim Herzog and Paul R. Smith, University of Frankfurt, Germany, Richard Stong, University of California, Los Angeles, CA, and the editors. We interpret $R(n)$ to be the number of ordered pairs of squarefree positive integers j, k such that $j + k = n$ and $(j, n) = (k, n) = 1$; for example, $R(2) = 1$, $R(3) = R(4) = R(5) = R(6) = 2$, $R(7) = R(8) = 4$, $R(9) = R(10) = 2$. We show that

$$R(n) \geq \{1 + o(1)\}\{12/\pi^2 - 1\}\phi(n) \tag{1}$$

for large n, so that $R(n) > \phi(n)/5$ for large n. To take care of values of n of moderate size we establish the following two additional facts:

$$R(n) > \phi(n)/200 \quad \text{for } n \geq 6000000, \tag{A}$$

$$R(n) \geq 2 \quad \text{for } 2 < n \leq 6000000. \tag{B}$$

Assertions (A) and (B) show that a constant c having the desired property exists.

We remark that actually the following asymptotic formula holds

$$R(n) = \alpha\phi(n)\prod_{p|n}(1 - 2/p^2)^{-1} + O(n^{\theta}), \tag{2}$$

where $\theta < 1$ and $\alpha = \prod\{(1 - 2/p^2): p \text{ prime}\}$. Formula (2) shows that $R(n) > 0.3\phi(n)$ for large n. Empirically the smallest value of $R(n)/\phi(n)$ appears to be $R(91)/\phi(91) = 5/18 = 0.2777\ldots$.

Lemma 1. *For $n \geq 3$ let*

$$g(n) = \sum_{\substack{k \leq n \\ (k,n)=1}} \mu^2(k).$$

Then $R(n) \geq 2g(n) - \phi(n)$.

Proof: By the definition of g there are $g(n)$ values of k between 1 and $n - 1$ inclusive such that k is squarefree and $(k, n) = 1$. By a change of variable in the sum defining $g(n)$, there are $g(n)$ values of k between 1 and $n - 1$ inclusive such that $n - k$ is squarefree and $(n - k, n) = 1$. Hence there are at least $2g(n) - \phi(n)$ values of k between 1 and $n - 1$ inclusive such that both k and $n - k$ are squarefree and relatively prime to n.

Lemma 2. *If $Q(x)$ denotes the numbers of squarefree positive integers not exceeding the positive number x and if $\omega(n)$ is the number of distinct prime factors of the positive integer n, then we have*

$$g(n) \geq 6\pi^{-2}\phi(n)\prod_{p|n}(1 - 1/p^2)^{-1} - \phi(n)(n^{-1/2} + n^{-1}) - Q(\sqrt{n})2^{\omega(n)-1}.$$

Proof: We require the preliminary result

$$\sum_{\substack{j \leq x \\ (j,n)=1}} 1 = x\phi(n)/n + E_n(x), \qquad |E_n(x)| < 2^{\omega(n)-1}, \tag{3}$$

which follows from the identity

$$\sum_{\substack{j \leq x \\ (j,n)=1}} 1 = \sum_{j \leq x} \sum_{d|(j,n)} \mu(d) = \sum_{d|n} \mu(d)\lfloor x/d \rfloor$$

$$= x\phi(n)/n + \sum_{d|n} \mu(d)(\lfloor x/d \rfloor - x/d)$$

and the remark that in the last sum there are exactly $2^{\omega(n)-1}$ values of d for which $\mu(d) = +1$ and exactly $2^{\omega(n)-1}$ values of d for which $\mu(d) = -1$. From (3) we get

$$g(n) = \sum_{\substack{k \leq n \\ (k,n)=1}} \sum_{d^2|k} \mu(d) = \sum_{\substack{d \leq \sqrt{n} \\ (d,n)=1}} \mu(d) \sum_{\substack{j \leq n/d^2 \\ (j,n)=1}} 1$$

$$= \sum_{\substack{d \leq \sqrt{n} \\ (d,n)=1}} \mu(d)\{\phi(n)/d^2 + E_n(n/d^2)\}$$

$$= \phi(n) \sum_{(d,n)=1} \mu(d)/d^2 - \phi(n) \sum_{\substack{d > \sqrt{n} \\ (d,n)=1}} \mu(d)/d^2$$

$$+ \sum_{\substack{d \leq \sqrt{n} \\ (d,n)=1}} \mu(d)E_n(n/d^2).$$

Since

$$\sum_{(d,n)=1} \mu(d)/d^2 = \prod_{p|n}(1 - 1/p^2) = 6\pi^{-2}\prod_{p|n}(1 - 1/p^2)^{-1}$$

and

$$\sum_{d>\sqrt{n}} |\mu(d)|/d^2 \le 1/n + \sum_{d>\sqrt{n}+1} 1/d^2 < 1/n + 1/\sqrt{n},$$

the inequality of the lemma follows.

Since

$$\sqrt{n}\,2^{\omega(n)} \le n^{3/4} \prod_{p|n} (2/p^{1/4}) \le n^{3/4} \prod_{p<17} (2/p^{1/4}) < 5n^{3/4} = o(\phi(n)),$$

Lemma 2 gives $g(n) \ge 6\pi^{-2}\phi(n) + o(\phi(n))$. Lemma 1 then gives (1).

We proceed to the inequality given in (A). We first note that $6/\pi^2 = 0.607927\ldots$. Thus if $n \ge 6000000$, Lemma 2 gives

$$g(n) \ge 0.60792\phi(n) \prod_{p|n} (1 - 1/p^2)^{-1}$$
$$- 0.00042\phi(n) - Q(\sqrt{n})2^{\omega(n)-1}. \tag{4}$$

Now, using the inequality $Q(x) \le 6/\pi^{-2}x + x^{1/2}$ proved in L. Moser & R. A. MacLeod, "The error term for the squarefree integers," *Canad. Math. Bull.* 9(1966), 303–306, we obtain

$$Q(\sqrt{n})/\sqrt{n} < 6/\pi^2 + 1/n^{1/4} < 6\pi^2 + 0.022 < 0.630 \tag{5}$$

for $n > 6000000$. Combining (4) and (5), we have

$$g(n) \ge 0.6075\phi(n) \prod_{p|n} (1 - 1/p^2)^{-1} - 0.630n^{1/2}2^{\omega(n)-1}$$
$$> 0.5025\phi(n) + 0.105\phi(n) \prod_{p|n} (1 - 1/p^2)^{-1} - 0.630n^{1/2}2^{\omega(n)-1}.$$

Now

$$\frac{0.630n^{1/2}2^{\omega(n)-1}}{0.105\phi(n)\Pi_{p|n}(1 - 1/p^2)^{-1}} = \frac{3}{n^{1/2}} \prod_{p|n} \frac{2(p+1)}{p}$$
$$\le 3n^{-1/4} \prod_{p|n} \{(2p+2)/p^{5/4}\}$$
$$\le 3n^{-1/4} \prod_{p<20} \{(2p+2)/p^{5/4}\}$$
$$< 3n^{-1/4}(16.49) < (6000000/n)^{1/4}.$$

Hence, if $n > 6000000$, we have $g(n) \ge 0.5025\phi(n)$ and so by Lemma 1

$$R(n) \ge 2g(n) - \phi(n) \ge \phi(n)/200.$$

Thus (A) is established.

We establish (B) by using a computer to verify that every n in [7,6000000] can be expressed in at least one way in the form $n = p + q$, where q is squarefree and p is a prime not dividing n and less than $\min(n/2, 100)$. (There are 25 primes not exceeding 100, at most seven of which can divide n when $n < 6000000$.) The editors wish to thank Kevin Ford for carrying out this computation.

Both Herzog and Smith and Stong showed how to reduce the amount of computation required by a judicious use of Lemmas 1 and 2.

Herzog and Smith included a complete proof of (2) in their solution.

No other solutions were received

E 3403 [1990,847]. *Proposed by Paul Erdős, Hungarian Academy of Sciences, Budapest.*

It is well known that the maximum size of a subset of $\{1, 2, \dots n\}$ containing no pair i, j with $i \mid j$ is $\lfloor (n+1)/2 \rfloor$. Prove that the maximum size of a subset of $\{1, 2, \dots, n\}$ containing no pair i, j with $i \mid 2j$ is $4n/9 + O(\log n)$ for large n.

Solution by Richard Stong, University of California, Los Angeles, CA. Let $\alpha(m)$ be the number of times 2 divides the positive integer m, i.e., let $2^{\alpha(m)}$ be the highest power of 2 dividing m. Define

$$T_n = \{m \in \mathbb{Z}^+ \colon n/3 < m \leq n \text{ and } \alpha(m) \text{ is even}\}.$$

We show that T_n contains no pair i, j with $i \mid 2j$ and that no larger subset of $\{1, 2, \dots, n\}$ has this property.

First suppose that $i, j \in T_n$, $i \neq j$, and $i \mid 2j$. Then $\alpha(i) \leq \alpha(j) + 1$. Since $\alpha(i)$ and $\alpha(j)$ are even, $\alpha(i) \leq \alpha(j)$ and so $i \mid j$; that is, the quotient j/i is an integer greater than 1. But, on the one hand, $T_n \subset (n/3, n]$, so that $j/i < 3$, and, on the other hand, $\alpha(i)$ and $\alpha(j)$ are both even, so that $j/i \neq 2$. Thus our supposition that $i \mid 2j$ is untenable. Hence T_n contains no pair i, j with $i \mid 2j$.

For any $k \in Z^*$ with $3 \nmid k$ and $\alpha(k)$ even, let

$$F_k = \{m \in \mathbb{Z}^+ \colon \text{either } m = 3^r \cdot k \text{ or } m = 2 \cdot 3^r \cdot k \text{ for some } r \geq 0\}.$$

Note that the sets F_k are disjoint and cover $\mathbb{Z}^+$, and also that if i and j are in F_k with $i < j$, then $i \mid 2j$. The latter fact shows that any subset of $\{1, 2, \dots, n\}$ containing no pair i, j with $i \mid 2j$ must intersect any F_k in at most one element. But the definition of T_n shows that T_n contains an element of $F_k \cap \{1, 2, \dots, n\}$ of the form $3^r \cdot k$ for each $k \leq n$ with $3 \nmid k$ and $\alpha(k)$ even. Thus a subset of $\{1, 2, \dots, n\}$ containing no pair i, j with $i \mid 2j$ cannot have more elements than T_n.

In order to obtain the assertion of the problem it therefore suffices to estimate $|T_n|$. For any interval $I \subset \mathbb{R}$ define $N(I)$ to be the number of odd integers in I. Then dividing by as many factors 4 as possible gives

$$|T_n| = \sum_{j=0}^{\infty} N\left(\left[2^{-2j}n/3, 2^{-2j}n\right]\right).$$

Consider replacing each term in the preceding sum by half the length of the corresponding interval. Doing so to all the terms with $j > \log_4 n$ introduces an error

$$\sum_{j > \log_4 n} 2^{-2j}n/3 = 4^{-\lfloor \log_4 n \rfloor} n/9 < 4/9.$$

Doing so with any term with $j \leq \log_4 n$ introduces an error of at most one. Therefore

$$\left| |T_n| - 4n/9 \right| < \log_4 n + 2,$$

which gives the assertion of the problem.

Editorial comment. If $f(n) = |T_n|$ is the maximum size of a subset of $\{1, 2, \dots, n\}$ containing no pair i, j with $i \mid 2j$, then the above solution shows that $f(n)$ is equal to the number of positive integers k such that $k \leq n$, $3 \nmid k$, and $\alpha(k)$ is even. By a

clever induction, J. L. Selfridge proved the sharp result that, on the one hand, we have

$$f(n) - 4n/9 \geq -(1/3)\log_4(n+1), \qquad (*)$$

with equality in $(*)$ if and only if $n + 1 = 2^{2r}$ $(r \geq 0)$, and that, on the other hand, if $n \neq 1, 7$ or 31, we have

$$f(n) - 4n/9 \leq (1/3)\log_4 n + C_5, \qquad (**)$$

where $C_5 = f(5) - 20/9 - (1/3)\log_4 5 = 0.39\ldots$. Equality holds in $(**)$ if and only if $n = 5 \cdot 2^{2r}$ $(r \geq 0)$. If we wish to include the exceptional cases $n = 1, 7$ and 31, we may write

$$-(1/3)\log_4(n+1) \leq f(n) - 4n/9 \leq (1/3)\log_4 n + 5/9$$

for all n. Selfridge's results show that $f(n) - 4n/9$ changes sign infinitely often.

Ossama A. Saleh and Terry J. Walters proved the following generalization of the assertion of the problem: If k is a fixed positive integer, the maximum size of a subset of $\{1, 2, \ldots, n\}$ containing no pair i, j with $i \mid 2^k j$ is

$$n/(3 - 3 \cdot 2^{-k-1}) + O(\log n).$$

Solved also by O. P. Lossers (The Netherlands), O. A. Saleh & T. J. Walters, J. L. Selfridge, and the proposer. One incorrect solution was received.

Flipping Tokens in Circles

E 3406 [1991,848]. *Proposed by Jeffrey Shallit, Dartmouth College, Hanover, NH.*

Consider three circles in the plane that intersect to form seven bounded regions. In each region there is a token that is white on one side and black on the other. At any stage the following two operations are permissible: (a) we can invert (flip over) all four tokens inside one of the three circles, or (b) we can invert those tokens showing black inside one of the three circles so that afterwards all tokens in that circle show white. From the starting configuration in which all tokens show white, can we reach the configuration in which all tokens show white except that the central region common to the three discs shows black?

Solution by Jyotirmoy Sarkar, Indiana University-Purdue University, Indianapolis, IN. The configuration cannot be reached.

Call a configuration "all-odds" if each of the three circles contains an odd number of black tokens. In particular the desired ending configuration (in which all regions but the central one show white) is all-odds. Since an operation of type (a) flips either two or four tokens in each of the three circles, it does not change the parity of the number of black tokens in any circle. On the other hand, an operation of type (b) results in an even number of black tokens in at least one circle and so the use of (b) at any stage precludes the possibility of ending up with an all-odds configuration. Hence an all-odds configuration can be the end result only when we have made merely operations of type (a) upon another all-odds configuration. Since the given initial configuration (all white) is not all-odds, the desired ending configuration cannot be reached from it.

Solved also by 45 readers and the proposer.

Closed Formulas for Certain Sums

E 3411 [1990,916]. *Proposed by Donald E. Knuth and Boris Pittel, Stanford University, Stanford, CA.*

Find a closed formula for

$$\sum_{k_1+k_2+\cdots+k_n=m} \frac{1}{k_1!\,(k_1!+k_2!)\cdots(k_1!+k_2!+\cdots k_n!)}$$

and

$$\sum_{k_1+k_2+\cdots+k_n=m} \frac{1}{2^{k_1}(2^{k_1}+2^{k_2})\cdots(2^{k_1}+2^{k_2}+\cdots+2^{k_n})}$$

where both sums are extended over all n-tuples of nonnegative integers with sum m.

Solution by Richard Stong, University of California, Los Angeles, CA. We require the identity of the following lemma.

Lemma. *For any positive numbers $x_1, x_2, \ldots, x_n$ we have*

$$\sum_{\sigma} \frac{1}{x_{\sigma(1)}(x_{\sigma(1)}+x_{\sigma(2)})\cdots(x_{\sigma(1)}+x_{\sigma(2)}+\cdots+x_{\sigma(n)})} = \frac{1}{x_1x_2\cdots x_n},$$

where the sum runs over all permutations σ of $\{1,2,\ldots,n\}$.

Proof: We proceed by induction on n, the lemma being trivial if $n=1$.

$$\begin{aligned}
&\sum_{\sigma} \frac{1}{x_{\sigma(1)}(x_{\sigma(1)}+x_{\sigma(2)})\cdots(x_{\sigma(1)}+x_{\sigma(2)}+\cdots+x_{\sigma(n)})}\\
&\quad= \sum_{k=1}^{n}\sum_{\{\sigma:\,\sigma(n)=k\}} \frac{1}{x_{\sigma(1)}(x_{\sigma(1)}+x_{\sigma(2)})\cdots(x_{\sigma(1)}+x_{\sigma(2)}+\cdots+x_{\sigma(n)})},\\
&\quad= \sum_{k=1}^{n} \frac{1}{x_1\cdots x_{k-1}x_{k+1}\cdots x_n(x_1+\cdots+x_n)} = \frac{1}{x_1x_2\cdots x_n},
\end{aligned}$$

where the second step follows by the inductive hypothesis. Thus the lemma is proved.

Since the given sums are symmetric in $k_1, k_2, \ldots, k_n$, applying the above lemma gives

$$\begin{aligned}
&\sum_{k_1+k_2+\cdots+k_n=m} \frac{1}{k_1!(k_1!+k_2!)\cdots(k_1!+k_2!+\cdots k_n!)}\\
&\quad= \frac{1}{n!}\sum_{k_1+k_2+\cdots k_n=m} \frac{1}{k_1!k_2!\cdots k_{s^n}!}\\
&\quad= \frac{1}{n!m!}\sum_{k_1+k_2+\cdots k_n=m} \binom{m}{k_1;k_2;\cdots;k_n} = \frac{n^m}{n!\,m!}
\end{aligned}$$

and

$$\sum_{k_1+k_2+\cdots k_n=m} \frac{1}{2^{k_1}(2^{k_1}+2^{k_2})\cdots(2^{k_1}+2^{k_2}+\cdots+2^{k_n})}$$
$$=\frac{1}{n!}\sum_{k_1+k_2+\cdots+k_n=m} 2^{-m}$$
$$=\frac{1}{n!2^m}\binom{m+n-1}{m}.$$

Solved also by 26 other readers and the proposer.

Only finitely many rows in Pascal's triangle consist exclusively of rth-power-free integers

E 3424 [1991,159]. *Proposed by Paul Erdős, Hungarian Academy of Science, Budapest.*

(i) Given an integer $r > 1$, prove that there is a positive integer n_r such that for every $n > n_r$, at least one of the binomial coefficients $\binom{n}{k}$, $1 \le k \le n-1$, is divisible by the r-th power of some prime.

(ii) Prove that n_2 can be taken as 23. (The binomial coefficients $\binom{n}{k}$, $1 \le k \le n-1$, are all squarefree when $n = 2, 3, 5, 7, 11, 23$.)

Solution by Charles Vanden Eynden, Illinois State University, Normal, IL. To prove (i) let p_m denote the mth prime and let $\alpha_1 = 1/p_1^r = 2^{-r}$ and $a_m = p_1 p_2 \dots p_{m-1}/p_m^r$ for $m > 1$. By Bertrand's Postulate (proved in Chapter 8 of Ivan Niven, Herbert S. Zuckerman, and Hugh L. Montgomery, *An Introduction to the Theory of Numbers*, Wiley, 1991) $p_{m+1} < 2p_m$ for all m. This gives

$$\frac{a_{m+1}}{a_m} = \left(\frac{p_m}{p_m+1}\right)^r p_m > 2^{-r}p_m > 2$$

for all sufficiently large m. Thus the sequence $\{a_m\}_{m=1}^{\infty}$ eventually increases rapidly, and hence there exists a positive integer t such that $a_m > 1$ for $m > t$.

We choose $n_r = p_t^r$. For $n > n_r$ let p_m be the smallest prime not dividing $n+1$. If $m > t$, we have $n + 1 \ge p_1 p_2 \cdots p_{m-1} > p_m^r$; if $m \le t$, we have $n > n_r \ge p_m^r$. Thus $n \ge p_m^r$ in either case. We now take $k = p_m^r - 1$, so that $1 \le k \le n-1$. Then

$$\binom{n}{k} = \frac{k+1}{n-k}\binom{n}{k+1} = \frac{p_m^r}{n+1-p_m^r}\binom{n}{k+1}.$$

Since $n+1-p_m^r$ is not divisible by p_m and $\binom{n}{k+1}$ is an integer, we have

$$p_m^r \mid \binom{n}{k}.$$

To obtain (ii) we note that when $r = 2$ the sequence $\{a_m\}$ begins $1/4, 2/9, 6/25, 30/49, 210/121$. Thus we may take $t = 4$ and $n_2 = 49$ in the above argument. The cases $n = 23, 25, \dots, 49$ can be handled easily by noting that (a) $\binom{31}{5}$, $\binom{43}{8}$ and $\binom{47}{21}$ are each divisible by 9 and (b) for all other values of n in the interval $[24, 49]$ at least one of the numbers $\binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \binom{n}{4}$ is divisible by either 4, 9, 25, or 49.

Editorial comment. Gerry Myerson observed that for $2 \le n \le 12$ the binomial coefficients $\binom{n}{k}$, $1 \le k \le n - 1$, are all squarefree if and only if n is prime. (See Example 39 in Richard K. Guy, "The Second Strong Law of Small Numbers", *Math. Magazine* **63**(1990) 3-20.) The present problem shows that this fails for $n = 13, 17, 19$, and all primes greater than 23.

Several solvers found explicit values for n_r. The best, found by Richard Stong, was 6^r.

Marijo Le Van showed that "for every $n > n_r$, at least one" in the statement of part (i) could be replaced by "for every $n > n_r(s)$, at least s" for any positive integer s.

Robert High and Thomas Honold each showed that the result follows quickly from the result of the later problem E3431 [1991,264].

Solved also by J. Bukor (Czechoslovakia), D. Callan, M. Dindos (Czechoslovakia), K. Ford (student), R. High, Th. Honold (Germany), M. LeVan, O. P. Lossers (The Netherlands), J. B. Muskat (Israel), R. Stong, N. Strauss, and the proposer.

The moving power of mathematical invention is not reasoning but imagination.

—*A. de Morgan*

REVIEWS

Edited by **Darrell Haile**
Indiana University, Bloomington, IN 47405

Mathematica in Action, by Stan Wagon, W. H. Freeman & Company, Inc., New York, 1991.

Exploring Mathematics with Mathematica, by Theodore W. Gray and Jerry Glynn, Addison-Wesley Advanced Book Program, Redwood City, CA, 1991.

Reviewed by **Bruce Solomon**

All of us have experienced how, after concentrating on lots of individual details, we suddenly grasp a theory on a level far above those details. All at once, the ideas become clear and satisfying; we behold a rich multi-dimensional structure of great beauty. But in the acquisition process we are like bugs on a TV screen, forced to reconstruct the communicator's mental image by crawling along it, line-by-line. It is rather miraculous that sometimes, after struggling long and hard over the stream of symbols, we actually manage to leap off the screen and perceive the big picture. And yet, no matter how many times we experience this, we can seldom do better than to communicate our own understanding by, again, encoding the image into thin, linear, one-dimensional trails of words and symbols, much as the picture tube paints its image. The bandwidth for mathematical communication is terribly pinched.

Exasperating as this is, there may never be a method for rapid, direct transfer of mathematical know-how from one mind to another. But I have had the opportunity, over the last couple years, to work with a "power tool" that promises to widen the bandwidth for some types of mathematical communication like nothing since moveable type. I refer to the *Mathematica* Notebook, available as part of the Front End written by Theo Gray for *Mathematica* on NeXT and Macintosh machines.

A *Mathematica* Notebook is an electronic mathematics document which one can read—but more important, interact with—on a computer screen. Textually, it offers many features of a word/outline processor and desktop publishing package. Graphically, it empowers both authors and their "reader-users" to rapidly create high quality 2- and 3-dimensional images. Most importantly, however, by dint of the natural interface it provides to *Mathematica*, the Notebook integrates these features into an efficient environment for rapid, well-documented mathematical exploration, be it symbolic, numerical, or algorithmic. By way of example, this entire book review was written as a *Mathematica* Notebook, printed out by *Mathematica*, and submitted to the *Monthly* in that form. Except for minor typesetting variations, what you have before you is a fairly accurate representation

of what a Notebook *looks like*. Note especially the integration of text, graphics, and *Mathematica* input/output on succeeding pages.

In granting control over a whole universe of examples that would otherwise be difficult or impossible to investigate, Mathematica Notebooks enable author and reader to cooperate much more actively than they can in a traditional book; experiments can be suggested, the user can easily and quickly try them, and a great deal more "discovery"—which is, of course, the primary joy of mathematics research—enters and accelerates the user's learning process. In fact, as Wagon writes in his preface, "so much can be done, that it may take a little time for our imaginations to catch up with the possibilities."

The books under review here, *Mathematica in Action* by Stan Wagon, and *Exploring Mathematics with Mathematica*, by Theo Gray and Jerry Glynn differ greatly in the relative balance they strike between Mathematics and *Mathematica*. But both seize the opportunity created by *Mathematica* Notebooks with gusto. Glynn/Gray do so more boldly: *Exploring Mathematics with Mathematica* really *is* a series of Notebooks, every byte of which is committed to the CD-ROM that comes with each copy of the book. The book was printed directly from the Notebooks using *Mathematica*, as Gray, primary designer of the Notebook Front End explains on page 8: "My feeling was that if there was no suitable format available for publishing electronic mathematical books, then it was about time we made one. Since I was working on Version 2.0 of *Mathematica* at the same time we were writing the book, I took the opportunity to make sure that *Mathematica* was such a format."

Of course Wagon, who used Version 1.2 of Mathematica, didn't have quite this opportunity when writing *his* book! Nevertheless, *Mathematica in Action* also breaks its mathematics out of the traditional theorem/proof/theorem/proof straitjacket, drawing the reader instead into the sort of Mathematica-mediated dialogue for which Notebooks are ideal.

* * *

Mathematica in Action visits a wide variety of topics, ranging from multivariable calculus, where the emphasis is on 3D graphics, to advanced undergraduate number theory, where it really shines brightly. There are treatments of the cycloid, with its interesting variational properties, complex Cantor sets and numerous other fractals, the lore of prime numbers, including an extended look at the Gaussian integers and an introduction to Riemann's zeta function, and much more. The book is full of good mathematical arguments and instructive *Mathematica* code, all very nicely, if tersely, written. It generally proceeds by using *Mathematica* to engage the reader in an active exploration of interesting mathematical material, exposing problems, raising questions, and suggesting exercises/experiments. Wherever possible Wagon works supporting theorems and proofs into his discussion in a pleasantly informal, but rigorous manner.

For instance, he dives immediately into his exploration of primes in Chapter One by introducing the Mathematica functions `Factorial[ ]` and `Mod[ ]`. After explaining the latter's syntax—*Mathematica* itself does so like this:

```
?Mod

Mod[m,n] gives the remainder on division of m by n.
The result has the same sign as n.
```

Wagon displays two calculations:

```
Mod[90!,91]        (* 91 = 13*7 *)
0
Mod[100!,101]      (* 101 is prime *)
100
```

Do these facts mean anything? As a geometer, I can admit without too much embarrassment that I puzzled over the second one until the text reminded me of Wilson's theorem: *p is prime if and only if* $(p-1)! \equiv -1 \ (mod\, p)$. Of course! I tried a few more examples and began to wonder: how fast would this theorem be at finding primes? I used it to `Select [ ]` all primes in the entire `Range [ ]` of integers between `101` and `201`, and timed the process:

```
Select[Range[101,201], (Mod[(# - 1)!,#] == # - 1)&]//
Timing
{0.15 Second,
 {101,103,107,109,113,127,131,137,139,149,
 151,157,163,167,173,179,181,191,193,197,199}}
```

Seems pretty fast. How does Mathematica's prime tester compare? Let's see:

```
Select[Range[101,201],PrimeQ]//Timing
{0.0333333 Second,
 {101,103,107,109,113,127,131,137,139,
 149,151,157,163,167,173,179,181,191,193,197,199}}
```

Much faster. In fact, Wagon explains that the "Wilson" test is unuseable for numbers having more than 50 digits—computing factorials of very large numbers is impractical. For curiosity's sake, though, how efficient is `PrimeQ` on numbers that big? Suppose we make a `Table [ ]` of 1000 random integers having 100 digits each, ask *Mathematica* to test each for primality, and `Select [ ]` the ones that pass:

```
bigNums = Table[Random[Integer,10^100,10^101], {i,1000}];
Select[bigNums,PrimeQ]//Timing
{536.0833333333334 Second,
 {201683051847648638098049801803777773356797013196749020\
  7433105140372806013530856602733085633410886803,

5248708346199953738592426002712756676662432600758383 61\
  0351208511810965788006314846926917256827961937,
```

```
43828861667756736674155750587485637966218480003026335 3\
  034490476427115902515030470412672754442586941 7,

810915893304467043168203571765817021961903443904558367\
  0994621972707023883783434294888889143867723729}}
```

Fast! 9 minutes to test 1000 hundred-digit integers for primality, with 4 hits. How many hits are expected according to the prime number theorem, which puts the density of primes near large `x` at approximately `1/Log[x]`? Here `x` ≈ `5*(10^100)`, so the density per thousand should be about

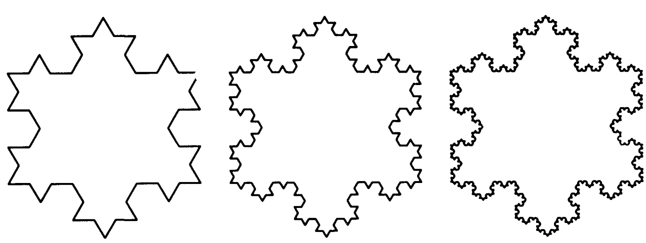
```
1000/(Log[5.] + 100Log[10.])

4.3128
```

Right on the money—just over 4 per thousand. Hmm ... would this be a good way to test random number generators in general?

I have digressed, but actually, that's the point! Digression, and exploration of otherwise inaccessible territory become almost inevitable when *Mathematica* mediates the communication. For me—and I suspect this will be true of many—the big picture comes into focus much more quickly when I can manipulate the message hands-on in this way.

As Wagon's book proceeds, he uses *Mathematica* in increasingly sophisticated ways. Much of the book is devoted to graphical methods, my favorite being the recursive, string-rewriting "turtle," which takes an alphanumeric string such as `"+f--f--f"`, rewrites it a specified number of times according to simple replacement rules like `"f" -> "f+f--f+f"`, and then interprets the result graphically by mapping `f` to a forward step, '`+`' to a left turn, and '`-`' to a turn right. If each turn alters the turtle's heading by $\pi/3$, for example, then the initial string above produces an equilateral triangle. One application of the replacement rule above turns it into a hexagram, and with 2, 3, and 4 applications of the rule, we find that a familiar sequence begins to materialize:

```
koch[n_]:=
recursiveTurtle[
  {"f" -> "f+f--f+f"}, "+f--f--f", n, N[Pi/3],
3.^(-n)
]
Show[GraphicsArray[Table[koch[i], {i,2,4}] ] ]
```

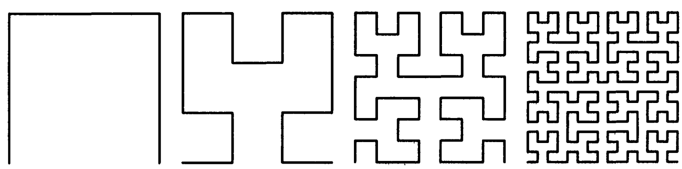


If instead, the initial string is `y` (a "do nothing" dummy symbol, as is `x` below), turns subtend 90 degrees, and the rewriting rules are

`"x" -> "-yf+xfx+fy-"` and `"y" -> "+xf-yfy-fx+"`,

then the results iterate toward a square-filling example due to Hilbert:

```
hilbert[n_]:=
  recursiveTurtle[
    {"x" -> "-yf+xfx+fy-", "y" -> "+xf-yfy-fx+"},
    "y", n, N[Pi/2], 2.^(-n), 4^n
  ]

Show[GraphicsArray[Table[hilbert[i], {i,1,4}] ] ]
```



In addition to putting this powerful turtle at our disposal—and clearly, there is a lot of room for exploration here—Wagon includes a healthy dose of serious mathematical discussion: computation of Hausdorff dimension, issues of convergence, and the lovely proof that the limit of the curve-sequence pictured above is, in fact, a continuous mapping from the interval *onto* the square.

Mathematica in Action is new kind of addition to the literature of Mathematics. Combining mathematical content and integrity with the interactive possibilities inherent in a good *Mathematica* Notebook, it is valuable both for its ideas, and as a stylistic precedent.

* * *

Next to Wagon's book, *Exploring Mathematics with Mathematica* by Gray and Glynn comes off as more of a *Mathematica* magic show. Great mathematical fun, certainly, but *Mathematica*—not Mathematics—takes center stage. This isn't surprising—unlike Wagon, neither Theo Gray nor Jerry Glynn is a research mathematician. But that's not really a drawback, either. Gray, as designer of the Notebook Front End is uniquely qualified to exploit the potential of that format, and he is tremendously creative in doing so here. Glynn is a math educator enthusiastically determined to bring the joy of math to the masses. It's a pleasure to see two non-mathematicians having such a ball with the subject, and in fact, they can teach us much about the opportunities Notebooks offer for mathematical communication.

Exploring Mathematica with Mathematics lightheartedly pushes the interactive concept implicit in Wagon's book to its classical Galilean limit, building chapters around casual dialogues between the authors. Still, the book makes a reasonable effort to back up its whiz-bang explorations with rigorous explanations, often by including separate discussions by "visiting mathematicians" Dan Grayson, of the Math Department at University of Illinois, Urbana-Champaign, and Jerry Kieper of Wolfram Research, Inc.

There are three main sections in the book, each containing several chapters which, on the CD-ROM version, are separate Notebooks. The first section deals with various simple phenomena arising out of iteration. For instance, they demonstrate the power of *Mathematica*'s `Nest[]` and `NestList[]` functions with a

familiar example from elementary number theory:

```
?NestList

NestList[f,expr,n] gives a list of the results of ap-
plying f to expr 0 through n times.

f[x_]:=1/(1+x);
NestList[f, x, 3]
```

$$\{x, \frac{1}{1+x}, \frac{1}{1+\frac{1}{1+x}}, \frac{1}{1+\frac{1}{1+\frac{1}{1+x}}}\}$$

```
NestList[f, 1, 12]
```

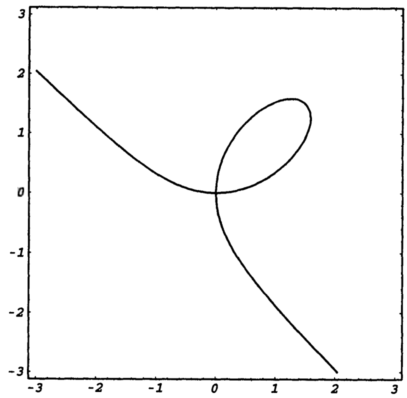
$$\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \frac{21}{34}, \frac{34}{55}, \frac{55}{89}, \frac{89}{144}, \frac{144}{233}, \frac{233}{377}\}$$

The numerators and denominators follow the Fibonacci sequence, and as is well known, the fractions themselves converge to the reciprocal of the Golden Ratio. But Gray & Glynn invent a wonderfully vivid way to illustrate this convergence, after computing the sequence `NestList[f,1,100]` to 50 decimal places. Namely, they apply *Mathematica*'s animation capability to display the expansions in rapid succession. One by one, like the flickering display on a slot machine, the decimal digits click into place!

This example illustrates one of the book's real strengths: familiar material takes on new life in these authors' hands. In the section on 'Sound and Graphics,' they really wax creative, though the emphasis is far more on "what can be done" rather than on the mathematics *per se*: we encounter intersecting surfaces with cutaways for better viewing, a dodecahedron suspended by threads joining the non-adjacent vertices of an enveloping iscosahedron, and 3D animations. I especially liked what they do with contour plots: they animate level-set diagrams for a one-parameter family of functions, and better still, show how `ContourPlot[]` can be used to "graph" implicit functions in the plane. For instance, here's how *Mathematica* renders the Folium of Descartes, defined in the 1974 edition of *CRC Standard Mathematical Tables* as the locus

$$f(x, y) = x^3 + y^3 - 3xy = 0.$$

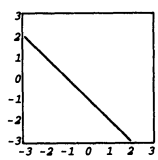
```
level[t_]:=
ContourPlot[x^3+y^3-3x*y, {x,-3,3},{y,-3,3},
  Contours->{t}, ContourShading->False,
  PlotPoints->50, ContourSmoothing->5,
  PlotLabel->"Folium of Descartes"
];
level[0]
```

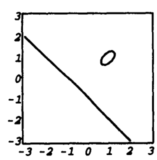


Folium of Descartes

Looks just like *CRC* said it would. But then, *CRC* couldn't animate a whole sequence of such pictures, showing the curve's changing topology as the function's value crosses zero. A book review can't quite do so either, but here's an approximation:

```
Show[
  GraphicsArray[
    Table[level[-1 + (3i + j)/4.], {i,0,2},{j,0,2}]
  ]
]
```


Folium of Descartes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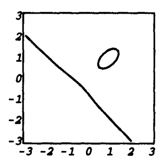
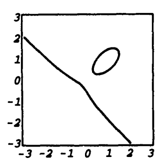
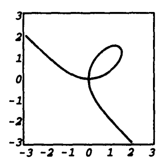

Folium of Descartes


Folium of Descartes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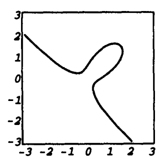
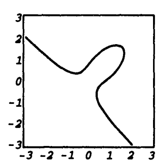
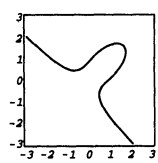
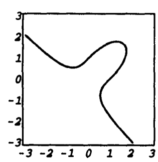

Folium of Descartes


Folium of Descartes


Folium of Descartes


Folium of Descartes

The availability of sound is entirely new with version 2.0 of *Mathematica*, and Glynn and Gray have quite a bit of fun with it. They investigate Shepard tones—the audio analogue of M. C. Escher's continuously rising staircase. They hear the difference between rational and irrational numbers by "listening" to their decimal expansions, each digit treated as a sampled amplitude. (Irrationals, having

aperiodic expansions, produce white noise; rationals produce distinct pitches.) They digitize a piece of Beethoven's Ninth Symphony (sounds beautiful), take its square root (after all—the piece *is* just a sequence of digits, i.e., a *big* number), and listen to *that*. What does it sound like? Well, just white noise—as Gray says, "this is a random number in every regard, except that when you square it, you get Beethoven's Ninth Symphony." And you haven't heard anything until you `Play[]` the amazing Riemann-Siegel function `RiemannSiegelZ[]`!

The book's last main section is called "Adventures in Mathematics." For me, its high points are the pretty section on the geometry of complex functions, which explores the images of polar and rectangular grids under analytic transformations, and a detective-like investigation of cyclotomic polynomials and their factorizations.

Despite the presence of these latter sections, though, *Exploring Mathematics with Mathematica* does not, on the whole, explore mathematics very deeply. What it does very well, however, is illustrate the tremendous power that *Mathematica* Notebooks make available for exploring, communicating and *presenting* mathematics. Gray & Glynn truly point the way toward bringing the beauty and fascination of mathematics to a much less sophisticated audience than has traditionally been "susceptible."

* * *

Before closing, a few words about equipment are in order. Unfortunately, Notebooks don't deliver their vast potential without a price. At the time of this writing, I believe a NeXT or a well-equipped Macintosh is required in order to run them well. Most of the plain code in either book will work on any *Mathematica* platform, but to run it comfortably undoubtedly requires considerable amounts of both memory and disk space. Wagon did the Right Thing and developed all his examples under *Mathematica* 1.2, and on a fairly modest machine: a Macintosh SE/30 with 8 megabytes of RAM. All the book's code is available on a Macintosh diskette for $5.00 from the author. I certainly had no trouble running any of the examples I tried on a NeXT.

Exploring Mathematics with Mathematica is a different story. One has the feeling that whenever the authors needed a more powerful machine to realize their ideas, they simply went out and bought one! Many of the book's examples are not terribly hardware-intensive, but numerous others require huge amounts of time to generate without access to the book's "pre-computed" electronic (CD-ROM) version. I found this to be true even on a monochrome NeXT 68040 with 16 megabytes of memory. The authors do warn readers about some—but by no means all—of these time-consuming examples.

As mentioned above, *Exploring Mathematics with Mathematica* in its entirety, plus a few little extras, comes on the CD-ROM disk included with every copy. At present, the disk may not do many users much good; CD-ROM drives are hardly standard equipment. I did manage to borrow a drive and download the whole book, but one doesn't put these notebooks onto floppies. Chapter Nine alone, with its color graphics, etc., spans 35 *mega*bytes. On the other hand, if you can get it, it's great to have the book available in its intended interactive format, especially for exploring the examples. Time-consuming sound and graphics needn't be recomputed by the user, and of course, it's nice to have access to all of the authors' code without having to type it in from the book.

From the software standpoint, *Exploring Mathematics with Mathematica* depends heavily on features specific to *Mathematica* 2.0. Many of its explorations will force owners of previous versions to modify the authors' approach. Examples involving sound will simply have to be skipped.

Gray and Glynn should not be faulted too heavily, however, for casting restraint to the wind and going for broke. When you're blazing a new trail, there's bound to be some rough spots. *Someone* had to be first.

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Our Apologies

The following diagram should have been included in Douglas Dunham's Review of *Visions of Symmetry: Notebooks, Periodic Drawings, and Related Work of M. C. Escher* by Doris Schattschneider in the January 1992 issue of the *Monthly*.

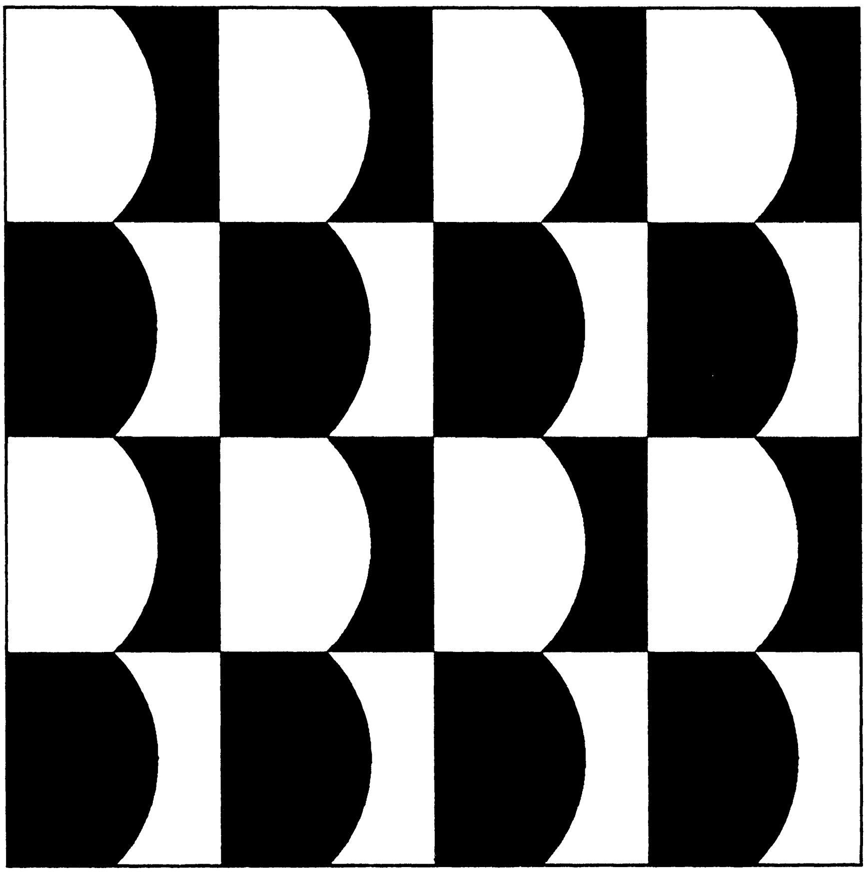

Figure 1. A periodic 2-color 2-motif pattern not of "Heaven and Hell" type.

TELEGRAPHIC REVIEWS

Edited by
Lynn Arthur Steen
with the assistance of
the Mathematics Departments of Carleton, Macalester, and St. Olaf Colleges

Telegraphic Reviews are designed to alert readers in a timely manner to new books and computer software appropriate to mathematics teaching and research. Special codes classify reviews by subject area and appropriate use:

T : Textbook	*P : Professional Reading*	*1-4: Semester*
C : Computer Software	*L : Undergraduate Library*	*** : Special Emphasis*
S : Supplementary Reading	*13: Grade Level*	*?? : Questionable*

Readers are advised that price information is subject to change. Selected books and software packages receive a second, more extensive review in the *Monthly*.

Books and software submitted for review should be sent to *Reviews Editor, American Mathematical Monthly, St. Olaf College, Northfield, Minnesota 55057.*

Mathematics Appreciation, S*, L*. *Game, Set and Math: Enigmas and Conundrums.* Ian Stewart. Penguin Books, 1989, viii + 191 pp, $9.95 (P). [ISBN: 0-14-013237-6] Paperback version of 1989 Basic Blackwell edition (TR, June-July 1990). Contains a collection of Stewart's columns translated from the French edition of *Scientific American.* LAS

Precalculus, T*(13: 2). *College Algebra with Trigonometry.* Paul K. Rees, Fred W. Sparks, Charles Sparks Rees. McGraw-Hill, 1991, xx + 724 pp, $42.85. [ISBN: 0-07-051737-1] Thoroughly readable. Has eight chapters in common with *College Algebra*, Rees and Sparks, 1990. Applications from a wide variety of disciplines are given. Some 6400 problems, most with answers. *Student's Solution Manual* and *Instructor's Resource Manual* available. DH

Education, P. *Kindergarten Book.* Grace Burton, *et al.* NCTM, 1991, viii + 24 pp, $9.50 (P). [ISBN: 0-87353-310-0] A series of hints for kindergarten to develop mathematical experiences consistent with the NCTM *Standards.* Focuses on patterns, number sense, data, geometry, and spatial sense. LAS

Education, P. *Epistemological Foundations of Mathematical Experience.* Ed: Leslie P. Steffe. Recent Res. in Psychology. Springer-Verlag, 1991, xvii + 312 pp, $45 (P). [ISBN: 0-387-97600-0] Twelve essays on the role of "reflective abstraction" in construction of mathematical knowledge. Essays span all educational levels, from primary school through college. Includes a composite list of references, name and subject of indices. LAS

History, P, L. *A History of Mathematics, Fifth Edition.* Florian Cajori. Chelsea, 1991, xi + 524 pp, $29.50. [ISBN: 0-8284-2303-6] Revisions since the *Fourth Edition* (TR, May 1986) are primarily in the chapter on Babylonian mathematics. An enduring brief classic, first published in 1919. LAS

Logic, S(17-18), P*, L. *Logic and Information.* Keith Devlin. Cambridge Univ Pr, 1991, xii + 308 pp, $34.50. [ISBN: 0-521-41031-4] A bold effort to restore logic as the science of "reasoning, thinking, and inference" by providing a "pre-mathematical" framework for a science of information. Building on a theory of "infons" (a "digitalization" of information) and "situations," logician Devlin writes with uncommon clarity for an interdisciplinary audience of linguists, computer scientists, philosophers, and mathematicians. LAS

Graph Theory, P. *Cycles and Bridges in Graphs.* Heinz-Jürgen Voss. Math. & Its Applic., V. 49. Kluwer Academic, 1991, xii + 271 pp, $112. [ISBN: 0-7923-0899-9] In-depth and advanced research on title topic. Builds from classic results for planar and

Hamiltonian graphs to latest research on separating cycles, cycle length and diagonals as a function of valency, and extremal results. JPH

Linear Algebra, S(13-15). *Ejercicios y problemas de álgebra lineal.* Jesús Rojo, Isabel Martín. Vector Ediciones (Carretera de Canillas, 134, 28043 Madrid), 1989, xi + 419 pp, (P). [ISBN: 84-86707-05-6] A course in linear algebra largely given as a set of exercises in each topic. Complete exposition of all solutions included. In Spanish. AD

Algebra, T(14-16: 1). *Elements of Modern Algebra, Third Edition.* Jimmie Gilbert, Linda Gilbert. PWS-Kent, 1992, xv + 364 pp, $40. [ISBN: 0-534-92888-9] Classical introductory course in groups, rings, integral domains, and fields ending with treatment of polynomials and algebraic field extensions. Plentiful exercises; many computational ones whose solutions are included. Little advanced material. (*First Edition*, TR, August-September 1984; *Second Edition*, TR, January 1989.) AD

Algebra, T*(16-17: 1, 2), S, P, L.** *Algebra.* Michael Artin. Prentice Hall, 1991, xviii + 618 pp. [ISBN: 0-13-004763-5] The culmination of several years of preparing supplementary notes for the standard abstract algebra course and the author's desire to incorporate "some concrete topics such as symmetry, linear groups, and quadratic number fields, and to shift the emphasis in group theory from permutation groups to matrix groups." The result is an innovative text that builds on concrete material (e.g., geometry), and combines linear algebra with groups, rings, and fields. Written for a mathematically mature undergraduate (say at the level of Herstein). There is more here than can be covered in a single year, however, much of it can be omitted without sacrificing the flavor. LCL

Calculus, S(13-14), C. *Discovering Calculus with HP-28 and the HP-48.* Robert T. Smith, Roland B. Minton. McGraw-Hill, 1992, x + 277 pp, $17.95 (P). [ISBN: 0-07-059179-2] A resource book on using the HP-28 and HP-48 calculators as tools for learning and applying elementary calculus. The first chapter is a useful, readable introduction to the machines themselves, with emphasis on graphics and functional manipulations. Remaining chapters investigate various calculus topics: limits, differentiation and applications, integration, series. Problem sets include both routine exercises and open-ended, "exploratory" problems—the latter usable as student "projects." Although exposition focuses, necessarily, on HP machines, many ideas are readily transferable to other platforms. PZ

Numerical Analysis, T(15-17: 1), L. *Scientific Computing and Differential Equations: An Introduction to Numerical Methods.* Gene H. Golub, James M. Ortega. Academic Pr, 1992, xi + 337 pp, $49.95. [ISBN: 0-12-289255-0] A revision of *Introduction to Numerical Methods for Differential Equations* by J.M. Ortega and W.G. Poole, Jr. Although focused on differential equations, most of the traditional topics in a first course in numerical analysis are covered. Introduces numerical methods for both ordinary and partial differential equations, but concentrates on ordinary differential equations, especially boundary value problems. AO

Functional Analysis, S(17-18). *Fundamentals of the Theory of Operator Algebras, Special Topics, Volume III: Elementary Theory—An Exercise Approach.* Richard V. Kadison, John R. Ringrose. Birkhäuser, 1991, xiv + 273 pp, $34.50. [ISBN: 0-8176-3497-5] Companion to *Volume I* (TR, April 1984) of same title, providing restatement and solution of each exercise in it. KS

Analysis, T(18), P, L. *Clifford Algebras and Dirac Operators in Harmonic Analysis.* John E. Gilbert, Margaret A.M. Murray. Stud. in Adv. Math., V. 26. Cambridge Univ Pr, 1991, vii + 334 pp, $75. [ISBN: 0-521-34654-1] Classical singular integral theory, representation theory, and analysis on manifolds are treated with a view to making this material accessible to clasically trained analysts. Topics include Clifford algebra theory, Hardy space theory and its extension to minimally smooth domains, representations of the spin and rotation groups, operators of Dirac type. Concludes with recent simplified proof of the local Atiyah-Singer index theorem. KS

Differential Geometry, S, L. *Differential Geometry.* Erwin Kreyszig. Dover, 1991, xiv + 352 pp, $8.95 (P). [ISBN: 0-486-66721-9] Republication of a 1959 monograph first published by the University of Toronto Press (TR, January 1970). Classical theory, pre-differential forms. Includes problems with answers in the back; also a

reference list of formulas, and a full index and references. A lot of good mathematics for the money. LAS

Geometry, S, P, L. *Geometry From Multiple Perspectives.* Arthur F. Coxford, Jr., *et al.* NCTM, 1991, vii + 72 pp, $14 (P). [ISBN: 0-87353-330-5] Innovative approaches to geometric shapes (triangles, quadrilaterals, polygons, solids, fractals), concepts (congruence, similarity, coordinates), and proof. Intended to help teachers implement ideas in the NCTM *Standards.* LAS

General Topology, P. *General Topology and Applications: Fifth Northeast Conference.* Eds: Susan J. Andima, *et al.* Lect. Notes in Pure & Appl. Math., V. 134. Marcel Dekker, 1991, xiii + 416 pp, $135 (P). [ISBN: 0-8247-8552-5] Proceedings of the Fifth Northeast Conference held June 15-17, 1989 at the College of Staten Island. Twenty-seven research papers, five from invited speakers; 80 participants. Index included. MC

Optimization, P. *Lecture Notes in Control and Information Sciences-163: The Autonomous Linear Quadratic Control Problem, Theory and Numerical Solution.* V. L. Mehrmann. Springer-Verlag, 1991, 177 pp, $29 (P). [ISBN: 0-387-54170-5] Research report and survey of the recent literature on the theory and numerical solutions of (discrete and continuous) autonomous optimal control problems with differential algebraic equation constraints. Develops techniques for solving by employing solutions of algebraic (or differential) Riccati equations; gives general algorithms ('expert system') for solutions for the control problems. RM

Stochastic Processes, S(18), P. *Numerical Solution of Markov Chains.* Ed: William J. Stewart. Pure & Appl., V. 8. Marcel Dekker, 1991, xvii + 704 pp, $145. [ISBN: 0-8247-8405-7] Papers from a Markov chain workshop covering most aspects of solving Markov models numerically. Topics include matrix generation techniques, generalized stochastic Petri nets, computation of stationary distributions (aggregation and disaggregation approaches, projection type methods, and conjugate gradient-based methods), recursive type methods, sensitivity analysis, the computation of transient solutions, bounds and approximations, computer communications models, and descriptions of relevant software packages. KB

Languages, P, L. *The C++ Programming Language, Second Edition.* Bjarne Stroustrup. Addison-Wesley, 1991, xi + 669 pp, (P). [ISBN: 0-201-53992-6] A guide to C++ written by the language's principal designer. Includes a tutorial introduction to the language, advice on using C++ for large-scale software projects, and the C++ reference manual. This edition reflects recent changes in the language definition. (*First Edition,* TR, January 1991.) AO

Computer Systems, P, L. *The Z-Mail Handbook.* Hanna Nelson. O'Reilly & Assoc, 1991, xxiii + 434 pp, $29.95 (P). [ISBN: 0-937175-76-5] Z-Mail is a turbocharged version of the standard UNIX "mail" system that provides a choice of three user interfaces: line mode (like traditional "mail"), full-screen mode (akin to the UNIX "vi" editor), and a graphics (GUI) mode (using the X-window system). This is a thorough, clear user manual for all three versions. LAS

Computer Systems, P*, L*. *The Joy of TeX: A Gourmet Guide to Typesetting with the TeX Macro Package, Second Edition.* M.D. Spivak. AMS, 1990, xxii + 309 pp, $38 (P). [ISBN: 0-8218-2997-1] Revisions from the *First Edition* (TR, May 1987) include many technical changes required to match Version 2.1 of the TeX macro package, particularly in options for the preprint style, whose expanded discussion is now given in Appendix A. LAS

Theory of Computation, T?(15-16: 1), S. *Computability Theory: Concepts and Applications.* Paul E. Dunne. Ser. in Comput. & Their Applic. Ellis Horwood, 1991, ix + 150 pp, $59. [ISBN: 0-13-161936-5] Basic introduction to computability theory, based on the Turing machine model, with emphasis on universality, undecidability, incompleteness. Some discussion of alternate models (Post systems, recursive functions). RM

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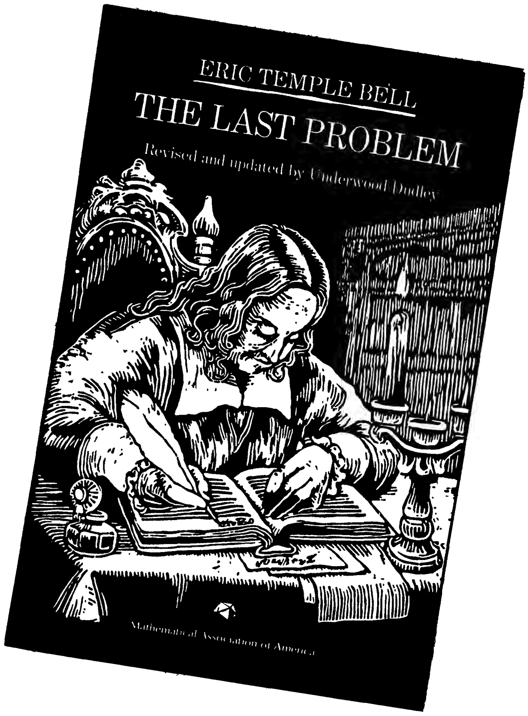

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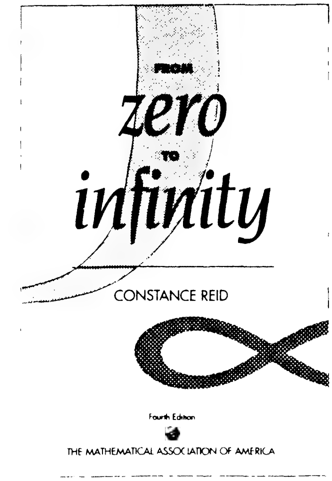

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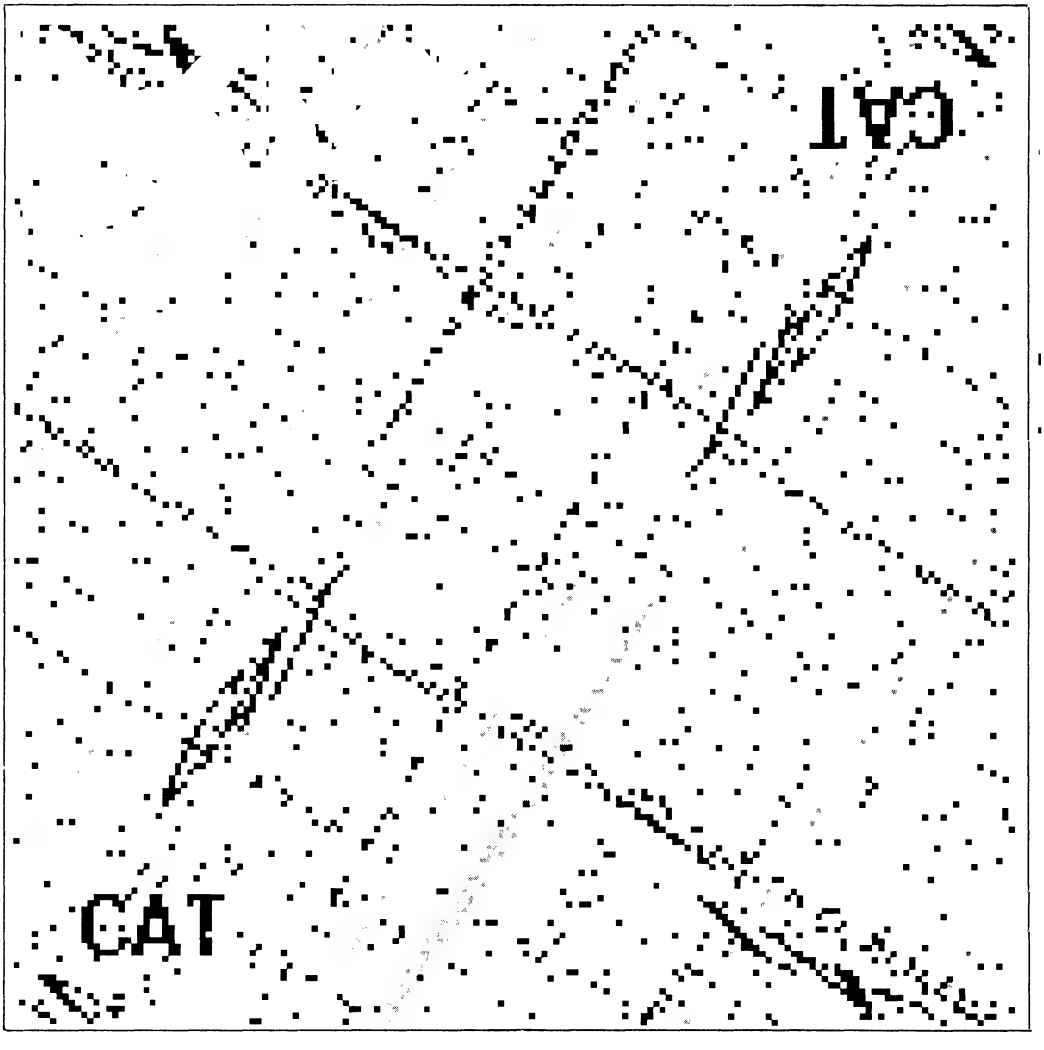

Discrete Cat Mappings (page 603)

NOTICE TO AUTHORS

The *Monthly* publishes articles, notes, and other features about mathematics and the profession. The readership of the *Monthly* is intended to include everybody who is mathematically inclined, including of course professional mathematicians and students of mathematics at all collegiate levels. While no single article or feature is likely to appeal to everyone, material should interest and be accessible to a large number of readers. This is the most important criterion for acceptance.

Articles may be expositions of old results or presentations of new ones. They may concern all of mathematics or one small area, a broad development or a single application, historical reminiscences or one important event. While some articles may contain the author's new research, the novelty of material and generality of the results is far less important than the clarity of exposition and general interest. Discussing one illuminating case of a well known result is far better than providing all the details of an obscure but new proposition. Articles in the *Monthly* are supposed to inform and to entertain; they are meant to be read rather than archived.

Notes are short and possibly informal articles. A note may concern a clever new proof of an old theorem, a novel way to present tired material, or a lively discussion of a philosophical (but still mathematical) issue. Also any topic is suitable, so long as it is related to mathematics. Because a note is short, the first few sentences are the most important part: They should explain the purpose and invite the reader in. Photographs or diagrams often will attract the reader's attention.

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Letters to the Editor, both for publication and for private reading, should be sent to the Editor at the address given above. Comments, including criticisms, are welcome, as are all suggestions for making the *Monthly* a lively, entertaining, and informative journal.

Cover: The 24 images of the word CAT under a simple automorphism of the torus. The automorphism is mixing, but not on the computer screen. The article by Dyson and Falk tells why.

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The [planning] process entails a mixture of priorities developed at different levels. Disciplinary priorities are articulated at the division level, filtered and coalesced at the directorate level, and refined at the agency level. At each step, the overlay of priorities developed outside the disciplinary context becomes stronger.

Judith Sunley, Director—Division of Mathematical Sciences, NSF

Dear Dr. Sunley:

I read your article in the April *Notices*—the one in which you encouraged mathematicians to give you some input. Well, I find these discussions about priorities and planning a little hard to understand—I suppose that comes from living in the midwest too long—but here I am with some input.

People seem to be pretty upset about next year's budget request, which asks for no increase for individual grant support and allocates what increases there are for special initiatives. I read that some people are saying the budget request is "regrettable" and they're calling the situation a "disaster" for mathematics. Those are tough words.

To make matters worse, you seem to be having a problem with your boss, Walter Massey, the Director of the National Science Foundation. On the one hand, you write: "There has been much discussion in the community in recent years about how to increase the number of investigators whose research is supported, with frequent suggestions that we decrease the size of awards if that is what it takes." On the other hand, Dr. Massey says (in the same *Notices*) that his "highest priority is to increase the support to individual investigators through larger grant size and extended award duration." He adds, "it does not require a mathematician to recognize that... increasing either size or the duration of grants will place pressure on the number of awards that we can make." (I guess he figures not all his readers are mathematicians because he goes on to spell out the details.)

Now you have a real problem here. Mathematicians are upset because more and more of them are not receiving grants. Not receiving grants makes people mad. Dr. Massey is upset because he wants bigger grants for longer periods, which means fewer mathematicians will get the chance for funding. And lacking the chance for funding makes the mathematicians even madder, and... well, you get the idea.

Is there some way out of this mess? I think so. Why not give *everyone* "a chance" for funding, just a chance. A lottery—it's worked for lots of states. Instead of agonizing over 2000 proposals each year, you can simply award say 6 million dollar grants to 10 lucky ticket holders. Charge a small fee for lottery tickets and you can raise more funds (and help reduce the deficit). Peer review? You can use referees to assign a number of lottery tickets to each proposal based on the rating. It will work, honest.

The winners (should we call them "Math Millionaires"?) are wildly enthusiastic, of course, and their universities can throw a lavish celebration (with the overhead). The losers are disappointed but not disheartened—there's always another chance next year (and your department won't count losing a lottery against you at salary time). Dr. Massey ought to be happy since it's pretty clear we're paying attention to his ideas. And *you*? You should be happy since the lottery means you can streamline that complicated planning process and maybe save a few bucks along the way.

Sincerely yours—**John Ewing**

Period of a Discrete Cat Mapping

Freeman J. Dyson and Harold Falk

1. INTRODUCTION. In studying the dynamics of a mechanical system one uses time averages and phase-space averages [**1**] to describe the evolution. The existence and properties of the averages are part [**2, 3, 4**] of ergodic theory. The latter theory is not restricted to mechanical systems described by Newton's laws of motion, but also deals with abstract dynamical systems such as the abstract dynamical system involving the following mapping [**4**].

Let (x, y) denote a point in the unit square. The mapping takes (x, y) to the new point

$$\begin{pmatrix} x' \\ y' \end{pmatrix} \equiv \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \pmod 1. \tag{1.1}$$

The mapping preserves area (measure $d\mu = dx\,dy$); is associated with a discrete-time flow on a torus; and provides an example of a hyperbolic toral automorphism [**4, 5**]. In an abstract sense the flow relates to the phase-space flow described by the Liouville Theorem [**6**].

Let $\vec{x}$ denote the initial point (x, y) and let $\vec{x}_n$ denote the image of $\vec{x}$ after n iterations of (1.1), $n = 0, 1, 2, \ldots$. The time average of a complex-valued function f, defined on the unit square and μ-integrable, is

$$\langle f(\vec{x}) \rangle_{\text{time}} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\vec{x}_n), \tag{1.2}$$

and the phase-space average of f is

$$\langle f \rangle = \int_{\text{unit square}} f(\vec{x})\, d\mu \tag{1.3}$$

Since phase-space averages are widely employed and play a prominent role in statistical mechanics, a natural question is: Is $\langle f \rangle$ equal to $\langle f(\vec{x}) \rangle_{\text{time}}$? The following concept of mixing [**4**] has been a useful tool in pursuing an answer to that question.

Let $\mathscr{A}$ denote a measurable subset of $\mathscr{M}$ ($\mathscr{M}$ is the unit square in our example, and $\mu(\mathscr{M}) = 1$). Let $\mathscr{A}_n$ denote the image of $\mathscr{A}$ after n iterations of the mapping (1.1). If for every pair of measurable subsets $\mathscr{A}$ and $\mathscr{B}$ of $\mathscr{M}$,

$$\lim_{n \to \infty} \mu(\mathscr{A}_n \cap \mathscr{B}) = \mu(\mathscr{A}) \mu(\mathscr{B}) / \mu(\mathscr{M}), \tag{1.4}$$

the mapping (more precisely, the dynamical system) is mixing.

For a mixing dynamical system view $\mathscr{A}$ as a two-dimensional ink droplet and $\mu(\mathscr{A})/\mu(\mathscr{M})$ as the "concentration" of ink in the unit square. Then after "many" iterations the ratio $\mu(\mathscr{A}_n \cap \mathscr{B})/\mu(\mathscr{B})$ (for $\mu(\mathscr{B}) \neq 0$) represents the concentration of ink in $\mathscr{B}$. According to (1.4), that concentration should also be

$\mu(\mathscr{A})/\mu(\mathscr{M})$. Thus, the ink drop has been somewhat uniformly "smeared" over the unit square.

The mixing property is heuristically demonstrated [**4**, **2**] by placing a picture of a cat in the unit square and then displaying several subsequent images resulting from the flow. The images show that the cat tends to become "smeared" over the unit square.

It has been shown [**4**] that the above hyperbolic toral automorphism is mixing, and mixing implies [**4**] that

$$\langle f(\vec{x})\rangle_{\text{time}} = \langle f\rangle, \quad \text{almost everywhere.} \tag{1.5}$$

A mapping having the above mathematical properties and connections with statistical mechanics has an "intellectual domain of attraction," and we were drawn in. This paper documents our pleasant experience.

The computer is a convenient device for demonstrating mappings, where the screen serves as a two-dimensional lattice of points (pixels). For the purpose of demonstration, consider a square lattice of points and denote the points by (x, y). Restrict x and y to the integer values $0, 1, \ldots, N-1$ with the operations of addition and multiplication performed (mod N). The mapping (1.1) is approximated by the mapping

$$\begin{pmatrix} x' \\ y' \end{pmatrix} \equiv \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \pmod{N} \tag{1.6}$$

where x and y are integers in $[0, 1, \ldots, N-1]$. N will typically be selected so as to make ample use of the capability of the screen; we take $N = 161$ as an example. Note that the computer deals precisely with the arithmetic operations of the mapping (1.6); the problem of round-off error does not arise.

Figure 1 displays "snapshots" of the early iterations of the mapping (1.6), starting with the initial "cat" configuration. The tendency to mix is evident, but one knows that the initial configuration must eventually return, since there are $2^{N\times N}$ possible configurations of the $N \times N$ pixels, where each pixel is either "on" or "off." However, for $N = 161$ the number $2^{N\times N}$ is large, and it was surprising to see the cat configuration return after only 24 iterations. This paper contains theorems which explain the observed periodicity.

It will be convenient to use the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{where } A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

and the Fibonacci sequence $u_0 = 0$, $u_1 = 1$, $u_2 = 1$, $u_3 = 2$, $u_4 = 3, \ldots, [u_{n+2} = u_{n+1} + u_n]$. Then the nth iteration of the mapping (1.6) is

$$A^{2n} = \begin{pmatrix} u_{2n-1} & u_{2n} \\ u_{2n} & u_{2n+1} \end{pmatrix} \quad (n = 1, 2, 3, \ldots). \tag{1.7}$$

For a given N the period m_N of the mapping (1.6) is the smallest positive integer n such that

$$\left.\begin{array}{ll} u_{2n} \equiv 0 & \pmod{N} \\ \text{and} & \\ u_{2n-1} \equiv 1 & \pmod{N}. \end{array}\right\} \tag{1.8}$$

[Note that (1.8) implies $u_{2n+1} \equiv u_{2n+2} \equiv 1 \pmod{N}$.] Thus, the period is related to the divisibility properties of Fibonacci numbers.

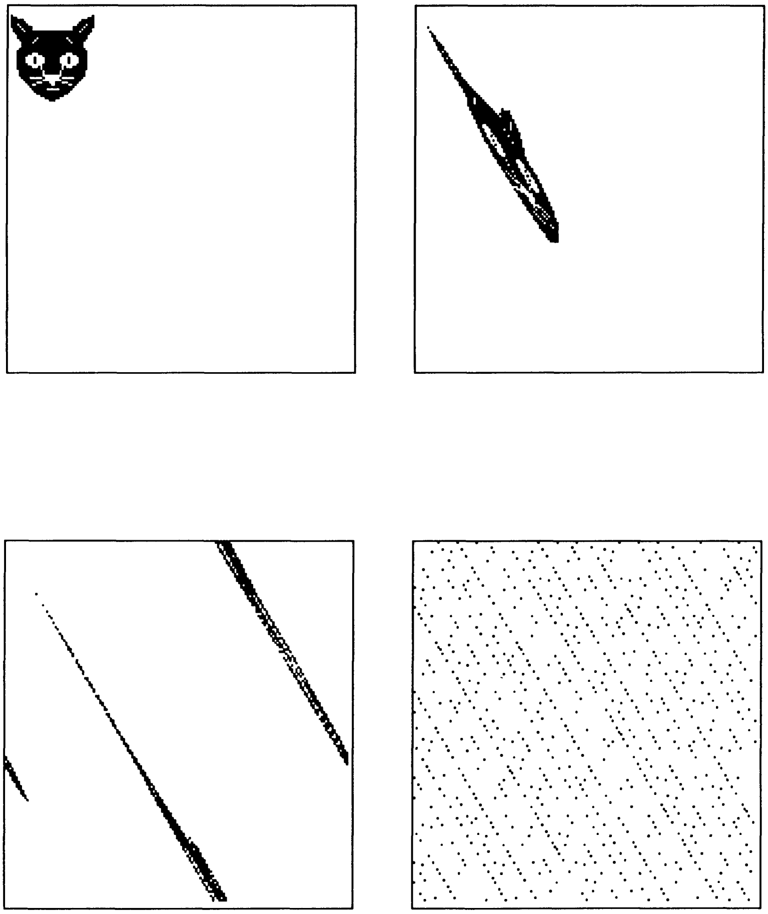

Figure 1. "Snapshots" of the initial "cat" configuration and of the images at $t = 1$, $t = 2$ and $t = 5$ under the mapping given by Eq. (1.6) for $N = 161$. That is, top row left to right: $t = 0$, $t = 1$; bottom row left to right: $t = 2$, $t = 5$.

We will use theorems contained in Hardy and Wright [7], and we refer to specific theorems as numbered in the fifth edition; e.g., HW Thm. 97 [7]. Two useful identities [8] are:

For any positive integers k, r

$$u_{k+r} = u_k u_{r+1} + u_{k-1} u_r \tag{1.9}$$

$$(-1)^k = u_{k+1} u_{k-1} - u_k^2. \tag{1.10}$$

These identities may be extended to all integer values k, r if one defines

$$u_{-k} = (-1)^{k-1} u_k \quad \text{for } k = 0, 1, 2, 3, \dots . \tag{1.11}$$

2. UPPER BOUNDS FOR THE PERIOD. Our first upper bound for the period is $m_N \le N^2/2$ for $N > 2$. Consequently, m_N does not grow exponentially with N.

To derive that bound we retrace the path of Vorob'ev [9] and write

$$u_n = \phi_n \pmod{N} \tag{2.1}$$

where ϕ_n is the least non-negative residue of u_n to modulus N. Consider the sequence of ordered pairs $\langle \phi_1, \phi_2 \rangle, \langle \phi_2, \phi_3 \rangle, \dots, \langle \phi_n, \phi_{n+1} \rangle, \dots$. There are at

most N^2 distinct pairs. Any set of $N^2 + 1$ pairs contains some equal ones among them.

Lemma 1 [9]. *The first pair that repeats in the above sequence is* $\langle 1, 1 \rangle$.

Proof: Assume the opposite; i.e., that the first repeated pair is $\langle \phi_k, \phi_{k+1} \rangle$, where $k > 1$. Let us find in the sequence a pair $\langle \phi_r, \phi_{r+1} \rangle$ $(r > k)$ such that $\phi_k = \phi_r$, $\phi_{k+1} = \phi_{r+1}$. From the definition of the Fibonacci numbers

$$\phi_{r-1} = \phi_{r+1} - \phi_r \tag{2.2}$$

$$\phi_{k-1} = \phi_{k+1} - \phi_k \tag{2.3}$$

so

$$\phi_{r-1} = \phi_{k-1} \tag{2.4}$$

and we have

$$\langle \phi_{r-1}, \phi_r \rangle = \langle \phi_{k-1}, \phi_k \rangle. \tag{2.5}$$

But $\langle \phi_{k-1}, \phi_k \rangle$ is situated earlier in the sequence than $\langle \phi_k, \phi_{k+1} \rangle$; therefore $\langle \phi_k, \phi_{k+1} \rangle$ is not the first pair that repeats itself. So the supposition $k > 1$ is wrong, and we must have $k = 1$. That proves the Lemma.

Theorem 1 [9]. *For any positive integer N at least one number divisible by N can be found among the first* N^2 *Fibonacci numbers.*

Proof: From the Lemma $\langle 1, 1 \rangle$ is the first pair that repeats itself. So $\langle \phi_t, \phi_{t+1} \rangle = \langle 1, 1 \rangle$ for some integer t such that $1 < t \leq N^2 + 1$. Thus

$$\phi_t \equiv 1 \pmod{N} \tag{2.6}$$

and

$$\phi_{t+1} \equiv 1 \pmod{N}. \tag{2.7}$$

But

$$u_{t-1} = u_{t+1} - u_t; \tag{2.8}$$

therefore,

$$\phi_{t-1} \equiv 0 \pmod{N}, \tag{2.9}$$

and the Theorem is proved.

Lemma 2. *For* $N > 2$ *if* $u_n \equiv 0 \pmod{N}$ *and* $u_{n+1} \equiv 1 \pmod{N}$, *then n must be even.*

Proof: The Lemma is equivalent to the statement that for $N > 2$ if $A^n \equiv 1 \pmod{N}$, then n is even. But the determinant $\det(A) = -1$, so $\det(A^n) = (\det A)^n = (-1)^n \equiv 1 \pmod{N}$. Hence n must be even.

Theorem 2. *For* $N > 2$ *the period* m_N *of the mapping* (1.6) *satisfies*

$$m_N \leq N^2/2. \tag{2.10}$$

Proof: From Lemma 1 and Theorem 1, the first reappearance of the pattern 0, 1, 1 in the sequence $\phi_0, \phi_1, \phi_2, \phi_3, \ldots, \phi_n, \phi_{n+1}, \ldots$ occurs for $\phi_{t-1}, \phi_t, \phi_{t+1}$, where $0 < t - 1 \leq N^2$. From Lemma 2, $t - 1$ must be even. From the definition of the period one has $2m_N = t - 1$. That proves the Theorem.

Numerical results for m_N indicate that the bound is rather loose; nevertheless, the bound establishes that m_N does not grow exponentially with N. The method which will be used subsequently to prove Theorem 3 also gives a stronger Theorem than Theorem 1; viz.,

Theorem 1′. *For any positive integer N, at least one Fibonacci number $u_n \equiv 0 \pmod{N}$ with $n \leq 2N$.*

Remark. We have $n \leq 12N/7$ except in cases $N = 6 \cdot 5^{\delta}$, $\delta = 0, 1, 2, \ldots,$ when $n = 2N$.

Remark. From Theorem 1′, for any positive integer N there is an $n \leq 2N$ such that $u_n \equiv 0 \pmod{N}$. Identity (1.9) then implies $u_{2n} \equiv 0 \pmod{N}$. One now may use Theorem 5 to write

$$m_N \leq 2n \leq 4N. \tag{2.11}$$

That is a substantial improvement over (2.10), but Theorem 3a is a little stronger still.

Next we give a much tighter upper bound for m_N. The bound, denoted by m^*, is always an integer multiple of the period m_N for the mapping (1.6). The bound is based on the following Theorem, which may be viewed as an extension of HW Thm. 180 [7].

Theorem 3. *Let p be a prime $\equiv \pm 1 \pmod{10}$. Then $A^{p-1} \equiv 1 \pmod{p}$. Let q be a prime $\equiv \pm 3 \pmod{10}$. Then $A^{q+1} \equiv -1 \pmod{q}$. For the prime 5, $A^{10} \equiv -1 \pmod{5}$; and for the prime 2, $A^{6} \equiv 1 \pmod{4}$.*

Application of Theorem 3 to the periodicity of the mapping (1.6) is made as follows.

Consider a positive integer $N > 1$ and write N in terms of its prime factors p and q, which were referred to in the above Theorem.

$$N = \left(\prod_{p|N} p^{\alpha}\right)\left(\prod_{q|N} q^{\beta}\right) 5^{\gamma} 2^{\delta} \tag{2.12}$$

where the notation $p|N$ means "p divides N." Since α will always be associated with p, and β with q, we will avoid the notation α_p and β_q.

As $A^{p-1} \equiv 1 \pmod{p}$, it follows from HW Thm. 78 [7] that $A^{(p-1)p^{\alpha-1}} \equiv 1 \pmod{p^{\alpha}}$. Further, the congruence $A^{q+1} \equiv -1 \pmod{q}$ implies $A^{2(q+1)} \equiv 1 \pmod{q}$, and HW Thm. 78 [7] gives $A^{2(q+1)q^{\beta-1}} \equiv 1 \pmod{q^{\beta}}$. Finally, the congruence $A^{10} \equiv -1 \pmod{5}$ implies $A^{2(10)5^{\gamma-1}} \equiv 1 \pmod{5^{\gamma}}$, and $A^{6} \equiv 1 \pmod{4}$ implies $A^{3 \cdot 2^{\delta-1}} \equiv 1 \pmod{2^{\delta}}$.

For a given N, the period of the mapping (1.6) was defined to be the smallest positive integer m_N such that $A^{2m} \equiv 1 \pmod{N}$. To find an upper bound m^* on m_N, compute the least common multiple [LCM]

$$2m^* = LCM\left[(p-1)p^{\alpha-1}, 2(q+1)q^{\beta-1}, 2(10)5^{\gamma-1}, (3)2^{\varepsilon}\right] \tag{2.13}$$

with

$$\varepsilon = \mathrm{Max}(\delta - 1, 1). \tag{2.14}$$

Each factor in (2.12) has a corresponding term in the LCM. Therefore (2.12) and

(2.13) imply

$$A^{2m^*} \equiv 1 \pmod{N}, \tag{2.15}$$

so that m^* is a multiple of m_N and

$$m_N \leq m^*. \tag{2.16}$$

In the particular case mentioned above, $N = 161 = 7 \cdot 23$; only the two primes $q = 7$ and $q = 23$ play a role, and $\beta = 1$ for each. Thus $m^* = 24$, equal to the value we found for m_N. Numerical results for m_N and m^* indicate that the inequality (2.16) is satisfied as an equality for most values of $N \leq 10^6$.

We call an integer N "primitive" if $m_N = m^*$. A primitive N is one whose period m_N achieves the upper bound, m^*. Thus, 161 is primitive. To our surprise we found that the great majority of small N are primitive. The first non-primitive N is 29, with $m_N = 7$, $m^* = 14$. We looked at three stretches of 100 values of N and found:

$1 \leq N \leq 100,$	96 are primitive,
$901 \leq N \leq 1000,$	84 are primitive,
$999901 \leq N \leq 1000000,$	82 are primitive.

So far as they go, these numbers suggest that the fraction of primitive N is tending to a limit substantially greater than 0.5 as $N \to \infty$. However, we conjecture that the opposite is true.

Conjecture. *The fraction of primitive integers not exceeding N has the asymptotic behavior*

$$F(N) \sim \frac{K}{\log\log\log N} \tag{2.17}$$

as $N \to \infty$, where

$$K = e^{-\gamma}\left(\frac{\log(10/3)}{\log 2}\right) = 0.975, \tag{2.18}$$

and γ is Euler's constant.

Since $\log\log\log 10^6 = 0.965$, our numerical data do not begin to test the validity of (2.17).

The argument leading to (2.17) is probabilistic and makes no claim to be rigorous. According to HW Thm. 436 [7], almost all integers not exceeding N have about

$$y = \log\log N \tag{2.19}$$

distinct prime factors, which will appear in the definition (2.13) of m^*. For N to be primitive it is necessary and sufficient that

$$A^{2m^*/s} \not\equiv 1 \pmod{N}, \tag{2.20}$$

for every prime s dividing $2m^*$. Now the matrix

$$B = A^{2m^*/s} \tag{2.21}$$

satisfies the congruence

$$B^s \equiv 1 \pmod{N}. \tag{2.22}$$

We wish to estimate the probability that $B \not\equiv 1 \pmod{N}$. If N is a p-prime, then s must be a divisor of $(N-1)$ and the congruence (2.22) has exactly s roots. We assume that each of the roots has equal probability s^{-1} of being (2.21). Then the probability that (2.20) holds is

$$1 - s^{-1}. \tag{2.23}$$

If N is a q-prime, then s must be a divisor of $2(q+1)$ and again the congruence (2.22) has s roots in the field generated by $A \pmod{N}$. If s is an odd prime, the estimate (2.23) holds as before. But for $s = 2$, we know from Theorem 3 that $B \equiv -1 \pmod{N}$ and therefore (2.20) holds with probability 1.

When N is composite, we assume that the probabilities for (2.20) to hold are independent for all primes s dividing $2m^*$. The probability for N to be primitive is then

$$F(N) = \left(1 - \tfrac{1}{2}(1 - Q)\right)\prod_{s>2}\left(1 - s^{-1}d_s\right), \tag{2.24}$$

where d_s is the probability that the odd prime s divides $2m^*$, and Q is the probability that the highest power of 2 in the LCM (2.13) belongs to one of the terms $2(q+1)$. Since each s has roughly y chances to divide one of the factors appearing in (2.13),

$$d_s = 1 - (1 - s^{-1})^y. \tag{2.25}$$

To estimate Q, we suppose that each term $(p-1)$ or $(q+1)$ appearing in (2.13) is divisible by 2^k with probability 2^{-k}, $k = 1, 2, 3, \ldots,$. For large N, the number of p-primes and q-primes will both be approximately

$$M = \tfrac{1}{2}y. \tag{2.26}$$

The probability that k_1 is the highest power of 2 dividing any $(p-1)$ is $r(k_1)$, and the probability that k_2 is the highest power of 2 dividing any $(q+1)$ is $r(k_2)$, where

$$r(k) = (1 - 2^{-k})^M - (1 - 2^{1-k})^M. \tag{2.27}$$

Q is the probability that

$$1 + k_2 \geq k_1. \tag{2.28}$$

Thus

$$\begin{aligned} Q &= \sum_{1+k_2 \geq k_1}\sum r(k_2)r(k_1) \\ &= \sum_k \left((1 - 2^{-k})^M - (1 - 2^{1-k})^M\right)(1 - 2^{-1-k})^M. \end{aligned} \tag{2.29}$$

For large M we may replace the sum over k by an integral over a continuous variable u given by

$$e^{-u} = 1 - 2^{-k}. \tag{2.30}$$

Then (2.29) becomes in the large-M limit

$$\begin{aligned} Q &= (\log 2)^{-1}\int_0^\infty (e^u - 1)^{-1}\left(e^{-(3/2)Mu} - e^{-(5/2)Mu}\right) du \\ &= \left(\log \tfrac{5}{3}/\log 2\right), \end{aligned} \tag{2.31}$$

and (2.24) becomes

$$F(N) = \left(\log \frac{10}{3} \Big/ \log 2\right) \prod_s (1 - s^{-1} d_s), \tag{2.32}$$

with the product extending over all primes s. A more exact analysis of the sum (2.29) shows that Q contains also an extravagantly small oscillating term

$$\sum_{k=1}^{\infty} A_k \cos\left(2\pi k (\log 2)^{-1} (\log\log\log N) + \delta_k\right), \tag{2.33}$$

with amplitude

$$A_k \sim \exp\left(-\pi^2 (\log 2)^{-1} k\right) \sim 10^{-6k} \tag{2.34}$$

which we shall neglect.

We return to (2.32) with d_s given by (2.25). The factors in the product can be crudely approximated by

$$\begin{aligned} d_s &= (1 - s^{-1}) && \text{for } s \le y, \\ d_s &= 1 && \text{for } s > y. \end{aligned} \tag{2.35}$$

The error in (2.35) is small when s is either small or large compared with y. The maximum error is of order y^{-1} for primes s in the neighborhood of y. The number of such primes is of order

$$(y/(\log y)). \tag{2.36}$$

Therefore, the fractional error introduced by (2.35) into the product (2.32) is of order $(\log y)^{-1}$. A more careful analysis shows that the leading term in the error is a factor

$$1 - \gamma (\log y)^{-1}, \tag{2.37}$$

where γ is Euler's constant. Neglecting this factor, we find from (2.32) and (2.35)

$$F(N) \sim \left(\log \tfrac{10}{3} / \log 2\right) \prod_{s \le y} (1 - s^{-1}). \tag{2.38}$$

Finally, HW Thm. 430 [7] (Mertens's Theorem) says

$$\prod_{s \le y} (1 - s^{-1}) \sim \frac{e^{-\gamma}}{\log y}, \tag{2.39}$$

and this with (2.18), (2.19), and (2.38) gives (2.17).

From (2.13) and (2.16) one may derive a simpler upper bound for m_N.

Theorem 3a.

$$m_N \le 3N. \tag{2.40}$$

Moreover, (2.40) *holds with equality if and only if*

$$N = 2 \cdot 5^{\gamma}. \tag{2.41}$$

For all N except for (2.41) *we have*

$$m_N \le 2N, \tag{2.42}$$

with equality only for

$$N = 5^{\gamma}, \qquad N = 6 \cdot 5^{\gamma}. \tag{2.43}$$

For all N except for (2.41) *and* (2.43) *we have*

$$m_N \le \frac{12}{7} N. \tag{2.44}$$

We could find smaller bounds with larger lists of exceptions, but beyond (2.44) it seems unprofitable to go.

Proof of (2.40)–(2.44). Consider the ratio

$$R = (m^*/N) \geq (m_N/N), \tag{2.45}$$

with N given by (2.12) and m^* by (2.13). The definition of an LCM gives

$$2R \leq \left(\prod_{p|N}(1-p^{-1})\right)\left(\prod_{q|N}[2(1+q^{-1})]\right) \cdot 4 \cdot 3 \cdot 2^{-k} \tag{2.46}$$

where the factor 4 appears if $\gamma \geq 1$, the factor 3 appears if $\delta \geq 1$, and k is the number of powers of 2 that appear redundantly in the various terms of (2.13). We wish to choose N to make R as large as possible. By (2.46), R will be increased by dropping all the p-primes from N. Since each q-prime gives a term in (2.13) divisible by 4, R will be increased by dropping all of the q-primes except one, and by dropping all except one power of 2. We are left with only the following simple choices for N giving possibly maximum values for R,

$$N = 5^\gamma, 5^\gamma \cdot 3^\beta, 5^\gamma \cdot 7^\beta, 2 \cdot 5^\gamma, 6 \cdot 5^\gamma, 2 \cdot 5^\gamma \cdot 7^\beta, \tag{2.47}$$

giving respectively

$$R = 2, 4/3, 8/7, 3, 2, 12/7. \tag{2.48}$$

This proves the inequalities (2.40), (2.42), (2.44) and proves that the cases of equality are at most (2.41) and (2.43). It remains to prove that equality holds, i.e., $m_N = m^*$, in the cases (2.41), (2.43).

The Lucas numbers v_k are related to the Fibonacci numbers by

$$v_k = u_{k-1} + u_{k+1}. \tag{2.49}$$

By (1.7) and (1.11), the matrix A generates Fibonacci and Lucas numbers by

$$A^{2k} + A^{-2k} = v_{2k} \tag{2.50}$$

$$A^{2k} - A^{-2k} = u_{2k} \cdot \sqrt{5}, \tag{2.51}$$

where $\sqrt{5}$ [in this section] stands for the matrix

$$\sqrt{5} = A + A^{-1} = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}, \tag{2.52}$$

whose square is 5. Now (2.50) and (2.51) give

$$v_{4k} = 5 \cdot u_{2k}^2 + 2, \tag{2.53}$$

$$u_{10k} = u_{2k}(1 + v_{4k} + v_{8k}) = 5 \cdot u_{2k}(1 + u_{2k}^2 + u_{4k}^2). \tag{2.54}$$

(2.54) implies

$$u_{10k} \equiv 0 \pmod 5, \tag{2.55}$$

$$u_{50k}/u_{10k} \equiv 5 \pmod{125}. \tag{2.56}$$

Thus u_{50k} is divisible by exactly one more power of 5 than u_{10k}. Now Theorem 3 with (2.51) shows that u_{2k} is periodic (mod 5) with period 10, so that

$$u_{2k} \not\equiv 0 \pmod 5 \quad \text{for } k \not\equiv 0 \pmod 5. \tag{2.57}$$

This with (2.54) implies

$$u_{10k} \not\equiv 0 \pmod{25} \quad \text{for } k \not\equiv 0 \pmod 5. \tag{2.58}$$

Together (2.55), (2.56), (2.57), and (2.58) imply

$$u_{2k} \equiv 0 \pmod{5^\gamma} \quad \text{if and only if } k \equiv 0 \pmod{5^\gamma}. \tag{2.59}$$

This means that for any N divisible by 5^γ, m_N is also divisible by 5^γ.

Consider in particular $N = 2 \cdot 5^\gamma$, which has m_N dividing $m^* = 6 \cdot 5^\gamma$. We have proved that m_N is divisible by 5^γ. Since N is divisible by 5 and by Theorem 3

$$A^{10} \equiv -1 \pmod{5}, \tag{2.60}$$

m_N must also be divisible by 2. Since N is even, m_N must be divisible by 3. Therefore $m_N = m^*$ and (2.40) holds with equality. The same argument shows that (2.42) holds with equality for N given by (2.43).

3. LOWER BOUNDS FOR THE PERIOD AND EXPLICIT VALUES FOR PARTICULAR CASES.

Theorem 4. *Both* $u_{4n} \equiv 0 \pmod{N}$ *and* $u_{4n-1} \equiv 1 \pmod{N}$ *if and only if*

$$u_{2n} \equiv 0 \pmod{N}. \tag{3.1}$$

Proof: The identities

$$u_{4n} = u_{2n} v_{2n}, \tag{3.2}$$

$$u_{4n-1} - 1 = u_{2n} v_{2n-1}, \tag{3.3}$$

imply the "if" part of the theorem immediately. The "only if" is equivalent to the statement that (v_{2n-1}, v_{2n}) are coprime, which is contained in HW Thm. 179 [7].

Theorem 5. *For* $N \geq 2$ *let* n *be the smallest positive integer such that* $u_{2n} \equiv 0$ $(\text{mod}\, N)$. *Then either* $m_N = n$ *or* $m_N = 2n$.

Proof: By Theorem 4, n is the smallest integer such that

$$A^{4n} \equiv 1 \pmod{N}, \tag{3.4}$$

while m_N is the smallest such that

$$A^{2m_N} \equiv 1 \pmod{N}. \tag{3.5}$$

Integers satisfying (3.4) are multiples of n, and integers satisfying (3.5) are multiples of m_N. Therefore, m_N is a multiple of n, and $2n$ is a multiple of m_N. The conclusion follows.

Theorem 6. *Given* $N = u_{2n}$ *with* $n = 2, 3, \ldots$; *there does not exist an* $N' > N$ *with even period,* $m_{N'} \leq 2n$.

We give the proof of Theorem 7; the proof of Theorem 6 is similar.

Theorem 7. *Given* $N = v_{2n-1}$ *with* $n = 2, 3, \ldots$; *there does not exist an* $N' > N$ *with odd period,* $m_{N'} \leq 2n - 1$.

Proof: Assume $m_{N'} = 2n' - 1 \leq 2n - 1$ so that

$$u_{4n'-2} \equiv 0 \pmod{N'} \tag{3.6}$$

and

$$u_{4n'-3} \equiv 1 \pmod{N'}. \tag{3.7}$$

Then from Theorem 4

$$u_{2n'-1} \equiv 0 \pmod{N'}. \tag{3.8}$$

But if $2n' - 1 \leq 2n - 1$, then

$$\begin{aligned} u_{2n'-1} &\leq u_{2n-1} < u_{2n-1} + 2u_{2n-2} \\ &= u_{2n} + u_{2n-2} \\ &= N < N'. \end{aligned} \tag{3.9}$$

That contradicts (3.8) and completes the proof of the Theorem.

Corollary. *For $N' > v_{2n-1}$ with $n = 2, 3, \ldots$; $m_{N'} > 2n$.*

Proof: Since $u_{2n} + u_{2n-2} > u_{2n}$, the condition $N' > u_{2n} + u_{2n-2}$ implies the condition $N' > u_{2n}$. By Theorem 6 there are no even periods $m_{N'} \leq 2n$, and by Theorem 7 there are no odd periods $m_{N'} \leq 2n - 1$. That proves the corollary.

The corollary provides a "staircase" lower bound for m_N as a function of N. This bound may be expressed in the following way.

Define

$$\begin{aligned} N(n) &= u_n \quad \text{for } n \text{ even} \\ &= v_n \quad \text{for } n \text{ odd} \end{aligned} \tag{3.10}$$

and let

$$\lambda_+ = (1 + \sqrt{5})/2. \tag{3.11}$$

Then for n even and $N > N(n)$, any even period

$$m_N > n > \left[\log\left(N(n)\sqrt{5}\right)\right]/\log \lambda_+ \tag{3.12}$$

and for n odd and $N > N(n)$, any odd period

$$m_N > n > [\log N(n)]/\log \lambda_+. \tag{3.13}$$

These results may be summarized in

Theorem 8. *For any integer N,*

$$m_N > \left[\log\left(N\sqrt{5}\right)\right]/\log \lambda_+ \quad \text{if } m_N \text{ is even}, \tag{3.14}$$

$$m_N > [\log N/\log \lambda_+] \quad \text{if } m_N \text{ is odd}. \tag{3.15}$$

In the context of chaos, others [**10**] have displayed an approximate recurrence of a digitized image of an appropriately selected subject; viz., Henri Poincaré. The importance of background fluctuations is pointed out in that article.

Theorem 9.

(a) *For $N = u_{2n}$,* $\quad m_N = 2n, (n > 1).$ (3.16)
(b) *For $N = u_{2n-1}$,* $\quad m_N = 4n - 2, (n > 2).$ (3.17)
(c) *For $N = v_{2n}$,* $\quad m_N = 4n.$ (3.18)
(d) *For $N = v_{2n-1}$,* $\quad m_N = 2n - 1.$ (3.19)
(e) *For $N = v_{2n} - 1$,* $\quad m_N = 6n.$ (3.20)
(f) *For $N = v_{2n} + 1$,* $\quad m_N = 3n.$ (3.21)

[Note, e.g., $N = 842, 843, 844$ yield $m_N = 42, 28, 21$, respectively.]

Proof: The proofs of each part are similar so we choose to select a few for detailed presentation and sketch the others. For part (a) since $u_{2n} \equiv 0 \pmod N$, we find from Theorem 4 that $2n$ satisfies the conditions defining m_N and is therefore a multiple of m_N. But $u_{2m_N-1} \equiv 1 \pmod N$ implies $u_{2m_N-1} > u_{2n}$ and $2m_N - 1 > 2n$. Therefore, $2n$ can only be m_N.

Part (c) is proved by using $u_{4n} = u_{2n}v_{2n} \equiv 0 \pmod N$ and Theorem 4 to establish that $4n$ is a multiple of m_N. From the Corollary following Theorem 7 one concludes that $m_N > 2n$. Consequently, $4n = m_N$.

Part (d) is proved by making use of the two identities $u_{4n-2} = u_{2n-1}v_{2n-1}$ and $u_{4n-1} - 1 = u_{2n}v_{2n-1}$ to show that $2n - 1$ is a multiple of m_N. But $u_{2m_N-1} \equiv 1$ $\pmod N$ implies $u_{2m_N-1} > N$, where $N = v_{2n-1} > u_{2n-1}$ so that $2n - 1 < 2m_N$. A multiple of m_N satisfying the latter condition can only be m_N itself. Parts (e) and (f) are proved by using the identity $u_{6n} = u_{2n}(v_{2n} - 1)(v_{2n} + 1)$ along with part (a). The proof of part (b) is a bit more involved. One uses (1.9) to obtain $u_{4n-2} \equiv 0 \pmod N$ and then one uses Theorem 4 to obtain $A^{8n-4} \equiv 1 \pmod N$ so m_N divides $4n - 2$. But $N > v_{2n-3}$, so the Corollary following Theorem 7 says $m_N > 2n - 2$. The only possibilities are $m_N = 4n - 2$ or $2n - 1$. Assume that $m_N = 2n - 1$. Then $u_{4n-3} \equiv 1 \pmod{u_{2n-1}}$, but $u_{4n-3} = 1 + u_{2n-2}v_{2n-1}$ so $u_{2n-2}v_{2n-1} \equiv 0 \pmod{u_{2n-1}}$. According to HW Thm. 179 [7], u_k and u_{k+1} are coprime, while u_k and v_k have at most one common factor 2. Thus, the congruence $u_{2n-2}v_{2n-1} \equiv 0 \pmod{u_{2n-1}}$ is possible only if $u_{2n-1} = 1$ or 2, $n = 1$ or 2. This explains why (b) fails for $n \leq 2$.

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Why *Do* We Teach Calculus?

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The chimera of a course in discrete mathematics to replace freshman calculus raised its head briefly in the early 1980s and drew forth the defenders of calculus. Ronald Douglas, Daniel Kleitman, Peter Lax, Saunders MacLane, and others [**1**] have eloquently defended the necessity of placing calculus at the heart of the college mathematics curriculum. The issue seems settled, witness the Committee on the Undergraduate Program in Mathematics (CUPM) report reprinted in *Reshaping College Mathematics* [**2**] which affirms their position. I agree, but we are not done. If we are to accomplish the systemic changes that are needed in undergraduate education, then we must be clear about *why* we teach calculus.

The CUPM recommendation "to make no substantive changes in the first semester of calculus" is wrong. This course is not adequate as it stands. Our students approach calculus with a mixture of trepidation and anticipation. They know that it is going to be hard, but they also expect that this will be the course that draws together the mathematics that they have learned and transforms it into an instrument for comprehending the world around us. We know that this tool exists, but our students usually miss it. They leave disillusioned and disappointed. This past year I taught Advanced Placement AB (first semester) Calculus at our local high school. It gave me time to reflect on and experiment with my own response to the question in the title. I have two answers.

The first is that calculus is used in a variety of contexts by many disciplines. If we mathematicians did not teach it, others would have to. That is the essence of Lax's article and the thrust of Douglas's. It is an answer that is widely given and is being acted upon. Physicists, engineers, and biologists are being brought into our discussion of calculus reform. Textbooks are using real applications, and there is now rich source material [**3**]. Our use of this material is often faulty—too frequently it is tacked on rather than incorporated into the motivation for the concept it is to convey—but there is effort and progress in reforming calculus in this direction.

But, the usefulness of calculus is not a sufficient answer to my question. There are topics from discrete mathematics—statistical analysis, linear programming—that are far more useful to most of our students. My second answer, the one that has radical consequences for the way we teach calculus, is that calculus lies at the foundation of our scientific world view. Modern scientific thought has been formed from the concepts of calculus and is meaningless outside this context. When I speak of science, I do not restrict myself to other disciplines. In a very significant respect, mathematics itself came into being with the development of calculus.

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Sitting at the core of any modern education, mathematicians gaze back to ancient Babylon, Egypt, and Greece and preen themselves, secure in the delusion of an exalted position that has endured through the ages. In fact, there was no chair of mathematics at Oxford until 1619, nor at Cambridge until 1662. To the gentry of the mid-seventeenth century, the advantage was to Cambridge. Anthony à Wood describes this period: "Here by the way it must be remembered that the generality of the people some years before did verily think that the most useful branches of mathematics were spells and her professors limbs of the devil [**4**]." Samuel Pepys graduated from Cambridge ignorant of the multiplication table [**5**]. John Wallis would write of mathematics in the 1630s and 1640s at Cambridge: "[They were] scarce looked upon as Academical studies, but rather Mechanical; as the business of Traders, Merchants, Seaman, Carpenters, Surveyors of Lands, or the like, and perhaps some Almanack Makers in London [**6**]."

What changed this attitude was Newton's *Philosophiæ Naturalis Principia Mathematica*. It captured the public imagination in its revelation, explanation, and prediction of the phenomena of celestial mechanics. Suddenly, mathematics was being applied to the secrets of nature wherever they lay. One is struck by the *exuberance* of eighteenth century mathematics. We teach calculus because it is important for an understanding of who we are as a society.

We do a tremendous disservice to our students in the first year of calculus if we do not convey this excitement. I began my high school class with a discussion of why *Principia* is so important and concluded it with the proof that Kepler's laws imply the law of gravity [**7**], a simple and elegant illustration of the power that arises from recognizing acceleration as the second derivative of position. I brought in simple differential equations at every opportunity and tried to introduce each new concept with its original purpose: Fermat was led to discover the derivative not because it gave him the slope of the tangent but because it identified local extrema; integration in the 1700s was about antidifferentiation, not finding areas and volumes.

History also tells me what I should not teach or, at the least, what I should approach with great caution: *anything* that follows Joseph Fourier's *Theory of the Propagation of Heat in Solid Bodies* of 1807. Euler, Lagrange, and Cauchy committed great errors in their ignorance of the analysis that was developed in the nineteenth century, but the first year of calculus is not the time to describe these potential pitfalls. I would rather a student share Euler's flare for manipulating series than memorize convergence tests. If we draw the line at 1807, then we do not need careful definitions of function, limit, and continuity. We can postpone the intermediate value theorem and satisfy our students with a heuristic understanding of the mean value theorem. I am willing to go over the line to admit the definite integral, introduced by Fourier in 1816, but a description of the Riemann integral is out of bounds.

A historical pedagogy should not be applied with rigidity. Differential forms make sense of vector calculus, but we cannot begin the study of vector calculus with differential forms and neither should we forget the effort required to achieve the modern sense of rigor in calculus or ignore the reasons that made it necessary. Here, I follow Henri Poincaré:

> The task of the educator is to make the child's spirit travel again where his fathers have passed, crossing certain stages rapidly but suppressing none of them. In this regard, the history of science must be our guide [**8**].

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Whoops!

In the April '92 issue of this MONTHLY, we announced Slowinski's discovery of the most recent Mersenne prime ($2^{756839}-1$) and declared it to have 227,831 digits. Several people have written in to point out the number is in fact larger than this. Attentive reader Charles Vanden Eynden was the first. He wrote a (polite) letter pointing out that every student in his elementary number theory class quickly calculated the number of digits as 227,832 since they realized the number of digits was one *greater* than the log base 10 of the number. The MONTHLY apologizes for the error.

Tape Counters

Richard L. Roth

The tape counter on many VCRs and audiocassette players is an example of a function, a practical function that at first may seem mysterious. Anyone who has played a VCR probably has noticed that the counter reading is not a simple linear function of the time. For example, the following data came from reading a 6 hour tape at one hour intervals:

Time (minutes)	Counter
60	1540
120	2669
180	3604
240	4422
300	5157
360	5831

If we let $f(t)$ denote the counter reading as a function of time we see that f is an increasing function, but its rate of increase slows down (that is, $f''(t)$ is negative). It's not obvious what kind of function $f(t)$ is. When the VCR operates, the tape moves past the heads at a constant speed k. As it is wound onto the take-up reel, the radius increases, and hence the reel turns more slowly. The number on the tape counter is proportional to the number of turns of the take-up reel. (Some of the newest models of VCR, however, have now replaced this kind of tape counter with one that gives the time elapsed.)

How do we determine the function $f(t)$? At a given time t, let s denote the length of tape which has been wound, r denote the radius of the tape on the take-up reel, n the number of turns the take-up reel has made, and θ denote the angle (in radians) through which the reel has turned. If the initial radius of the tape on the take-up reel is r_0 and the thickness of the tape is b, then it is easy to see that $\theta = 2\pi n$ and $r = r_0 + nb$. See Figure 1.

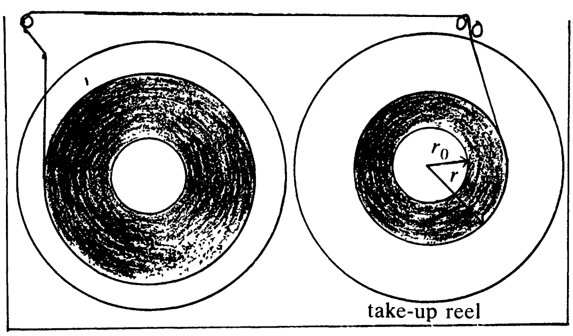

Figure 1. Side view of a videocassette.

We make the assumption that b is very small compared to r at any time, and hence that a winding of the tape may be approximated by a circle of radius r. Then

for a small rotation $\Delta\theta$, the length of the tape wound is $\Delta s = r\Delta\theta$. Hence

$$s = \int_0^\theta r\, d\theta = \int_0^n (r_0 + nb)2\pi\, dn \tag{1}$$

and we see that

$$s = \pi b n^2 + 2\pi r_0 n. \tag{2}$$

Formula (2) for the tape length s is of course applicable in any problem of winding tape, rope, or ribbon on a spool or roll where the thickness of the tape is small relative to the radius of the spool. It isn't necessary to use calculus to derive formula (2) and on the other hand, one can also derive an exact (but more complicated) formula for s; see Box 1.

Finding a Formula for s

Formula (2) can also be derived easily without calculus using the following geometric approach. A side view of the reel shows that the area of the tape wound on the reel is that of a "washer" of outer radius r and inner radius r_0, hence equals $\pi r^2 - \pi r_0^2$. But this same area also equals the length of the wound tape s times the tape thickness b. Thus

$$sb = \pi r^2 - \pi r_0^2 = \pi\left[(r_0 + nb)^2 - r_0^2\right] = 2\pi r_0 nb + \pi n^2 b^2$$

Dividing by b yields equation (2).

One can obtain the exact value of s by using polar coordinates to study the curve. We have $r = r_0 + nb = r_0 + (b/2\pi)\theta = g(\theta)$ as the equation in polar coordinates. Since the formula for arc length in polar coordinates is $ds = \sqrt{[g(\theta)]^2 + [g'(\theta)]^2}\, d\theta$, s is given by the formula

$$s = \int_0^\theta \sqrt{r^2 + \left(\frac{b}{2\pi}\right)^2}\, d\theta. \tag{3}$$

If we drop the term $(b/2\pi)^2$ in equation (3) (since b is very small compared to r), and simplify, we get the equation $s = \int_0^\theta r\, d\theta$ which was used in equation (1). It is possible to evaluate the integral in (3) by standard techniques, but the resulting function would be much more complicated than that described in equation (2). [Note that the informal geometric proof of (2) given above is justified by assuming that the wound tape consists of concentric circles instead of being a spiral.]

Box 1

Now in a VCR, the tape moves at a constant speed k so we know $s = kt$ for some constant k. The counter reading m is a constant multiple of the number of turns n; that is $m = cn$ for some constant c. (I have found VCR's where apparently $c = 2$ or 4, for example). Substituting into equation (2) yields

$$kt = \pi b \frac{m^2}{c^2} + 2\pi r_0 \frac{m}{c}$$

and hence

$$t = \left[\frac{\pi b}{c^2 k}\right] m^2 + \left[\frac{2\pi r_0}{ck}\right] m = Am^2 + Bm \tag{4}$$

which is a quadratic function whose graph is part of a parabola passing through the origin. To find the function $f(t)$ (the counter function), we simply invert the

function described in (4) to get

$$m = f(t) = \frac{-B + \sqrt{B^2 + 4At}}{2A} \tag{5}$$

Thus the function $f(t)$ is a modified square root function and its graph is the upper part of a parabola opening to the right and passing through the origin. It's a naturally occurring inverse function, something that should interest our students.

Since it's hard to get accurate values for the constants such as b and r_0, the easiest way to calculate A and B is simply to use two test values in equation (4) and solve the simultaneous equations for A and B. For example using $t = 60$, $m = 1540$ and $t = 240$, $m = 4422$ yields $A = 5.31334E - 06$ and $B = 3.07785E - 02$. I used these values in equation (5), and checking minute by minute, found that the formula for m matched the readings with a discrepancy of at most ± 2. I have also found that different tapes give different readings, even when they are the same brand and type. (For example when $t = 240$, besides the reading of 4422 on the tape described in this example, I have found readings $m = 4310$ and $m = 4370$ on different tapes.)

What happens if the tape is being wound onto a reel which is turning at a constant speed? Substitute $n = kt$ into equation (2). You'll see that the tape length is now growing as a quadratic function of time.

Using formula (4) or (5) you can generate a handy reference table for use with your VCR. What happens, however, if someone resets the counter in the middle of your tape? Or if you start with a tape that has been played part way through? The table can't be used, but you can still estimate how far the tape has been played by using the derivative dm/dt.

Differentiating equation (5) yields

$$\frac{dm}{dt} = \frac{1}{\sqrt{B^2 + 4At}}. \tag{6}$$

Solving for t gives

$$t = \frac{\dfrac{1}{(dm/dt)^2} - B^2}{4A}. \tag{7}$$

In addition if we differentiate equation (4) with respect to m and use $dm/dt = 1/(dt/dm)$ we find

$$\frac{dm}{dt} = \frac{1}{2mA + B} \tag{8}$$

which can also be inverted to express m in terms of dm/dt.

If we can estimate dm/dt then we can use these formulas to estimate t and m. A simple-minded way to get a rough estimate for dm/dt is to run the VCR for one minute ($\Delta t = 1$) and calculate Δm from the counter. There are also more sophisticated methods which involve getting several values of the function and then using elementary numerical analysis to estimate the derivative.

Be warned, however: dm/dt changes rapidly at the beginning of the tape, and much more slowly at the end. For the particular tape used in this example, we

computed the following data from equation (7).

dm/dt	t
30	8
26	15
22	53
18	101
14	195
13	234
12	282
11	344

It follows that near the end of the tape, where dm/dt changes slowly, we need to know it more accurately in order to approximate t or m. We might let the tape run for ten minutes, for example, and divide Δm by 10 to get an estimate with one decimal place. A stopwatch might also be used for more careful estimates.

Addendum. It has come to my attention that some of the material in this paper has previously appeared in an article by Arnold J. Insel: "Cassette Tape: Predicting Recording Time," the UMAP Journal, Vol. V, No. 2, 1984, pp. 200–214.

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THE LESTER R. FORD AWARDS FOR 1990

The 1990 recipients of the Lester R. Ford awards for mathematical exposition in the American Mathematical Monthly were announced at the 1991 summer meetings of the MAA in Orono, ME. The awards were given to

Joyce Justicz (Emory University), Edward Scheinerman (Johns Hopkins University), and Peter Winkler (Emory University) for their article *Random Intervals*, in the December, 1990 issue.

Marcel Berger (IHES in Paris) for his article *Convexity* in the special geometry issue, October, 1990.

Ronald Graham (AT&T Bell Laboratories) and Frances Yao (Xerox PARC) for their article *A Whirlwind Tour of Computational Geometry* in the special geometry issue, October, 1990.

Strange Series and High Precision Fraud

J. M. Borwein and P. B. Borwein

INTRODUCTION. Five of the following twelve series approximations are exact. The remaining seven are not identities but are approximations that are correct to at least 30 digits. One in fact is correct to over 18,000 digits and another to in excess of a billion digits. The reader is invited to separate the true from the bogus. (For answers see the end of the introduction.) Most of these series are easily amenable to high precision calculation in one's favorite high precision environment, such as Maple or MACSYMA, and provide examples of "caveat computat." Things are not always as they appear.

Sum 1

$$\sum_{n=1}^{\infty} \frac{a(2^n)}{2^n} \doteq \frac{1}{99}$$

where $a(n)$ counts the number of odd digits in odd places in the decimal expansion of n. ($a(901) = 2$, $a(210) = 0$, $a(811) = 1$, here the 1st digit is the 1st to the left of the decimal point.)

Sum 2

$$\sum_{n=1}^{\infty} \frac{a(n)}{10^n} \doteq \frac{10}{99}$$

where $a(n)$ is as above.

Sum 3

$$\sum_{n=1}^{\infty} b(n)\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) \doteq \frac{25\pi^2}{297}$$

where $b(n)$ counts the number of odd digits in n ($b(901) = 2$, $b(811) = 2$, ($b(406) = 0$).

Sum 4

$$\sum_{n=1}^{\infty} \frac{c(n)}{2^n} \doteq \frac{511}{8184}$$

where $c(n) := 32c_1(n) - c_2(n)/32$, and $c_1(n)$ counts the number of nines in n, while $c_2(n)$ counts the number of eights in n ($c(8199) = 32 \cdot 2 - 1/32$).

Sum 5

$$1+\sum_{n=1}^{\infty}\frac{(-1)^{\delta(n)}}{16^{\delta(n)}(D(n))^4}\doteq 2\frac{(e^{\pi/2}-e^{-\pi/2})}{\pi^2}$$

where $\delta(n)$ is the number of ones in the binary expansion of n and $D(n)$ is the product $\prod_i \max\{i\delta_i(n), 1\}$ where $\delta_i(n)$ is the ith binary digit of n ($\delta(1011_2) = 3$, $D(1011_2) = 4 \cdot 2 \cdot 1 = 8$).

Sum 6

$$\sum_{n=1}^{\infty}\frac{e(n)}{n(n+1)}\doteq\frac{10}{99}\log 10$$

where $e(n)$ "reflects" n through the decimal point ($e(123) = .321$, $e(90140) = .04109$).

Sum 7

$$\sum_{n=1}^{\infty}\frac{b(2^n)}{2^n}\doteq\frac{1}{9}$$

where $b(n)$ counts the number of odd digits in n (as in Sum 3).

Sum 8

$$\sum_{n=1}^{\infty}\frac{e(2^n)}{2^n}\doteq\frac{3166}{3069}$$

where $e(n)$ counts the number of even digits in n.

Sum 9

$$\sum_{n=1}^{\infty}\frac{\lfloor n\tanh\pi\rfloor}{10^n}\doteq\frac{1}{81}$$

where $\lfloor\ \rfloor$ is the greatest integer function ($\lfloor 3.7\rfloor = 3$).

Sum 10

$$\sum_{n=1}^{\infty}\frac{\left\lfloor ne^{\pi\sqrt{163/9}}\right\rfloor}{2^n}\doteq 1280640$$

Sum 11

$$\sum_{-\infty}^{\infty}\frac{1}{10^{(n/100)^2}}\doteq 100\sqrt{\frac{\pi}{\log 10}}$$

Sum 12

$$\left(\frac{1}{10^5}\sum_{n=-\infty}^{\infty}e^{-(n^2/10^{10})}\right)^2\doteq\pi$$

These sums break into four types. Sums 2, 3, 4, 5, and 6 are all specializations of generating functions for digit sums, more-or-less of the type:

$$\prod_{n=0}^{\infty}(1+q^{2^n}) = \sum_{n=0}^{\infty} x^{\delta(n)}q^n \tag{1.1}$$

where $\delta(n)$ counts the number of ones in the binary expansion of n. These are treated in section 2. See also [**14**].

Sums 1 and 7 are related to a problem independently due to E. Levine (*College Math Journal*, Vol. 19, number 5, 1989) and to D. Bowman and T. White (*Amer. Math. Monthly*, Vol. 96 1989, p. 743), which asks if

$$\sum_{n=0}^{\infty}\frac{g(2^n)}{2^n} = \frac{2}{9}$$

where $g(n)$ counts the number of digits ≥ 5 in n. The key to the solution we provide is due to our colleague A. C. Thompson. See section 3.

The sums 8, 9 and 10 revolve around the fact that

$$\sum_{n=0}^{\infty} w^{\lfloor n\alpha\rfloor}q^n$$

has a particularly attractive and rapidly convergent generating function that is related to the continued fraction expansion of α. This is essentially an observation of Mahler's [**11**], though the development we offer in section 4 is quite distinct. See also [**10**], [**3**]. This is closely related to problem #E3353 in the *MAA Monthly* due to H. Diamond [**6**].

The last section deals with series like Sums 11 and 12. There are consequences of the fact that $f(t) := \sum_{n=-\infty}^{\infty} e^{-n^2 t\pi}$ is a modular form and satisfies a simple functional equation linking $f(t)$ and $f(1/t)$.

The fradulent series are: Sum 2 (correct to 99 digits), Sum 4 (correct to 240 digits), Sum 8 (correct to 30 digits), Sum 9 (correct to 267 digits), Sum 10 (correct to at least half a billion digits), Sum 11 (correct to at least 18,000 digits), and Sum 12 (correct to at least 42 billion digits).

GENERATING FUNCTIONS—PART ONE. Many digit sums are generated by the following type of argument.

Example 2.1. Let $b(n)$ count the number of odd digits in n base 10 (as in Sums 3 and 7). Then for $|q| < 1$,

$$\begin{aligned}\sum_{n=0}^{\infty} x^{b(n)}q^n &= \prod_{n=0}^{\infty}\bigl(1 + xq^{10^n} + q^{2\cdot 10^n} + xq^{3\cdot 10^n} + q^{4\cdot 10^n} + xq^{5\cdot 10^n} + q^{6\cdot 10^n}\\ &\qquad + xq^{7\cdot 10^n} + q^{8\cdot 10^n} + xq^{9\cdot 10^n}\bigr)\\ &=: \prod_{n=0}^{\infty} r(x, q^{10^n}).\end{aligned} \tag{2.1}$$

To see this, observe that in the expansion of the product each power of q^m arises in exactly one way. This is just the unique expansion of m base 10. The coefficient of q^m is just a product of x's, one for each odd digit in m. If we differentiate (2.1) with respect to x as is legitimate since $b(n) = O(n)$ and the derivatives converge

uniformly, we get

$$\frac{\sum_{n=0}^{\infty} b(n) x^{b(n)-1} q^n}{\sum_{n=0}^{\infty} x^{b(n)} q^n} = \sum_{n=0}^{\infty} \frac{q^{10^n} + q^{3\cdot 10^n} + q^{5\cdot 10^n} + q^{7\cdot 10^n} + q^{9\cdot 10^n}}{1 + xq^{10^n} + q^{2\cdot 10^n} + \cdots + q^{8\cdot 10^n} + xq^{9\cdot 10^n}} \tag{2.2}$$

and at $x := 1$

$$\frac{\sum_{n=0}^{\infty} b(n) q^n}{(1-q)^{-1}} = \sum_{n=0}^{\infty} \frac{q^{10^n} + q^{3\cdot 10^n} + \cdots + q^{9\cdot 10^n}}{1 + q^{10^n} + q^{2\cdot 10^n} + \cdots + q^{9\cdot 10^n}} \tag{2.3}$$

$$= \sum_{n=0}^{\infty} \frac{q^{10^n}}{1 + q^{10^n}}$$

$$=: \sum_{n=0}^{\infty} R(q^{10^n})$$

where the second last equality follows on factoring each term. It is apparent from this representation for example that

$$\sum_{n=0}^{\infty} b(n) q^n = \frac{1}{1-q}\left(\frac{q^1}{1+q^1} + \frac{q^{10}}{1+q^{10}}\right) + O(q^{100}). \tag{2.4}$$

We need the following observation which we encapsulate as Lemma 2.1.

Lemma 2.1. *Suppose $R(q)$ is a non-negative, measurable function on* $[0,1]$. *If* $b > 1$ *and*

$$f(q) := \sum_{n=0}^{\infty} R(q^{b^n}) \qquad |q| < 1$$

then

$$\int_0^1 \frac{f(q)}{q}\, dq = \frac{b}{b-1} \int_0^1 \frac{R(q)}{q}\, dq.$$

Proof:

$$\int_0^1 \frac{f(q)}{q}\, dq = \int_0^1 \sum_{n=0}^{\infty} \frac{R(q^{b^n})}{q}\, dq$$

$$= \sum_{n=0}^{\infty} \int_0^1 \frac{R(q^{b^n})}{q}\, dq$$

$$= \sum_{n=0}^{\infty} \int_0^1 \frac{S(q^{b^n}) q^{b^n}}{q}\, dq$$

where $S(q) := R(q)/q$.

Now set $u = q^{b^n}$ and observe that

$$\int_0^1 \frac{f(q)}{q}\, dq = \sum_{n=0}^{\infty} \int_0^1 \frac{S(u)}{b^n}\, du$$

and the lemma is proved. (The interchange of sum and integral is just the monotone convergence theorem.) ■

From (2.3) we have

$$\sum_{n=0}^{\infty} b(n)q^{n-1}(1-q) = \sum_{n=0}^{\infty} \frac{R(q^{10^n})}{q} \tag{2.5}$$

and with Lemma 2.1,

$$\sum_{n=1}^{\infty} b(n)\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{10}{9}\int_0^1 \frac{1}{1+q}\,dq$$

or

$$\sum_{n=1}^{\infty} \frac{b(n)}{n(n+1)} = \frac{10}{9}\log 2. \tag{2.6}$$

Indeed this process iterates, in the sense that we can keep dividing by q and integrating in (2.5). This yields with some effort the following

Sum 13. For k a positive integer

$$\sum_{n=1}^{\infty} b(n)\left(\frac{1}{n^k} - \frac{1}{(n+1)^k}\right) = \frac{10^k}{10^k - 1}\alpha(k)$$

where, α is the alternating zeta function,

$$\alpha(s) := (1 - 2^{1-s})\zeta(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}.$$

Note that Sum 3 is just the $k := 2$ case of the above, while $k := 1$ gives (2.6).

A direct derivation of Sum 13 valid for non-integer k can be based on the fact that:

$$\alpha(s)\sum_{n=1}^{\infty} \alpha_n n^{-s} = \sum_{n=1}^{\infty} b_n n^{-s}$$

if and only if

$$\sum_{n=1}^{\infty} \alpha_n \frac{x^n}{1+x^n} = \sum_{n=1}^{\infty} b_n x^n.$$

This identity is now coupled with (2.3). See [**18**].

Example 2.2. The generating function for q, the number of odd digits in odd places (as in Sums 1 and 2), is given by

$$\sum_{n=0}^{\infty} x^{a(n)}q^n = \prod_{n=0}^{\infty} r\left(x, q^{10^{2n}}\right)$$

where

$$r(x,q) := \left(1 + xq + q^2 + xq^3 + q^4 + \cdots + xq^9\right)$$
$$\cdot\left(1 + q^{10} + q^{2\cdot 10} + q^{3\cdot 10} + \cdots + q^{9\cdot 10}\right)$$

and leads, as in (2.3), to the series

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{1-q}\sum_{n=0}^{\infty} \frac{q^{100^n}}{1+q^{100^n}}. \tag{2.7}$$

Sum 2 now appears on taking $q := \frac{1}{10}$ and using the first term of the above expansion. It is apparent that the remainder is positive of size very close to $\frac{1}{9} \cdot 10^{-99}$. This gives the nature of the estimate in Sum 2.

In similar fashion

$$\sum_{n=0}^{\infty} A_k(n) q^n = \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{q^{(10^k)^n}}{1 + q^{(10^k)^n}} \tag{2.8}$$

is the generating function for the number of odd digits in the 1st, $(k+1)$th, $(2k+1)$th places of k. So with $k = 10$, for example

$$\sum_{n=0}^{\infty} \frac{A_{10}(n)}{10^n} = \frac{10}{99} + \varepsilon_n \tag{2.9}$$

where $0 < |\varepsilon_n| < \frac{10}{9} \cdot 10^{-10^{10}}$, and the above approximation is correct to over a billion digits. ■

Example 2.3. The number of times the digit $i > 0$ occurs in n has generating function

$$\sum_{n=0}^{\infty} g(n) q^n = \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{q^{i \cdot 10^n}}{1 + q^{10^n} + \cdots + q^{9 \cdot 10^n}}.$$

So the generating function for $c(n)$ in Sum 4 is just

$$\sum_{n=0}^{\infty} c(n) q^n = \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{32 q^{9 \cdot 10^n} - \dfrac{q^{8 \cdot 10^n}}{32}}{1 + q^{10^n} + \cdots + q^{9 \cdot 10^n}}.$$

At $q := \frac{1}{2}$, the second term vanishes to give

$$\sum_{n=0}^{\infty} \frac{c(n)}{2^n} = \frac{1}{1-q} \left(\frac{32 q^9 - \dfrac{q^{8 \cdot 10^n}}{32}}{1 + \cdots + q^9} \right) + O(q^{800})$$

$$= \frac{511}{8184} + \varepsilon$$

where $\varepsilon < 10^{-241}$.

Example 2.4. The generating function which reverses digits, as in Sum 6, is

$$\sum_{n=0}^{\infty} x^{e(n)} q^n = \prod_{n=0}^{\infty} \left(1 + x^{1/10^{n+1}} q^{10^n} + \cdots + x^{9/10^{n+1}} q^{9 \cdot 10^n}\right). \tag{2.10}$$

So

$$\sum_{n=0}^{\infty} e(n) q^n = \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{\dfrac{1}{10^{n+1}} q^{10^n} + \cdots + \dfrac{9}{10^{n+1}} q^{9 \cdot 10^n}}{1 + q^{10^n} + \cdots + q^{9 \cdot 10^n}} \tag{2.11}$$

and as in Lemma 2.1

$$\sum_{n=1}^{\infty} \frac{e(n)}{n(n+1)} = \frac{10}{99} \log 10.$$

There are very many analogues of these results. All have variations in different bases. The binary digit counting functions δ has generating function

$$\sum_{n=0}^{\infty} x^{\delta(n)} q^n = \prod_{n=0}^{\infty} (1 + xq^{2^n}) \tag{2.12}$$

and

$$\sum_{n=0}^{\infty} \delta(n) q^n = \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{q^{2^n}}{1+q^{2^n}} \tag{2.13}$$

whence

$$\sum_{n=1}^{\infty} \frac{\delta(n)}{n(n+1)} = 2 \log 2. \tag{2.14}$$

(See the Putnam examinations of 1981, 1984 and 1987.) As in Example 2.1 we have Sum 14.

Sum 14. Let $\delta(n)$ denote the sum of the binary digits of n. Then

$$\sum_{n=1}^{\infty} \delta(n)\left(\frac{1}{n^k} - \frac{1}{(n+1)^k}\right) = \left(\frac{2^k}{2^k - 1}\right)\alpha(k)$$

where $\alpha(k)$ is the alternating zeta function.

The sum of the decimal digits of n denoted $s(n)$ has generating function

$$\sum_{n=0}^{\infty} x^{s(n)} q^n = \prod_{n=0}^{\infty} (1 + xq^{10^n} + x^2 q^{2\cdot 10^n} + \cdots + x^9 q^{9\cdot 10^n}) \tag{2.15}$$

from which we deduce that

$$\sum_{n=1}^{\infty} \frac{s(n)}{n(n+1)} = \frac{10}{9} \log 10. \tag{2.16}$$

Loxton and van der Poorten **[10]** and Mahler **[11]** treat transcendence questions for functions, with power series expansions at zero which satisfy functional equations. From these results, one knows that if f, holomorphic at zero and not an algebraic function, satisfies a function equation of the form

$$f(q^m) = f(q) + R(q) \tag{2.17}$$

where m is an integer and R is a rational function, then $f(\alpha)$ is transcendental for algebraic α. From this we deduce that the exact answers in Sum 2, Sum 4 and Sum 8, are transcendental. This can also be deduced easily from Roth's Theorem **[8]**.

GENERATING FUNCTIONS—PART TWO. A second type of digit function arises as follows.

Example 3.1. Let $\delta(n)$ as before, denote the sum of the binary digits of n, and let $\rho(n) := \prod\{S_i\colon i\text{th binary digit of } n \neq 0\}$ and $\rho(0) := 1$, where S_i is a given sequence and the product is taken over those binary digits of n which equal one. Then formally

$$\sum_{n=0}^{\infty} \frac{x^{\delta(n)} q^n}{\rho(n)} = \prod_{n=0}^{\infty} \left(1 + \frac{x}{S_{n+1}} q^{2^n}\right) \tag{3.1}$$

and

$$\sum_{n=0}^{\infty} \frac{x^{\delta(n)}}{\rho(n)} = \prod_{n=0}^{\infty} \left(1 + \frac{x}{S_{n+1}}\right).$$

Example 3.2. Let $\delta(n)$ denote the sum of the binary digits of n, and let

$$D(n) = \prod i$$

where the product is taken over those i where the ith binary digit of n is non-zero (as in Sum 5). So, if $0 < n_1 < n_2 < \cdots < n_k$,

$$D(2^{n_1} + 2^{n_2} + \cdots + 2^{n_k}) = (n_1 + 1)(n_2 + 1) \cdots (n_k + 1).$$

Then as in Example 3.1, starting with

$$F_q(x) := x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2} q^{2^{n-1}}\right) = x \prod_{n=0}^{\infty} \left(1 - \frac{x^2}{(n+1)^2} q^{2^n}\right) \tag{3.2}$$

we have, for $|x| < 1$,

$$F_1(x) = \frac{\sin \pi x}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^{\delta(n)} x^{2\delta(n)+1}}{[D(n)]^2} \tag{3.3}$$

and at $x := \frac{1}{2}$

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^{\delta(n)}}{4^{\delta(n)}[D(n)]^2}. \tag{3.4}$$

Similarly, starting with

$$\frac{(\sin \pi x)(\sinh \pi x)}{\pi^2} = x^2 \prod_{n=1}^{\infty} \left(1 - \frac{x^4}{n^4}\right) \tag{3.5}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{\delta(n)} x^{4\delta(n)+2}}{[D(n)]^4},$$

we have, at $x := \frac{1}{2}$,

$$2\left(\frac{e^{\pi/2} - e^{-\pi/2}}{\pi^2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^{\delta(n)}}{16^{\delta(n)}[D(n)]^4}, \tag{3.6}$$

which is Sum 5.

Example 3.3. Let $t(n) := \sum i$, where the sum is taken over the non-zero digits on n base 2. So $t(1011_2) = 4 + 0 + 2 + 1 = 7$. Then

$$\prod_{n=0}^{\infty} (1 - x^{n+1} q^{2^n}) = \sum_{n=0}^{\infty} (-1)^{\delta(n)} x^{t(n)} q^n. \tag{3.7}$$

So

$$\sum_{n=0}^{\infty} (-1)^{\delta(n)} x^{t(n)} = \prod_{n=1}^{\infty} (1 - x^n) = \sum_{-\infty}^{\infty} (-1)^n x^{(3n+1)n/2} \tag{3.8}$$

on using Euler's pentagonal number theorem [2] and on integrating, from zero to one,

$$\sum_{n=0}^{\infty} \frac{(-1)^{\delta(n)}}{t(n)+1} = \sum_{-\infty}^{\infty} \frac{2(-1)^n}{3n^2+n+2}. \tag{3.9}$$

4. CONTINUED FRACTION EXPANSIONS. The identities of this section are based on the two functions

$$G_\alpha(z,w) := \sum_{n=1}^{\infty} z^n w^{\lfloor n\alpha \rfloor} \tag{4.1}$$

and

$$F_\alpha(z,w) := \sum_{n=1}^{\infty} z^n \sum_{m=1}^{\lfloor n\alpha \rfloor} w^m \tag{4.2}$$

where α is a non-negative real number and $\lfloor n\alpha \rfloor$ is the integer part of $n\alpha$, while z and w are complex with modulus so as to ensure convergence. The function F_α was studied by Mahler [**11**] and is obviously related to G_α by

$$F_\alpha(z,w) + \frac{w}{1-w} G_\alpha(z,w) = \frac{zw}{(1-z)(1-w)} \tag{4.3}$$

for $|z|, |w| < 1$. Van der Poorten [**10**] comments that Mahler's paper has been largely overlooked. In [**3**] we explore these matters further. Note that for positive z and w, F_α is strictly increasing as a function of α.

For irrational α we will use the infinite continued fraction approximations generated by

$$\begin{array}{llll} \text{(a)} & p_{n+1} := p_n a_{n+1} + p_{n-1} & p_0 := a_0 = \lfloor \alpha \rfloor, & p_{-1} := 1 \\ \text{(b)} & q_{n+1} := q_n a_{n+1} + q_{n-1} & q_0 := 1, & q_{-1} := 0 \end{array} \tag{4.4}$$

for $n \geq 0$ where

$$\begin{aligned} \alpha &= [a_0, a_1, \ldots, a_n, a_{n+1}, \ldots] \\ &= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \end{aligned}$$

so that each a_i is integral, $a_0 \geq 0$ and $a_n \geq 1$ for $n \geq 1$. Then for $n \geq 0$ p_{2n}/q_{2n} increases to α while p_{2n+1}/q_{2n+1} decreases to α and

$$\frac{1}{q_n(q_n + q_{n+1})} < \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}. \tag{4.5}$$

All of this is standard and may be found in [**8**], [**9**], or [**16**]. We will avoid using finite continued fractions which arise only for rational α. Let us write $q_n\alpha - p_n$ as ε_n. By (4.5) and (4.4)

$$|\varepsilon_{n+1}| < \frac{1}{q_n + q_{n+1}} < |\varepsilon_n| < \frac{1}{q_{n+1}} \leq 1.$$

A key lemma is:

Lemma 4.1. *For irrational $\alpha > 0$ and n, N in* $\mathbf{N}$

(a) $$\lfloor n\alpha + \varepsilon_N \rfloor = \lfloor n\alpha \rfloor \qquad \text{for } n < q_{N+1}$$

(b) $$\lfloor n\alpha + \varepsilon_N n \rfloor = \lfloor n\alpha \rfloor + (-1)^N \quad \text{for } n = q_{N+1}.$$

Proof: Suppose N is even (the odd case is entirely parallel). Then $\varepsilon_N > 0$ and (a) fails when

$$n\alpha + \varepsilon_N > m > n\alpha \quad \text{for some } m \text{ in } \mathbf{N}. \tag{4.6}$$

As $\alpha > p_N/q_N$, we have an integer k with

$$(n + q_N)\varepsilon_N > mq_N - np_N = k > 0.$$

If $k \geq 2$ then $n + q_n > 2/\varepsilon_N > 2q_{N+1}$ and $n > q_{N+1}$.

If $k = 1$ we have

$$p_N q_N - q_N p_N = 0, \qquad p_{N+1}q_N - q_{N+1}p_N = 1,$$

so that the linear Diophantine equation $mq_N - np_N = 1$ has general solution $m = p_{N+1} + sp_N$, $n = q_{N+1} + sq_N$ for s integer. However, $n + q_N > 1/\varepsilon_N > q_{N+1}$ so that s is non-negative. This establishes (a). For $n = q_{N+1}$ we have

$$q_{N+1}\alpha < p_{N+1} < q_{N+1}\alpha + \varepsilon_N < p_{N+1} + 1$$

since $p_{N+1} > q_{N+1}\alpha$ and $0 < \varepsilon_{N+1} + \varepsilon_N < 1$. This yields (b). ∎

Theorem 4.1.

(a) *For rational $\alpha = p/q$ (reducible or irreducible)*

$$(1 - z^q w^p)G_\alpha(z,w) = \sum_{j=1}^{q} z^j w^{\lfloor jp/q \rfloor}.$$

(b) *For irrational α and $N > 0$*

$$(1 - z^{q_N} w^{p_N})G_\alpha(z,w)$$
$$= \sum_{n=1}^{q_n} z^n w^{\lfloor n\alpha \rfloor} + (-1)^N \left(\frac{w-1}{w}\right) z^{q_N} w^{p_N} z^{q_{N+1}} w^{p_{N+1}} + R_N(z,w)$$

with

$$|R_N(z,w)| \leq |1 - w| \frac{|z|^{q_{N+1}+q_N+1}}{1 - |z|}.$$

Proof:

(a) $$G_\alpha(z,w) = \sum_{k=0}^{\infty} \sum_{j=1}^{q} z^{qk+j} w^{kp + \lfloor j(p/q) \rfloor}$$
$$= \sum_{k=0}^{\infty} (z^q w^p)^k \sum_{j=1}^{q} z^j w^{\lfloor j(p/q) \rfloor},$$

(b) $$(1 - z^{q_N} w^{p_N})G_\alpha(z,w) - \sum_{n=1}^{q_N} z^n w^{\lfloor n\alpha \rfloor}$$
$$= \sum_{n=1}^{\infty} z^{n+q_N}\{w^{\lfloor (n+q_N)\alpha \rfloor} - w^{p_N + \lfloor n\alpha \rfloor}\}$$
$$= \sum_{n=1}^{\infty} z^{n+q_N} w^{\lfloor n\alpha \rfloor + p_N}\{w^{\lfloor n\alpha + \varepsilon_N \rfloor - \lfloor n\alpha \rfloor} - 1\}.$$

By the proof of Lemma 4.1, the first non-zero term in this last expression is $(-1)^N(w-1)/w)z^{q_N+q_{n+1}}w^{p_N+p_{N+1}}$ while the other terms are dominated by $|z|^n|1-w|$ with $n > q_N + q_{N+1}$. ∎

For fixed $\alpha > 0$ we write

$$P_N := \sum_{n=1}^{q_N} z^n w^{\lfloor n\alpha \rfloor}, \qquad Q_N := 1 - z^{q_N}w^{p_N}$$

and observe that Theorem 4.1 shows that

$$G_\alpha - \frac{P_N}{Q_N} = (-1)^N\left(\frac{w-1}{w}\right)\frac{z^{q_N}w^{p_N}z^{q_{N+1}}w^{p_{N+1}}}{Q_N} + O(z^{q_N+q_{N+1}}+1) \quad (4.7)$$

for α irrational (while $G_\alpha = P_N/Q_N$ for rational α). Thus as a function of z P_N/Q_N is the main diagonal Padé approximation to G_α of order q_N.

Corollary 4.1. *For irrational* $\alpha > 0$

$$G_\alpha(z,w) = \frac{zw^{p_0}}{1-zw^{p_0}} - \frac{1-w}{w}\sum_{n=0}^{\infty}(-1)^n\frac{z^{q_n}w^{p_n}z^{q_{n+1}}w^{p_{n+1}}}{(1-z^{q_n}w^{p_n})(1-z^{q_{n+1}}w^{p_{n+1}})}. \quad (4.8)$$

Proof: Let $A_N := P_{N+1}Q_N - Q_{N+1}P_N$. Then A_N is a polynomial of degree at most $q_{N+1}+q_N$ in z. From (4.7) we see that

$$\frac{P_{N+1}}{Q_{N+1}} - \frac{P_N}{Q_N} = \frac{A_N}{Q_NQ_{N+1}} = (-1)^N\left(\frac{w-1}{w}\right)\left\{\frac{z^{q_N}w^{p_N}z^{q_{N+1}}w^{p_{N+1}}}{Q_NQ_{N+1}}\right\}.$$

On summing from zero to infinity we produce (4.8). ∎

This is derived by Mahler for $\alpha \in (0,1)$ in **[11]**.

Corollary 4.2. *For irrational* $\alpha > 0$ *and for* $w \neq 1$

$$F_\alpha(z,w) = \frac{zw}{(1-z)(1-w)}\frac{1-w^{p_0}}{1-zw^{p_0}} + \sum_{n=0}^{\infty}\frac{(-1)^n z^{q_n}w^{p_n}z^{q_{n+1}}w^{p_{n+1}}}{(1-z^{q_n}w^{p_n})(1-z^{q_{n+1}}w^{p_{n+1}})}. \quad (4.9)$$

In particular, for $w = 1$, *the spectrum of* α [7] *is generated by*

$$\sum_{n=1}^{\infty}\lfloor n\alpha\rfloor z^n = \frac{p_0 z}{(1-z)^2} + \sum_{n=0}^{\infty}(-1)^n\frac{z^{q_n}z^{q_{n+1}}}{(1-z^{q_n})(1-z^{q_{n+1}})}. \quad (4.10)$$

Proof: Equation (4.9) follows from (4.8) and (4.3). Equation (4.10) is now obtained by letting w tend to 1. ∎

If F_N denotes the truncation of the right-hand side of (4.9)

$$\frac{zw}{(1-z)(1-zw^{p_0})}\left(\frac{1-w^{p_0}}{1-w}\right) + \sum_{n=0}^{N-1}(-1)^n\frac{z^{q_n}w^{p_n}z^{q_{n+1}}w^{p_{n+1}}}{(1-z^{q_n}w^{p_n})(1-z^{q_{n+1}}w^{p_{n+1}})}$$

we observe that (4.7) and (4.3) show that

$$F_N = \frac{\left(\frac{w}{1-w}\right)[zQ_N - (1-z)P_N]}{(1-z)(1-z^{q_N}w^{p_N})} \quad (4.11)$$

and some manipulation shows that, for $q_N > 1$, the numerator may be rewritten as

$$B_N := wz \sum_{n=1}^{q_N} z^n \left(\frac{w^{\lfloor (n+1)\alpha \rfloor} - w^{\lfloor n\alpha \rfloor}}{1-w} \right) + wz \left(\frac{1 - w^{p_0}}{1-w} \right) (1 - z^{q_N} w^{p_N}) \quad (4.12)$$

so that B_N is a very simple integer polynomial in w and z (of degree $q_N + 1$ in z), while

$$F_\alpha - F_N = O(z^{q_N + q_{N+1}}).$$

Note that F_N is especially simple for $w := 1$ and $0 < \alpha < 1$.

Example 4.1. (a) Let $\alpha := \pi/2$ in (4.11) or (4.10). As

$$\frac{\pi}{2} = [1, 1, 1, 31, \dots]$$

we have $p_0 = 1$, $p_1 = 2$, $p_2 = 3$, $p_3 = 11$, $p_4 = 344$ and $q_0 = 1$, $q_1 = 1$, $q_2 = 2$, $q_3 = 7$, $q_4 = 219$. Thus

$$\begin{aligned} F_{\pi/2}(z,1) &= \sum_{n=1}^{\infty} \left\lfloor \frac{\pi}{2} n \right\rfloor z^n \\ &= \frac{z}{(1-z)^2} + \frac{z^2}{(1-z)^2} - \frac{z^3}{(1-z)(1-z^2)} \\ &\quad + \frac{z^9}{(1-z^2)(1-z^7)} - \frac{z^{226}}{(1-z^7)(1-z^{219})} + \dots \end{aligned}$$

and the approximation F_4 is also expressible as

$$\frac{z(z^7 + z^6 + 2z^5 + z^4 + 2z^3 + z^2 + 2z + 1)}{(1-z^7)(1-z)}$$

and has an error like z^{226}. In particular

$$\sum_{n=1}^{\infty} \frac{\left\lfloor \frac{\pi}{2} n \right\rfloor}{2^n} \doteq \frac{339}{127}$$

with error less than 10^{-68}.

(b) Sum 9 follows from using (4.10) for $\tanh(\pi) = [0, 1, 267, \dots]$. This produces

$$\sum_{n=1}^{\infty} \lfloor n \tanh \pi \rfloor z^n = \frac{z^2}{(1-z)^2} - \frac{z^{269}}{(1-z)(1-z^{268})} + \cdots.$$

(c) Sum 10 follows similarly from (4.10) with one of our favorite transcendental numbers $\alpha := e^{\pi\sqrt{163/9}} = [640320, 1653264929, \dots]$.

(d) Let $\alpha := \log_{10}(2) = [0, 3, 3, 9, \dots]$. Then (4.11) with $N := 3$, $z := \frac{1}{2}$ and $w := 1$ gives

$$\sum_{n=1}^{\infty} \frac{\lfloor n \log_{10}(2) \rfloor}{2^n} \doteq \frac{146}{1023}$$

to 30 places since $q_0 = 1$, $q_1 = 3$, $q_2 = 10$, $q_3 = 93$. Thus, as the number of even digits in 2^n is $\lfloor n \log_{10}(2) \rfloor + 1$ less the number of odd digits in 2^n, the "false" Sum 8 follows from Sum 7 and this "false" identity. In fact, see below, Sum 8 is transcendental while Sum 7 is rational. ■

Other lovely approximations follow from

$$\log_{10}(6) = [0,1,3,1,1,32,\dots]$$
$$\tanh(1) = [1,3,7,9,11,\dots]$$
$$\frac{e-1}{2} = [0,1,6,10,14,\dots]$$

and other simple transcendental numbers. Thus

$$\sum_{n=1}^{\infty} \frac{\lfloor n\zeta(3)\rfloor}{2^n} \doteq \frac{64}{31}$$

to 30 places.

Example 4.2. Many other related sums can be derived from (4.8) and (4.9). We indicate some classes.

(a) For irrational $\alpha > 0$

$$G_\alpha(1,w) = \sum_{n=1}^{\infty} w^{\lfloor n\alpha\rfloor} = \left(\frac{1-w}{w}\right) F_{1/\alpha}(w,1),$$

and more generally

$$G_\alpha(z,w) = \left(\frac{1-w}{w}\right) F_{1/\alpha}(w,z).$$

This follows either from the elementary identity in **[11]**

$$F_\alpha(z,w) + F_{\alpha^{-1}}(w,z) = \frac{zw}{(1-z)(1-w)} \tag{4.13}$$

or from Theorem 2 in **[13]**, when $z = 1$.

(b) Letting $w := -1$ in (4.9) produces a Lambert-like series for $\Sigma_{\lfloor n\alpha\rfloor \text{ odd}} z^n$. As an example,

$$\sum \left\{ \frac{1}{2^n} \middle| \text{length}(2^n) \text{ even} \right\} \doteq \frac{114}{1025}$$

to 30 places.

(c) Observe that

$$\sum_{k=0}^{M} \frac{(-1)^k \binom{M}{R} G_\alpha(z,w^k)}{(1-w)^M} = \sum_{n=1}^{\infty} \left(\frac{1-w^{\lfloor n\alpha\rfloor^M}}{1-w}\right) z^n$$

so that on letting w tend to unity we obtain the approximation

$$\sum_{n=1}^{\infty} \lfloor n\alpha\rfloor^M z^n = \frac{\Delta_N^M(z)}{(1-z)(1-z^{q_N})^M} + O(z^{q_N+q_{N+1}})$$

where Δ_N^M is an integer polynomial in z of degree $MqN + 1$. In particular

$$\begin{aligned}\sum_{n=1}^{\infty} \lfloor n\alpha\rfloor^2 z^n = \sum_{n=0}^{\infty} &\frac{z^{q_n+q_{n+1}}}{(1-z^{q_n})^2(1-z^{q_{n+1}})^2} \\ &\times\{(2p_n + 2p_{n+1} - 1) - z^{q_n}z^{q_{n+1}} \\ &\quad -(2p_n - 1)z^{q_{n+1}} - (2p_{n+1} - 1)z^{q_n}\}\end{aligned}$$

for $0 < \alpha < 1$, α irrational. Thus

$$\sum_{n=1}^{\infty} \frac{(\text{length}(6^n))^2}{6^n} \doteq \frac{196669}{37303}$$

to 88 places.

(d) Similarly, if w is a primitive Nth root of unity

$$\frac{1}{N} \sum_{k=1}^{N} G_\alpha(z, w^k)\bar{w}^{Mk} = \sum_{\lfloor n\alpha \rfloor \equiv M \pmod N} z^n$$

[compare (b)]. Thus

$$\sum_{3 \mid \lfloor n \log_{10} 2 \rfloor} \frac{1}{3^n} \doteq \frac{3554}{7381}$$

to 50 places.

(e) Let $w := e^{i\theta}$(θ real) in (4.9). We obtain

$$\sum_{n=1}^{\infty} \cos(\lfloor n\alpha \rfloor) z^n = \frac{\sum_{n=1}^{q_N} \cos(\lfloor n\alpha \rfloor \theta) z^n - \sum_{n=1}^{q_N} \cos(p_N - \lfloor n\alpha \rfloor \theta) z^{n+q_N}}{1 - 2z^{q_N}\cos(p_N\theta) + z^{2q_N}}$$
$$+ O(z^{q_N + q_{N+1}}),$$

with a similar expression for sin replacing cos. ■

The rational counterpart to (4.13) is

$$F_{p/q}(z, w) + F_{q/p}(w, z) = \frac{zw}{(1-z)(1-w)} + \frac{z^q w^p}{1 - z^q w^p}, \tag{4.14}$$

for p and q relatively prime.

We consider $F(\alpha) := F_\alpha(z, w)$ as a function of α, and observe that $F(\alpha)$ is continuous at each irrational. Moreover, $\lim_{\alpha \downarrow p/q} F(\alpha) = F(p/q)$. Thus, on using (4.13) and (4.14) $\lim_{a \uparrow p/q} F(\alpha) = F(p/q) - z^q w^p/(1 - z^q w^p)$. In consequence, F is discontinuous at every rational and $F(1) - F(0) = \sum_{0 < p/q < 1}\{F(p/q) - F(\frac{p}{q} -)\}$ so that dF is a "pure jump measure" on the rationals in [0, 1]. [This observation was made by H. Diamond.] Explicitly the jumps are expressed as

$$J := \sum_{s=1}^{\infty} \sum_{\substack{1 \le r \le s \\ (r,s)=1}} \frac{z^s w^r}{1 - z^s w^r}$$
$$= \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} z^{sk} \sum_{\substack{(r,s)=1 \\ 1 \le r \le s}} w^{rk}. \tag{4.15}$$

Now, on setting $n = sk$, this yields

$$\sum_{n=1}^{\infty} z^k \left\{ \sum_{s/n} \sum_{\substack{(r,s)=1 \\ 1 \le r \le s}} w^{(r/s)n} \right\}.$$

Equation (16.2.3) in [**8**] applies with $F(w) := w^n$ and shows that the bracketed term is just $\sum_{m=1}^{n} w^m$. Hence $J = \sum_{n=1}^{\infty} z^n \sum_{m=1}^{n} w^m = F_1(z, w)$ as claimed. This is valid for $|z| < 1$, $|w| \le 1$. ■

We have also shown, using Theorem 4.1(a) and $\lfloor n\alpha \rfloor = \lfloor n(p_N/q_N) \rfloor$ for $n < q_N$, that for $0 < \alpha < 1$

$$F_N = \begin{cases} F_{p_N/q_N} & N \text{ even} \\ F_{p_N/q_N} - \dfrac{z^{q_N} w^{p_N}}{1 - z^{q_N} w^{p_N}} & N \text{ odd.} \end{cases} \tag{4.16}$$

Clearly F: $Q \to Q$. In **[10]**, **[11]** (4.9) is used to obtain transcendence estimates by functional equation methods. For $w := \pm 1$ and $z := 1/b$, $b = 2, 3, 4, \ldots$ we can get very accessible estimates for F_α or G_α from Roth's theorem **[2]**, **[9]**, **[15]**.

First, observe that Corollary 4.2 shows F_α is irrational when α is irrational and w, z are rational. It is convenient to introduce

$$s := s(\alpha) = \limsup_{n \to \infty} a_n.$$

Thus s is infinite when α has unbounded continued fraction coefficients. For b and w as above, we have from (4.12)

$$0 < \left| F(\alpha) - \frac{P_N}{Q_N} \right| \le O\left(\frac{1}{b^{q_N + q_{N+1}}} \right) \le O\left(\frac{1}{Q_N^{(1 + q_{N+1}/q_N)}} \right) \tag{4.17}$$

for integers P_N and $Q_N := (b - 1)(b^{q_n} - w^{p_N})$. Hence, Roth's theorem shows $F(\alpha)$ is transcendental when

$$\limsup_{n \to \infty} \frac{q_{N+1}}{q_N} > 1,$$

and clearly α is Liouville when $s(\alpha) = \infty$. Since almost all numbers have unbounded coefficients, $F(\alpha)$ is Liouville in almost all cases and F maps Liouville numbers to Liouville numbers as they have $s =$ infinity. When $s(\alpha)$ is finite, we have $q_{N+1} \le s q_N + q_{N-1} \le (s + 1) q_N$ eventually and so infinitely often

$$q_{N+1} \ge s q_N + q_{N-1} \ge \frac{s^2 + s + 1}{s + 1} q_N$$

and (4.17) shows $F(\alpha)$ is approximable to order at least $(s + 1) + (1/(s + 1)) \ge 5/2$. If $s = 1$ then α is equivalent to $(\sqrt{5} + 1)/2$. In every other case $F(\alpha)$ is approximable to order $10/3$. In summary $F(\alpha)$ is never algebraic, indeed never has the expected rate of rational approximation and is usually Liouville (**[2]**, **[8]**, **[15]**). In fact almost all irrationals have only finitely many solutions to

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2 (\log q)^{1+}}.$$

Example 4.3. (a) Arguing similarly from Example 4.2 we see that for almost all α,

$$\sum_{n=1}^{\infty} \frac{p(\lfloor n\alpha \rfloor)}{b^n}$$

is a Liouville number, for any integer polynomial p.

It is hard to find explicit numbers with unbounded continued fraction coefficients but e and $\tanh(1)$ are two examples:

$$\sum_{n=1}^{\infty} \frac{p(\lfloor ne \rfloor)}{b^n}$$

is Liouville for all p and b.

(b) Correspondingly, $\sum_{n=1}^{\infty} p(\lfloor n\alpha \rfloor)/b^n$ is approximable to order at least

$$\frac{1 + s(\alpha)}{\deg(p)}.$$

■

For irrational $0 < \alpha < 1$, $F_\alpha(z,w)$ may be computed entirely from the continued fraction expansion via

$$F_\alpha(z,w) = \sum_{n=0}^{\infty} (-1)^n \frac{z_n z_{n+1}}{(1 - z_n)(1 - z_{n+1})}$$

where $z_{n+1} := z_n^{a_{n+1}} z_{n-1}$, $z_0 := z$, $z_{-1} := w$. This follows from (4.9) and an easy induction.

We conclude with some remarks about iterates of $F(\alpha) := \sum_{n=1}^{\infty} \lfloor n\alpha \rfloor 2^{-n}$. For $\alpha = p/q$ $(0 < \alpha < 1)$ we have

$$F_\alpha(z,w) = zw \frac{\sum_{n=1}^{q} \left(\left\lfloor (n+1)\frac{p}{q} \right\rfloor - \left\lfloor n\frac{p}{q} \right\rfloor \right) w^{\lfloor n(p/q) \rfloor} z^n}{(1 - z)(1 - z^q w^p)} \tag{4.18}$$

either by direct computation or from (4.11) and (4.16). We now set $z := \frac{1}{2}$, $w := 1$ and observe that

$$F\left(\frac{p}{q}\right) + F\left(1 - \frac{p}{q}\right) = 1 + \frac{1}{2^q - 1}.$$

In particular $F(\frac{1}{2}) = \frac{2}{3}$. Moreover, (4.18) shows that

$$F\left(1 - \frac{1}{q}\right) = 1 - \frac{1}{2^q - 1}.$$

Let $q_0 := 2$ and $q_{n+1} := 1/(2^{q_n} - 1)$ to deduce that

$$F^{(n)}\left(\frac{1}{2}\right) = 1 - \frac{1}{q_{n+1}}$$

and so converges to 1. Similar analysis shows that

$$F\left(\frac{1}{q}\right) = \frac{2}{2^q - 1} < \frac{1}{2^{q-2}},$$

and so that

$$F^{(n)}\left(\frac{1}{3}\right) \to 0, \quad \text{because } F^{(2)}\left(\frac{1}{3}\right) = \frac{18}{127} < \frac{1}{7}.$$

Note that $\alpha \geq \frac{1}{2}$ implies $F^{(n)}(\alpha) \geq F^{(n)}(\frac{1}{2})$ and $\alpha < \frac{1}{2}$ implies $F^{(n+1)}(\alpha) \to 0$ for $0 \leq \alpha < \frac{1}{2}$. For rational α, the entire sequence is rational, otherwise it is entirely transcendental, usually Liouville.

5. RATIONAL DIGIT SUMS.

This section is based on the following Lemma whose proof we owe to A. C. Thompson.

Lemma 5.1. *For* $0 < q < 1$ *and integer* $m > 1$

$$q = \sum_{n=1}^{\infty} \frac{\lfloor m^n q \rfloor \pmod{m}}{m^n}. \tag{5.1}$$

Proof: Consider the base m expansion of q

$$q = \sum_{k=1}^{\infty} \frac{a_k}{m^k} \qquad 0 \leq a_k < m$$

where when ambiguous we take the terminating expansion. Then

$$m^n q = \sum_{k=1}^{n-1} m^{n-k} a_k + a_n + \theta_n$$

for some θ_n in $[0, 1[$. Thus a_n is the remainder of $\lfloor m^n q \rfloor$ modulo m, and (5.1) follows. ■

Let $F(q) := \sum_{n=1}^{\infty} c_n q^n$ be any formal power series.

Theorem 5.1. *For* $0 < q < 1/\limsup_{n \to \infty} |c_n|^{1/n}$,

$$F(q) = \sum_{n=1}^{\infty} \frac{f(n)}{m^n}$$

where

$$f(n) = \sum_{k \geq 1} c_k (\lfloor m^n q^k \rfloor \bmod m).$$

Proof: From Lemma 5.1

$$F(q) = \sum_{k=1}^{\infty} c_k q^k = \sum_{k=1}^{\infty} c_k \sum_{n=1}^{\infty} \frac{[m^n q^k] \bmod m}{m^n}$$

$$= \sum_{n=1}^{\infty} \frac{f(n)}{m^n}$$

on exchanging order of summation, as is valid within the radius of convergence of F. ■

Theorem 5.1 can be extended so as to replace m^n by $\prod_{k=1}^{n} r_k$ where r_k are integers ≥ 2, and where the remainder is computed modulo r_n.

If we specialize Theorem 5.1 to the case where $q := 1/b$ and b is an integer divisible by m we may observe that $\lfloor m^n / b^k \rfloor \bmod m$ coincides with the coefficient (mod m) of b^k in the base b expansion of m^n (the $(k+1)^{th}$ digit).

Specializing further so that $m := 2$ and b is even we have

$$F\left(\frac{1}{b}\right) = \sum_{n=1}^{\infty} \frac{f_b(n)}{2^n} \tag{5.2}$$

where

$$f_b(n) := \sum \left\{ c_k | \, 2^n \text{ has } (k+1)^{\text{th}} \text{ digit odd base } b \right\}.$$

Example 5.1. (a) Let $F(q) := q/(1-q)$. Then $f_b(n)$ counts the number of odd digits in 2^n base b. Sum 7 is established on setting $b := 10$.

(b) Sum 1 corresponds to taking $F(q) = q^2/(1-q^2)$ and $q = 1/10$.

(c) Let $F(q) = q/(1-q-q^2)$. Now F is the generating function of the Fibonacci numbers ($F_1 = 1$, $F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$). Again with $q := 1/10$, we

obtain for

$$f(n) := \sum \left\{F_k | 2^n \text{ has } (k+1)^{\text{th}} \text{ digit odd}\right\},$$

as in Bowman and White [4], that

$$\sum_{n=1}^{\infty} \frac{f(n)}{2^n} = \frac{10}{89}.$$

The generating function for F_k^2 is $\dfrac{q - q^2}{1 - 2q - 2q^2 + q^3}$ and so for

$$f(n) := \sum \left\{F_k^2 | 2^n \text{ has } (k+1)^{\text{th}} \text{ digit odd}\right\}$$

$$\sum_{n=1}^{\infty} \frac{f(n)}{2^n} = \frac{90}{781}.$$

(d) Let

$$F(q) = \sum_{n=1}^{\infty} q^{n^2} = \frac{\theta_3(q) - 1}{2}.$$

Then

$$\sum_{n=1}^{\infty} \frac{f(n)}{2^n} = \frac{\theta_3\left(\frac{1}{10}\right) - 1}{2}$$

where $f(n)$ counts the number of odd digits of 2^n in square positions (the second, fifth, tenth digits etc.).

(e) If we apply Theorem 5.1 to $F(q) := q/(1-q)$ with $b := 10$ and $m := 5$ we deduce that again

$$\sum_{n=1}^{\infty} \frac{f(n)}{5^n} = \frac{1}{9}$$

where $f(n)$ sums the digits (mod 5) of 5^n base 10 (e.g. $f(3125) = 6$). ■

6. THETA FUNCTION EXAMPLES. The underlying identity for this section is really just a modular transformation of $\theta_3(q) := \sum_{n=-\infty}^{\infty} q^{n^2}$. (See [2].)

Lemma 6.1. *For $\alpha, \beta > 0$ with $\alpha\beta = 2\pi$*

$$\sqrt{\alpha}\left[\sum_{n=-\infty}^{\infty} e^{-\alpha^2 n^2/2}\right] = \sqrt{\beta}\left[\sum_{n=-\infty}^{\infty} e^{-\beta^2 n^2/2}\right].$$

Example 6.1. From the Lemma, with $s = 2/\beta^2$ so $\alpha^2 = 2\pi^2 s$

$$\sqrt{\pi s} - \sum_{n=-\infty}^{\infty} e^{-n^2/s} = 2\sqrt{\pi s}\, e^{-\pi^2 s} + O(e^{-\pi^2 4s}) \tag{6.1}$$

$$\sim 2\sqrt{\pi s}\, 10^{-(4.2863\ldots)s}.$$

Now with $s := 10^{10}$ we get

$$\left|\sqrt{\pi} - \left(\frac{1}{10^5} \sum_{n=-\infty}^{\infty} e^{-n^2/10^{10}}\right)\right| \leq 10^{-4.2 \cdot 10^{10}}, \tag{6.2}$$

which is Sum 12.

If we set

$$s = \frac{1}{\log 10^{1/N}} = \frac{N}{\log 10}$$

we get

$$\sqrt{\frac{N\pi}{\log 10}} - \sum_{-\infty}^{\infty} \frac{1}{10^{n^2/N}} \sim 2 \cdot \sqrt{\frac{N\pi}{\log 10}}\, 10^{-(1.861\cdots)N} \tag{6.3}$$

and with $N := 10^4$ we get Sum 11.

Similarly we have

$$\sqrt{\frac{q\pi}{\log q}} - \sum_{-\infty}^{\infty} \frac{1}{q^{n^2/q}} \sim 2\sqrt{\frac{q\pi}{\log q}}\, e^{-\pi^2 q/\log q}. \tag{6.4}$$

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The Logarithmic Binomial Formula

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1. INTRODUCTION. The algebra $\mathscr{P}$ of polynomials in a single variable x provides a simple setting in which to do the "polynomial" calculus. One of the nicest features of $\mathscr{P}$ is that it is closed under both differentiation and antidifferentiation. Furthermore, within the algebra $\mathscr{P}$, we have the well-known binomial formula

$$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k x^{n-k}, \qquad n \in \mathbb{Z}, \quad n \geq 0 \tag{1}$$

which may have been known as early as A.D. 1100 in the works of Omar Khayyam. (Euclid knew the formula for $n = 2$ around 300 B.C.). To be sure, the formula, as we know it today, was stated by Pascal in his *Traite du Triangle Arithmetic* in 1665.

Now suppose we wish to include the negative powers of x in our setting. One possibility is to combine the positive and negative powers of x, by working in the algebra $\mathscr{A}$ of Laurent series of the form

$$\sum_{k=-\infty}^{n} a_k x^k.$$

This algebra is certainly closed under differentiation, and there is even a binomial formula for *negative* integral powers

$$(x+a)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^k x^{n-k}, \qquad n \in \mathbb{Z}, \quad n < 0. \tag{2}$$

due to Newton (1676), which converges for $|x| > |a|$.

Recall that the binomial coefficients are defined for integers satisfying $n \geq k \geq 0$, or $k \geq 0 > n$, by

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

where $k! = k(k-1)\cdots 1$.

The algebra $\mathscr{A}$ does suffer from one drawback, however. It is not closed under antidifferentiation, since there is no Laurent series $f(x)$ with the property that $Df(x) = x^{-1}$. To correct this problem, we must introduce the logarithm $\log x$. Doing so produces some rather interesting consequences, and it is the purpose of this paper to explore some of those consequences.

In particular, we will be led to some fascinating new functions, first studied by Loeb and Rota in 1989, who called them *harmonic logarithms*. We will also be led to a generalization of the binomial formulas (1) and (2), which holds for *all* integers n. This generalization is called the *logarithmic binomial formula*.

2. THE HARMONIC LOGARITHMS. Our setting will be the set L of all finite linear combinations, with real coefficients, of terms of the form $x^i(\log x)^j$, where i

is any integer, and j is any nonnegative integer. That is, L is the real vector space with basis $\{x^i(\log x)^j | i, j \in \mathbb{Z},\ j \geq 0\}$. Under ordinary multiplication, L becomes an algebra over the real numbers. Furthermore, the formula

$$Dx^i(\log x)^j = ix^{i-1}(\log x)^j + jx^{i-1}(\log x)^{j-1} \tag{3}$$

shows that L is closed under differentiation, and the formulas

$$D^{-1}x^i(\log x)^j = \frac{1}{i+1}x^{i+1}(\log x)^j - \frac{j}{i+1}D^{-1}x^i(\log x)^{j-1}, \qquad i \neq -1$$

$$D^{-1}x^{-1}(\log x)^j = \frac{1}{j+1}(\log x)^{j+1} \tag{4}$$

can be used to give an inductive proof showing that L is closed under antidifferentiation. In fact, we can characterize L as follows.

Proposition 2.1. *The algebra L is the smallest algebra that contains both x and x^{-1}, and is closed under differentiation and antidifferentiation.* ∎

Formulas (3) and (4) indicate that, while the basis $\{x^i(\log x)^j\}$ may be suitable for studying the algebraic properties of L, it is not ideal for studying the properties of L that are related to the operators D and D^{-1}. To search for a more suitable basis for L, let us take another look at how the derivative acts on powers of x. If we let

$$\lambda_n^{(0)}(x) = \begin{cases} x^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

then

$$D\lambda_n^{(0)}(x) = n\lambda_{n-1}^{(0)}(x)$$

for all integers n. Thinking of the functions $\lambda_n^{(0)}(x)$ as a doubly infinite sequence

$$\begin{array}{ccccccccc} \cdots & \lambda_{-3}^{(0)}(x) & \lambda_{-2}^{(0)}(x) & \lambda_{-1}^{(0)}(x) & \lambda_0^{(0)}(x) & \lambda_1^{(0)}(x) & \lambda_2^{(0)}(x) & \lambda_3^{(0)}(x) & \cdots \\ \cdots & 0 & 0 & 0 & 1 & x & x^2 & x^3 & \cdots \end{array}$$

we see that applying the derivative operator D has the effect of shifting one position to the left, and multiplying by a constant.

If we introduce the notation

$$\lfloor n \rceil = \begin{cases} n & \text{for } n \neq 0 \\ 1 & \text{for } n = 0 \end{cases}$$

then the functions $\lambda_n^{(0)}(x)$ are uniquely defined by the following properties.

1) $\lambda_0^{(0)}(x) = 1$
2) $\lambda_n^{(0)}(x)$ has no constant term for $n \neq 0$
3) $D\lambda_n^{(0)}(x) = \lfloor n \rceil \lambda_{n-1}^{(0)}(x)$

Notice that the antiderivative behaves nicely on the functions $\lambda_n^{(0)}(x)$, except when applied to $\lambda_{-1}^{(0)}(x)$. With the understanding that D^{-1} produces no arbitrary constant terms, we can write

$$D^{-1}\lambda_n^{(0)}(x) = \begin{cases} \dfrac{1}{n+1}\lambda_{n+1}^{(0)}(x) & \text{for } n \neq -1 \\ 0 & \text{for } n = -1. \end{cases}$$

At this point, we have only the nonnegative powers of x. However, we can obtain the negative powers of x by introducing a second row of functions $\lambda_n^{(1)}(x)$, starting with $\lambda_0^{(1)}(x) = \log x$, and using conditions similar to 1)–3). In particular, the conditions

4) $\lambda_0^{(1)}(x) = \log x$
5) $\lambda_n^{(1)}(x)$ has no constant term
6) $D\lambda_n^{(1)}(x) = \lfloor n \rceil \lambda_{n-1}^{(1)}(x)$

uniquely define a doubly infinite sequence of functions $\lambda_n^{(1)}(x)$

$\cdots$	$\lambda_{-3}^{(1)}(x)$	$\lambda_{-2}^{(1)}(x)$	$\lambda_{-1}^{(1)}(x)$	$\lambda_0^{(1)}(x)$	$\lambda_1^{(1)}(x)$	$\lambda_2^{(1)}(x)$	$\lambda_3^{(1)}(x)$	$\cdots$
$\cdots$	x^{-3}	x^{-2}	x^{-1}	$\log x$	$x(\log x - 1)$	$x^2(\log x - 1 - \frac{1}{2})$	$x^3(\log x - 1 - \frac{1}{2} - \frac{1}{3})$	$\cdots$

Observing the pattern in these functions, it is not hard to determine the general form of $\lambda_n^{(1)}(x)$.

Proposition 2.2. *The functions $\lambda_n^{(1)}(x)$, uniquely defined by conditions 4)–6) above, are given by*

$$\lambda_n^{(1)}(x) = \begin{cases} x^n(\log x - h_n) & \text{for } n \geq 0 \\ x^n & \text{for } n < 0 \end{cases}$$

where

$$h_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

for $n > 0$ and $h_0 = 0$. ∎

Notice that the behavior of D^{-1} on the functions $\lambda_n^{(1)}(x)$ is even nicer than it is on the functions $\lambda_n^{(0)}(x)$, for assuming no arbitrary constant, we have for *all* n,

$$D^{-1}\lambda_n^{(1)}(x) = \frac{1}{\lfloor n+1 \rceil}\lambda_{n+1}^{(1)}(x).$$

The vector space formed using the functions $\lambda_n^{(0)}(x)$ and $\lambda_n^{(1)}(x)$ as a basis contains both the positive and negative powers of x, and is closed under differentiation and antidifferentiation, but it is not an algebra. For instance, the functions $(\log x)^t$, for $t > 1$, are not in this vector space. This prompts us to enlarge our class of functions still further.

Definition. For all integers n and nonnegative integers t, we define the *harmonic logarithms* $\lambda_n^{(t)}(x)$ of *order* t and *degree* n as the unique functions satisfying the following properties.

1) $\lambda_0^{(t)}(x) = (\log x)^t$
2) $\lambda_n^{(t)}(x)$ has no constant term, except that $\lambda_0^{(0)}(x) = 1$
3) $D\lambda_n^{(t)}(x) = \lfloor n \rceil \lambda_{n-1}^{(t)}(x)$ ∎

This definition allows us (at least in theory) to construct the harmonic logarithms by starting each row (that is, the harmonic logarithms of a fixed order), at $\lambda_0^{(t)}(x) = (\log x)^t$. We then differentiate to get $\lambda_n^{(t)}(x)$ for $n < 0$, and antidifferentiate to get $\lambda_n^{(t)}(x)$ for $n > 0$.

In fact with the understanding that D^{-1} produces no arbitrary constants, we can write

$$\lambda_n^{(t)}(x) = a_{n,t}D^{-n}(\log x)^t$$

where the $a_{n,t}$ are constants. These constants can easily be determined using the definition of harmonic logarithm. It turns out that $a_{n,t}$ does not depend on t, and that $a_{n,t} = \lfloor n \rceil !$, where the latter are defined by

$$\lfloor n \rceil ! = \begin{cases} n! & \text{for } n \geq 0 \\ \dfrac{(-1)^{-n-1}}{(-n-1)!} & \text{for } n < 0 \end{cases}$$

Loeb and Rota have called $\lfloor n \rceil !$ the *Roman factorial*. The notation $\lfloor n \rceil !$ was suggested by Donald Knuth. Thus, we have

Proposition 2.3. *The harmonic logarithms have the form*

$$\lambda_n^{(t)}(x) = \lfloor n \rceil ! D^{-n}(\log x)^t. \qquad \blacksquare$$

Many of the well-known properties of the ordinary factorials carry over to the numbers $\lfloor n \rceil !$. Some of the more important of these properties are listed in Box 1.

Proposition 2.3 can be used to derive an explicit formula for the harmonic logarithms. However, since we do not need this formula yet, and since it is a bit involved, we prefer to postpone it until later. We should mention now, however, that the harmonic logarithms $\lambda_n^{(t)}(x)$ do form a basis for the algebra L.

Properties of the numbers $\lfloor n \rceil !$

1) $\lfloor n \rceil ! = \lfloor n \rceil \lfloor n-1 \rceil !$

2) $\dfrac{\lfloor n \rceil !}{\lfloor n-k \rceil !} = \lfloor n \rceil \lfloor n-1 \rceil \cdots \lfloor n-k+1 \rceil$, for $k > 0$

3) $\lfloor n \rceil ! \lfloor -n-1 \rceil ! = (-1)^{n+(n<0)}$, where $(n<0)$ is 1 if $n<0$ and 0 if $n \geq 0$.

Box 1

Using the definition of harmonic logarithm, along with Property 2 in Box 1, we get

$$D^k \lambda_n^{(t)}(x) = \frac{\lfloor n \rceil !}{\lfloor n-k \rceil !} \lambda_{n-k}^{(t)}(x)$$

which shows that the higher derivatives behave on all harmonic logarithms in the same way as they behave on the powers of x.

From the definition of $\lfloor n \rceil !$, it seems a natural step to generalize the binomial coefficients by setting

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil = \frac{\lfloor n \rceil !}{\lfloor k \rceil ! \lfloor n-k \rceil !}$$

for *all* integers n and k. Loeb and Rota have called the numbers $\left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rceil$ the *Roman coefficients*. The notation $\left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rceil$ was also suggested by Knuth, and is read "Roman n choose k."

The Roman coefficients agree with the ordinary binomial coefficients whenever the latter are defined. That is, whenever $n \geq k \geq 0$, or $k \geq 0 > n$, we have

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil = \binom{n}{k}.$$

On the other hand, we also have, for example

$$\left\lfloor {n \atop -1} \right\rceil = \left\lfloor {n \atop n+1} \right\rceil = \frac{1}{\lfloor n+1 \rceil} \quad \text{and} \quad \left\lfloor {0 \atop k} \right\rceil = \frac{(-1)^{k+(k>0)}}{\lfloor k \rceil}$$

showing that the Roman coefficients are not always integers, nor are they always nonnegative. Perhaps the most interesting question about these coefficients is "What, if anything, do they count, or measure?" The temptation to think that they do count, or measure, something is further enforced by their algebraic properties, which in many cases are direct generalizations of those of the ordinary binomial coefficients. Box 2 contains a small sampling.

Properties of the numbers $\left\lfloor {n \atop k} \right\rceil$

1) For all integers n, k and r,

$$\left\lfloor {n \atop k} \right\rceil = \left\lfloor {n \atop n-k} \right\rceil \quad \text{and} \quad \left\lfloor {n \atop k} \right\rceil \left\lfloor {k \atop r} \right\rceil = \left\lfloor {n \atop r} \right\rceil \left\lfloor {n-r \atop k-r} \right\rceil.$$

2) (Pascal's formula) For any two distinct, nonzero integers n and k,

$$\left\lfloor {n \atop k} \right\rceil = \left\lfloor {n-1 \atop k} \right\rceil + \left\lfloor {n-1 \atop k-1} \right\rceil.$$

3) (Knuth's rotation/reflection law)

$$(-1)^{k+(k>0)} \left\lfloor {-n \atop k-1} \right\rceil = (-1)^{n+(n>0)} \left\lfloor {-k \atop n-1} \right\rceil.$$

Box 2

3. THE LOGARITHMIC BINOMIAL FORMULA. Now let us turn to the logarithmic binomial formula. For any positive real number a, we can expand the function $\lambda_n^{(t)}(x+a)$ in a Taylor series that is valid for $|x| < a$

$$\lambda_n^{(t)}(x+a) = \sum_{k=0}^{\infty} \frac{\left[D^k \lambda_n^{(t)}(x)\right]_{x=a}}{k!} x^k = \sum_{k=0}^{\infty} \left\lfloor {n \atop k} \right\rceil \lambda_{n-k}^{(t)}(a) x^k.$$

Thus, we have the following logarithmic binomial theorem.

Proposition 3.1. (*Logarithmic binomial theorem*) *For all integers n,*

$$\lambda_n^{(t)}(x+a) = \sum_{k=0}^{\infty} \left\lfloor {n \atop k} \right\rceil \lambda_{n-k}^{(t)}(a) x^k$$

valid for $|x| < a$. ■

Boxes 3–5 describe the logarithmic binomial formula of orders one and two.

The First Order Logarithmic Binomial Formula

Let $t = 0$. We have $\lambda_{n-k}^{(0)}(a) = a^{n-k}$ for $n \geq k$, and $\lambda_{n-k}^{(0)}(a) = 0$ for $n < k$. Furthermore, since $\left\lfloor {n \atop k} \right\rceil = \binom{n}{k}$ when $n \geq k \geq 0$, the logarithmic binomial formula is

$$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} x^k$$

which is equivalent to the classical binomial formula (1).

Box 3

The Second Order Logarithmic Binomial Formula of Negative Degree

Let $t = 1$ and $n < 0$. Since

$$\lambda_n^{(1)}(x) = \begin{cases} x^n(\log x - h_n) & \text{for } n \geq 0 \\ x^n & \text{for } n < 0 \end{cases}$$

the logarithmic binomial formula is

$$(x+a)^n = \sum_{k=0}^{\infty} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor a^{n-k} x^k.$$

Interchanging the roles of x and a, and noting that $\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor = \binom{n}{k}$ when $k \geq 0 > n$, we get the classical binomial formula (2).

Box 4

The Second Order Logarithmic Binomial Formula of Nonnegative Degree

When $t = 1$ and $n \geq 0$, the logarithmic binomial formula gives some interesting new results. Extending the definition of the harmonic logarithms of order 1, when $n > 0$, by taking

$$\lambda_n^{(1)}(0) = \lim_{x \to 0+} \lambda_n^{(1)}(x) = 0$$

we have

$$\lambda_n^{(1)}(x+a) = \sum_{k=0}^{\infty} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor \lambda_{n-k}^{(1)}(a) x^k$$

which is valid

1) For $|x| < a$, when $n < 0$,
2) For $|x| \leq a$, $x \neq -a$, when $n = 0$,
3) For $|x| \leq a$, when $n > 0$, where $\lambda_n^{(1)}(0) = 0$.

Taking $a = 1$ leads to a nice expansion of the function $(x+1)^n \log(x+1)$ when $n > 0$

$$(x+1)^n \log(x+1) = \sum_{k=0}^{n} \binom{n}{k}(h_n - h_{n-k})x^k + \sum_{k=n+1}^{\infty} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor x^k$$

valid for $|x| \leq 1$, where the left side is equal to 0 for $x = -1$.

Taking $x = -1$ in this expression, we get the following beautiful summation (for $n > 0$)

$$\sum_{k=0}^{\infty} (-1)^k \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor = \frac{(-1)^{n+1}}{n}.$$

Box 5

4. AN EXPLICIT FORMULA FOR THE HARMONIC LOGARITHMS. Although the harmonic logarithms are ideally suited to differentiation and antidifferentiation, their expression in terms of powers of x and $\log x$ is not so simple.

Proposition 4.1. *The harmonic logarithms $\lambda_n^{(t)}(x)$ are given by the formula*

$$\lambda_n^{(t)}(x) = x^n \sum_{j=0}^{t} (-1)^j (t)_j c_n^{(j)} (\log x)^{t-j}$$

where $(t)_j = t(t-1)\cdots(t-j+1)$, $(t)_0 = 1$ and where the constants $c_n^{(j)}$ are uniquely determined by the initial conditions

$$c_n^{(0)} = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases} \quad \text{and} \quad c_0^{(j)} = \begin{cases} 1 & \text{for } j = 0 \\ 0 & \text{for } j \neq 0 \end{cases}$$

and the recurrence relation (*for* $j > 0$)

$$nc_n^{(j)} = c_n^{(j-1)} + [n]c_{n-1}^{(j)}. \qquad \blacksquare$$

The numbers $c_n^{(j)}$ are known as the *harmonic numbers*, and have some rather fascinating properties as shown, for example, in Boxes 6–8. Notice the intriguing pattern in the first few harmonic numbers of positive degree n (in Box 7). It is also interesting to contrast the asymptotic behavior of the harmonic logarithms of positive and negative orders (in Boxes 7 and 8).

Some values of the harmonic numbers $C_n^{(j)}$

$$
\begin{array}{cc}
 & \begin{array}{c} n = 0 \\ \downarrow \end{array} \\
j = 0 \rightarrow &
\left[\begin{array}{cccccccccccccc}
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
\cdots & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 1 & \frac{3}{2} & \frac{11}{6} & \frac{25}{12} & \frac{137}{60} & \cdots \\
\cdots & -\frac{137}{60} & -\frac{25}{12} & -\frac{11}{6} & -\frac{3}{2} & -1 & 0 & 0 & 1 & \frac{7}{4} & * & * & * & \cdots \\
\cdots & -\frac{15}{8} & -\frac{35}{24} & -1 & -\frac{1}{2} & 0 & 0 & 0 & 1 & \frac{15}{8} & * & * & * & \cdots \\
\cdots & -\frac{17}{24} & -\frac{5}{12} & -\frac{1}{6} & 0 & 0 & 0 & 0 & 1 & \frac{31}{16} & * & * & * & \cdots \\
\cdots & -\frac{3}{24} & -\frac{1}{24} & 0 & 0 & 0 & 0 & 0 & 1 & \frac{63}{32} & * & * & * & \cdots \\
\cdots & -\frac{1}{120} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{127}{64} & * & * & * & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}\right]
\end{array}
$$

Columns sum to n Columns approach n

Box 6

The harmonic numbers of positive degree $n > 0$

1) $c_n^{(1)} = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n}$

$$c_n^{(2)} = 1 + \frac{1}{2}\left(1 + \frac{1}{2}\right) + \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \cdots + \frac{1}{n}\left(1 + \frac{1}{2} + \cdots \frac{1}{n}\right)$$

$$c_n^{(3)} = 1 + \frac{1}{2}\left[1 + \frac{1}{2}\left(1 + \frac{1}{2}\right)\right] + \frac{1}{3}\left[1 + \frac{1}{2}\left(1 + \frac{1}{2}\right) + \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right)\right] + \cdots$$

$$\cdots + \frac{1}{n}\left[1 + \frac{1}{2}\left(1 + \frac{1}{2}\right) + \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \cdots + \frac{1}{n}\left(1 + \frac{1}{2} + \cdots \frac{1}{n}\right)\right].$$

2) In general, for $n > 0$ and $j > 0$, we have

$$c_n^{(j)} = \sum_{i=1}^{n} \frac{1}{i} c_i^{(j-1)}.$$

3) For $n > 0$,

$$c_n^{(j)} = \sum_{i=1}^{n} \binom{n}{i} (-1)^{i-1} i^{-j}.$$

4) (Asymptotic behavior) For each $n > 0$, the sequence $c_n^{(j)}$ forms a nondecreasing sequence in j which is strictly increasing for $n > 1$. Furthermore, we have for each $n \geq 0$,

$$\lim_{j \to \infty} c_n^{(j)} = n.$$

Box 7

The harmonic numbers of negative degree $n < 0$
1) For $n < 0$, $$c_n^{(j)} = (-1)^j \lfloor n \rceil ! s(-n, j).$$ Where the numbers $s(n, j)$ are the famous Stirling numbers of the first kind, defined for all nonnegative integers n and j, by the condition $$x(x-1)\cdots(x-n+1) = \sum_{j=0}^{n} s(n, j) x^j.$$ 2) (Asymptotic behavior) For each $n < 0$, we have $c_n^{(j)} = 0$ for $j > -n$, and so only a finite number of the $c_n^{(j)}$ are nonzero. Furthermore, their *sum* (not limit) is $$\sum_{j=0}^{\infty} c_n^{(j)} = \sum_{j=0}^{-n} c_n^{(j)} = n.$$

Box 8

5. CONCLUDING REMARKS. We have merely scratched the surface in the study of the algebra L and its differential operators. For example, the harmonic logarithms $\lambda_n^{(t)}(x)$ have a very special relationship with the derivative operator, spelled out in the definition of these functions. Loeb and Rota show that there are other, at least formal, functions that bear an analogous relationship to other operators, such as the forward difference operator Δ defined by $\Delta p(x) = p(x+1) - p(x)$. The functions associated with the operator Δ are denoted by $(x)_n^{(t)}$ and called the *logarithmic lower factorial functions*. In general, the sequences $p_n^{(t)}(x)$ associated with various operators can be characterized in several ways, for example as sequences of *logarithmic binomial type*, satisfying the identity

$$p_n^{(t)}(x+a) = \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} p_k^{(0)}(a) p_{n-k}^{(t)}(x).$$

The properties of the Roman coefficients seem to indicate that they are a worthy generalization of the binomial coefficients. (This is not to suggest that there may not be other worthy generalizations.) As mentioned earlier, it would be a further confirmation of this fact to discover a nice combinatorial, or probabilistic, interpretation of these coefficients.

For further details on the matters discussed in this paper, with complete proofs, we refer the interested reader to reference 5.

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Calculating Sums of Infinite Series

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1. INTRODUCTION. Most calculus textbooks leave the impression that the convergence or divergence of many infinite series $\Sigma_{n=1}^{\infty} a_n$ can be decided by appealing to appropriate tests, but except in special cases it is difficult to calculate the sum with precision, when the series converges. Numerical analysts have developed many quite satisfactory methods for calculating sums of infinite series, and as part of an increased emphasis on numerical methods some of these techniques belong in a modern introductory calculus course.

Leibniz's alternating series test provides a truncation error bound $|S - S_n| < a_{n+1}$ for a decreasing alternating series. (See **[3]** or **[6]** for a better one, assuming slightly stronger hypotheses.) Such an error bound yields an effective method of calculating the sum of the series with a given precision: just compute S_n, where n is large enough to guarantee that this partial sum differs from the exact sum S by less than the specified error tolerance. Our purpose in this note is to show how with only a little more effort the proofs of the common tests used to show convergence of *positive* series can be extended to give truncation error bounds.

We will construct two decreasing sequences $\{L_n\}, \{U_n\}$ with $\lim L_n = 0$ and $\lim U_n = 0$, such that $L_n < S - S_n < U_n$ for all n.[1] Such a pair of sequences $(\{L_n\}, \{U_n\})$ will be called an *error-bounding pair* for the series. An error-bounding pair traps the sum S in a sequence of intervals $[S_n + L_n, S_n + U_n]$ whose lengths $U_n - L_n$ converge to zero. The error-bounding pair we find will depend not only on the series but also on which of three common tests was used to establish its convergence: the integral, limit comparison or ratio test.

Example 1. How many terms of the series

$$\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^2 - 3}$$

would be needed to find its sum to within $\varepsilon = 0.001$, using an appropriate error-bounding pair?

Solution. The limit comparison test, using the comparison series $\Sigma(1/n^{3/2})$, is appropriate for establishing convergence of this series, and we'll show later, in Example 3, that, therefore,

$$L_n = \frac{2}{\sqrt{n+1}} \quad \text{and} \quad U_n = \left(\frac{1}{1 - \frac{3}{n^2}}\right)\left(\frac{2}{\sqrt{n}}\right)$$

[1]To simplify the treatment of series whose summation index starts at an arbitrary value n_0, we make the convention that $S_n = a_{n_0} + a_{n_0+1} + \cdots + a_n$. Thus $S - S_n = \Sigma_{k>n} a_k$.

form an error-bounding pair for our original series. So if

$$\frac{U_n - L_n}{2} \leqslant 0.001,$$

which can be shown with a calculator to be equivalent to $n \geqslant 67$, then we'll know that the midpoint

$$M_n = S_n + \frac{U_n + L_n}{2}$$

of the interval $[S_n + L_n, S_n + U_n]$ will differ from the sum S by less than 0.001. One calculates $S_{67} = 2.5845$, $M_{67} = 2.8280$. FIGURE 1 shows how much more rapidly the upper and lower estimates $\{S_n + L_n\}$ and $\{S_n + U_n\}$ converge to the sum than does the sequence $\{S_n\}$, for this series.

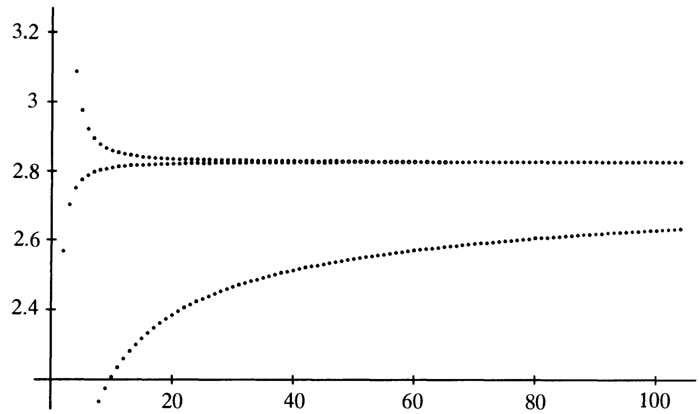

Figure 1.

The upper estimate $S_n + U_n$ is conceptually most important since it is an upper bound for the sum of the series; the lower estimate simply improves the lower bound from S_n to $S_n + L_n$.

2. DETERMINATION OF ERROR-BOUNDING PAIRS.

The integral test.

Theorem 1. *If $\sum_{n=1}^{\infty} a_n$ has been shown to converge by the integral test, and if $I_n = \int_n^{\infty} f(x)\,dx$, then $(\{I_{n+1}\}, \{I_n\})$ is an error-bounding pair for the series $\sum_{n=1}^{\infty} a_n$.*

Proof: (See [4] for a somewhat different argument.)

Let $a_n = f(n)$, where $f(x)$ is a positive decreasing continuous function. Then $\int_1^n f(x)\,dx < \int_1^{n+1} f(x)\,dx < S_n$, so $0 < b_n < c_n$ where $b_n = S_n - \int_1^{n+1} f(x)\,dx$ and $c_n = S_n - \int_1^n f(x)\,dx$. An argument which goes back to Euler shows that the sequence $\{c_n\}$ is decreasing: $c_{n+1} - c_n = a_{n+1} - \int_n^{n+1} f(x)\,dx < 0$. Thus $\{c_n\}$ converges to a limit $\gamma_f \geqslant 0$. It follows, then, that if either $\{S_n\}$ or the sequence $\{\int_1^n f(x)\,dx\}$ converges, so does the other, and $\gamma_f = S - I$, where $I = \int_1^{\infty} f(x)\,dx$.

Essentially the same argument shows that the sequence $\{b_n\}$ is increasing and, since $0 < c_n - b_n = \int_n^{n+1} f(x)\,dx < a_n$, it follows that $\{b_n\}$ also converges to γ_f.

Now since f is positive, if we set $I_n = \int_n^\infty f(x)\,dx$ then $I_1 > I_2 > I_3 > \cdots$ and, if the series converges, $\lim I_n = 0$. Also

$$\begin{aligned} S - S_n &= S - I + I - \int_1^n f(x)\,dx + \int_1^n f(x)\,dx - S_n \\ &= \gamma_f + I_n - c_n \\ &= I_n - (c_n - \gamma_f) < I_n \quad \big(\text{because } \{c_n\} \text{ decreases to its limit } \gamma_f\big). \end{aligned}$$

Similarly $S - S_n > I_{n+1}$ because $\{b_n\}$ increases to γ_f. This completes the proof.

Remark. If the integral test shows the series $\Sigma_{n=1}^\infty a_n$ to be divergent, but $\lim a_n = 0$, the inequality $b_n < \gamma_f < c_n$ above yields

$$\int_1^n f(x)\,dx + \gamma_f < S_n < \int_1^{n+1} f(x)\,dx + \gamma_f,$$

which describes the rate of divergence of the sequence of partial sums. For example for the harmonic series $\Sigma_1^\infty 1/n$ we get $\log n + \gamma < S_n < \log(n+1) + \gamma$, where $\gamma \approx 0.577\ldots$ is Euler's constant.

Example 2. Estimate the sum of the series $\Sigma_{n=1}^\infty 1/n^2$, with error less than 10^{-4}.

Solution. Here $I_n = 1/n$, and we just need $(I_n - I_{n+1})/2 = 1/2n(n+1) < 10^{-4}$, or $n \geqslant 71$. Using a computer we find $M_{71} = 1.644935$. Since it is known that the exact sum of this series is $\zeta(2) = \pi^2/6$, we can check our result: $\zeta(2) = 1.644934\ldots$. So the sum $\zeta(2)$ falls almost exactly in the middle of the interval $[S_{71} + L_{71}, S_{71} + U_{71}]$.

Remark. The Euler-Maclaurin formula gives the asymptotic series

$$c_n - \gamma_f = \frac{1}{2} f(n) + \frac{B_2}{2!} f'(n) + \frac{B_4}{4!} f^{(3)}(n) + \cdots + \frac{B_{2m}}{(2m)!} f^{(2m-1)}(n) + \cdots$$

which can be combined with our formula $S = S_n + I_n - (c_n - \gamma_f)$ above to give much more accurate error-bounding pairs than $(\{I_{n+1}\}, \{I_n\})$ associated with the integral test. (The B_n here are the Bernoulli numbers.) We will give examples of this more advanced technique in §3.

The limit comparison test.

Theorem 2. *Suppose* $\lim_{n\to\infty} a_n/b_n = L$, *where* $0 < L < \infty$, *and suppose that we have found an error-bounding pair* $(\{L_n\}, \{U_n\})$ *for the series* $\Sigma_{n=1}^\infty b_n$. *Then*

1) *If* $\{a_n/b_n\}$ *decreases to its limit* L, $(\{LL_n\}, \{(a_n/b_n)U_n\})$ *is an error-bounding pair for the series* $\Sigma_{n=1}^\infty a_n$.
2) *If* $\{a_n/b_n\}$ *is increasing*, $(\{(a_n/b_n)L_n\}, \{LU_n\})$ *is an error-bounding pair.*

Proof: By assumption $L_n < \Sigma_{k>n} b_k < U_n$ for any n. In case 1), setting

$$B_n = \frac{a_n}{b_n} U_n,$$

we have

$$\frac{B_n}{U_n} = \frac{a_n}{b_n}$$

which decreases to L, so

$$S - S_n = \sum_{k>n} a_k = \sum_{k>n} \frac{B_k}{U_k} b_k < \frac{B_n}{U_n} \sum_{k>n} b_k < B_n.$$

Similarly $S - S_n > L\Sigma_{k>n} b_k > LL_n$.

The proof of case 2) is entirely similar.

Example 3. We return to complete Example 1, finding the error-bounding pair as described in the preceding theorem.

Solution. Here

$$a_n = \frac{\sqrt{n}}{n^2 - 3}$$

and we choose the p-series with

$$b_n = \frac{1}{n^{3/2}},$$

finding

$$\lim \frac{a_n}{b_n} = \lim \frac{n^2}{n^2 - 3} = \lim \frac{1}{1 - \dfrac{3}{n^2}} = 1.$$

We note that $\{a_n/b_n\}$ *decreases* to its limit $L = 1$. Now the series $\Sigma_{n=1}^{\infty} b_n$ converges, by the integral test, and (using the notation of Theorem 1)

$$I_n = \frac{2}{\sqrt{n}},$$

so the pair $(\{2/\sqrt{n+1}\}, \{2/\sqrt{n}\})$ is an error-bounding pair for $\Sigma_{n=1}^{\infty} b_n$. Thus by Theorem 2 the pair $(\{L_n\}, \{U_n\})$ in Example 1 is an error-bounding pair for this series.

(Note that if the series were instead $\Sigma_{n=1}^{\infty} \sqrt{n}/(n^2 + 3)$, our argument would be the same except that the factor $1/(1 + 3/n^2)$ *increases* with n, with limit 1. Thus we would take

$$L_n = \left(\frac{1}{1 + \dfrac{3}{n^2}}\right)\left(\frac{2}{\sqrt{n+1}}\right)$$

and

$$U_n = \frac{2}{\sqrt{n}}$$

in this case.)

Remark. The comparison test is easy. If $(\{L_n\}, U_n\})$ is an error-bounding pair for a convergent series $\Sigma_{k=1}^n b_k$ and $0 < a_k \le b_k$ for all k, then $0 < \Sigma_{k>n} a_k \le \Sigma_{k>n} b_k < U_n$, so $(\{0\}, \{U_n\})$ is an error-bounding pair for the series $\Sigma_{k=1}^n a_k$.

The ratio test.

Theorem 3. *Suppose that* $\lim(a_{n+1}/a_n) = r < 1$. *Let*

$$A_n = \frac{a_{n+1}}{1 - \dfrac{a_{n+1}}{a_n}}$$

and

$$B_n = a_n\left(\frac{r}{1-r}\right).$$

If n is large enough that $a_{n+1}/a_n < 1$, and

1) *if $\{a_{k+1}/a_k\}$ is decreasing for $k > n$, then the pair of sequences $(\{B_n\}, \{A_n\})$ is an error-bounding pair for the series Σa_n.*
2) *If $\{a_{k+1}/a_k\}$ is increasing for $k > n$, then $(\{A_n\}, \{B_n\})$ is an error-bounding pair for the series.*

Proof: Our argument requires only a slight modification of the standard proof of the ratio test. The two cases are similar, so we just treat case 1). Let n be large enough that $a_{n+1}/a_n = \rho$ is less than 1 and $a_{k+1}/a_k < \rho$ for all $k > n$. Then adding the inequalities $a_{n+1} = \rho a_n$, $a_{n+2} < \rho^2 a_n$, $a_{n+3} < \rho^3 a_n, \ldots$ gives $\Sigma_{k>n} a_k < a_n \Sigma_{k=1}^\infty \rho^k$, or $S - S_n < a_n(\rho/(1-\rho)) = A_n$.

On the other hand $a_{k+1}/a_k > r$ for all $k \geqslant n$, so adding the inequalities $a_{n+1} > ra_n$, $a_{n+2} > r^2 a_n$, $a_{n+3} > r^3 a_n, \ldots$ gives $S - S_n > a_n \Sigma_{k=1}^\infty r^k = B_n$.

Remark. A similar argument leads to error-bounding pairs associated with the root test.

Example 4. Find the sum of the series $\Sigma_{n=1}^\infty n^2/n!$ with error $< 10^{-6}$.

Solution. Here

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n^2},$$

which *decreases* to its limit $r = 0$, and is less than 1 for all $n > 1$. So here $B_n = 0$ and

$$A_n = \frac{(n+1)^2}{(n+1)!\left(1 - \dfrac{n+1}{n^2}\right)} = \frac{(n+1)}{n!\left(1 - \dfrac{n+1}{n^2}\right)}.$$

We must choose n large enough that $(A_n - B_n)/2 \leqslant 10^{-6}$. Trial and error gives $n \geqslant 11$, so we know the sum S lies in the interval $[S_{11}, S_{11} + A_{11}] = [5.43656332, 5.43656366]$.

3. IMPROVED ERROR-BOUNDS.

We analyze the very slowly converging series

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2},$$

using the first few terms of the Euler-Maclaurin summation formula [**1**, p. 256]

$$c_n - \gamma_f = \frac{1}{2}f(n) + \frac{B_2}{2!}f'(n) + \frac{B_4}{4!}f^{(3)}(n) + \cdots + \frac{B_{2m}}{(2m)!}f^{(2m-1)}(n) + \cdots$$

to get a better error-bounding pair than Theorem 1 provides. Here

$$I_n = \frac{1}{\log n},$$

and since we showed in the proof of Theorem 1 that

$$S - S_n = I_n - (c_n - \gamma_f),$$

$$S = S_n + I_n - \frac{1}{2}f(n) - \frac{B_2}{2!}f'(n) - \frac{B_4}{4!}f^{(3)}(n) - \cdots .$$

Using the values $B_2 = \frac{1}{6}$, $B_6 = \frac{-1}{30}$ and computing the first three derivatives of f,

$$S = S_n + \frac{1}{\log n} - \frac{1}{2n(\log n)^2} + \frac{2 + \log n}{12n^2(\log n)^3}$$
$$- \frac{12 + 18\log n + 11(\log n)^2 + 3(\log n)^3}{360n^4(\log n)^5} + \cdots .$$

Now it can be shown [**1**, p. 257] that the series on the right alternately underestimates and overestimates S, the absolute value of the truncation error being smaller than that of the first neglected term. Thus if

$$U_n = \frac{1}{\log n} - \frac{1}{2n(\log n)^2} + \frac{2 + \log n}{12n^2(\log n)^3}, \quad \text{and}$$

$$L_n = U_n - \frac{12 + 18\log n + 11(\log n)^2 + 3(\log n)^3}{360n^4(\log n)^5}$$

then $(\{L_n\}, \{U_n\})$ is an error-bounding pair for our series. The table below gives the numerical results.

n	S_n	$S_n + L_n$	$S_n + U_n$	$U_n - L_n$
2	1.040684	2.0981380	2.1315150	0.0333769
5	1.524159	2.1097414	2.1097752	0.0000337
10	1.684585	2.1097427	2.1097434	6.368 * 10^ − 7

Note that the partial sum S_{10} is still far from the sum of the series, but the corresponding upper and lower estimates differ from the sum by less than 10^{-6}. This is impressive since, in view of the inequality $S - S_n > L_n$, even with a large value of n such as 10^{1000} we have $S - S_n > 0.000434$, so it would be impossible to calculate a partial sum with n large enough to make S_n differ from S by less than 10^{-6}.

Remark. Recall that when the limit comparison test is used to prove convergence, the error-bounding pair for Σa_n given in Theorem 2 depends upon an error bounding pair for the comparison series. If the comparison series is shown to be convergent by using the integral test, then using the Euler-Maclaurin summation formula to get an improved error-bounding pair for the comparison series will lead to a better pair for the original series Σa_n as well. As an example, consider series $\Sigma_{n=2}^{\infty}\sqrt{n}/(n^2-3)$ once again. Proceeding as in Example 3, but applying the Euler-Maclaurin summation formula with $f(x)=x^{-3/2}$, we are led to the improved error-bounding pair

$$U_n = \frac{2}{\sqrt{n}} - \frac{1}{2n^{3/2}} + \frac{1}{8n^{5/2}},$$

$$L_n = U_n - \frac{7}{384n^{9/2}}$$

for the comparison series $\Sigma_{n=2}^{\infty}1/n^{3/2}$. Then Theorem 2 gives the pair

$$L'_n = L_n, \qquad U'_n = U_n\left(\frac{1}{1-\dfrac{3}{n^2}}\right).$$

The results are a significant improvement on the estimates found earlier in Example 1:

n	S_n	$M_n = S_n + \dfrac{L_n+U_n}{2}$	$M'_n = S_n + \dfrac{L'_n+U'_n}{2}$	$\dfrac{U'_n-L'_n}{2}$
10	2.2075475	2.8245867	2.8341285	0.0095421
50	2.5464838	2.8279194	2.8280884	0.0001690
100	2.6284725	2.8279738	2.8280037	0.0000299

Recall that M'_n differs from the sum of the series by less than $\dfrac{U'_n-L'_n}{2}$.

CONCLUSION. Calculating the sums of a few infinite series chosen from their textbook, using the methods outlined above, will introduce students to the important notion of the rate of convergence of a sequence. Such computer-aided explorations can bring new life to a moribund topic in the calculus course.

The two articles [**1**], [**2**] will provide the interested reader with an entertaining discussion of the Euler-Maclaurin summation formula, and further references.

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L^p Arithmetic

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For every positive real number p and measure μ, let $L^p = L^p(\mu)$ denote the usual space of μ-measurable real-valued functions whose p-th power is absolutely integrable. We will also allow p, q to assume the values $0, \infty$. The meaning of L^∞ is the standard one, i.e. the set of μ-measurable functions which are equal μ-a.e. to a bounded function. We take L^0 to mean the set of μ-measurable functions which vanish outside a set of finite measure. Given two exponents p and q, define the sum and product of the associated function spaces by:

$$L^p + L^q = \{f + g | f \in L^p, g \in L^q\}$$

$$L^p \cdot L^q = \{f \cdot g | f \in L^p, g \in L^q\}.$$

Some properties of these operations are evident from those of the corresponding operations for individual functions. For example, both addition and multiplication are associative and commutative. However, difficulties arise if we try to prove other properties, such as distributivity of multiplication over addition. The individual function argument in this case just shows that every element of $L^p \cdot (L^q + L^s)$ is also in $(L^p \cdot L^q) + (L^p \cdot L^s)$; the other inclusion is not trivial. We would like to have a description of the operations between L^p spaces which makes the solution of this and similar problems more straightforward. One of the obstacles is that we don't really know yet what the domain of our operations is.

Question 1. *Is the sum of two L^p spaces also an L^p space? What about the product?*

Even if the answer is no, we would like to know what spaces are obtained by applying the operations a finite number of times to given L^p spaces, and what interesting properties, if any, the operations have on the resulting domain of definition. So, we are also asking:

Question 2. *More generally (and less precisely), is there an L^p arithmetic?*

In the present note, methods based on those used in [1], in particular the simple technique of decomposing functions into "upper and lower parts", will be applied to find answers to these questions. We will prove a characterization of $L^p + L^q$ in terms of upper and lower parts, and use it to obtain measure-independent identities describing behavior of the indices p and q under addition and multiplication. Though the results are elementary, they seem not to be well known. We believe that L^p arithmetic is an interesting aspect of the beautiful theory of L^p spaces which deserves to be examined.

1. PRODUCTS AND SUMS OF L^p SPACES. Let's begin on a positive note, by observing that Hölder's inequality implies that the *product* indeed behaves according to our wildest L^p-arithmetic dreams:

Proposition 1.1.

$$L^p \cdot L^q = L^{pq/(p+q)}.$$

For convenience, from now on we will write $p\|q$ instead of $pq/(p+q)$ (this notation is motivated by the formula for the electrical resistance of the "parallel" interconnection of resistors of value p and q). By convention, $p\|\infty = p$ and $p\|0 = 0$.

Proof: No generality is lost by assuming that all of the functions involved are non-negative.

Consider first the case in which neither p nor q equals 0 or ∞. If $g \in L^p$ and $h \in L^q$, then by Hölder's inequality we see that

$$\int (g \cdot h)^{pq/(p+q)} \leq \left(\int g^p\right)^{q/(p+q)} \cdot \left(\int h^q\right)^{p/(p+q)} < \infty,$$

so that $g \cdot h \in L^{pq/(p+q)}$.

For the other inclusion, assume $f \in L^{pq/(p+q)}$ and let $g = f^{q/(p+q)}$, $h = f^{p/(p+q)}$. Then $f = g \cdot h$, $g \in L^p$, and $h \in L^q$. This completes the proof for the case in which neither one of p, q equals 0 or ∞.

If $p = 0$ and if $g \in L^p$, $h \in L^q$, then since $g(x) \cdot h(x) = 0$ for all x such that $g(x) = 0$, we see that $g \cdot h$ vanishes outside a set of finite measure and so $g \cdot h \in L^0 = L^{p\|q}$. Conversely, if $f \in L^0 = L^{p\|q}$, then letting S be the set of all x such that $f(x) \neq 0$, we see that the characteristic function C_S of S belongs to L^q, and so $f = f \cdot C_S \in L^0 \cdot L^q$.

The final case is that in which one of p, q, say p, equals ∞. If $g \in L^\infty$ and $h \in L^q$, then clearly $g \cdot h \in L^q = L^{\infty\|q}$, because $g \cdot h$ is bounded by some constant real multiple of h a.e. Conversely, assuming that $f \in L^{\infty\|q} = L^q$, we may express f as the product of the constant function 1 with f itself, thus showing that $f \in L^\infty \cdot L^q$. ∎

On the other hand, the *sum* of L^p spaces is problematic.

Example. Consider the functions f, g defined on the positive real line by:

$$f(x) = \begin{cases} x^{-1/2}, & \text{if } x < 1 \\ 0, & \text{otherwise.} \end{cases} \qquad g(x) = \begin{cases} x^{-1/3}, & \text{if } x > 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then, since f and g are supported on disjoint sets, the sum $f + g$ is in L^p precisely for those p for which both f and g are in L^p. However, f belongs to L^p only if $p < 2$, while g is in L^p only if $p > 3$. Thus, no L^p space contains $f + g$.

We conclude that "true L^p arithmetic" as suggested above in Question 1, is impossible. That is to say, we have a

Tentative Answer. *NO.*

But, as the reader might have guessed by now, we won't give up so easily! The obvious way to make sure our collection of spaces is closed under sum and product is to enlarge it by including all possible sums and products. Fortunately, for the present L^p case this may be achieved in just one step, as we will now show.

2. L_q^p FUNCTIONS AND THEIR UPPER AND LOWER PARTS.

Definition 2.1. *Suppose p and q are non-negative extended real numbers. If $p \leq q$, let $L_q^p = L^p + L^q$. Notice that $L^p = L_p^p$ for all p.*

It turns out that the collection of all L_q^p spaces, $p \leq q$, in contrast to its subcollection consisting of the L^p spaces, *is* closed under sums and products; thus, any space obtainable from L^p spaces by a finite number of applications of sum and product operations is in fact equal to a sum of at most two L^p spaces. The proof of this fact relies on a characterization of L_q^p functions in terms of "upper and lower parts".

Definition 2.2. *If f is a measurable function and if $f = g + h$, with $g \in L^p$ and $h \in L^q$, we say that the pair (g, h) is a* (p, q)-decomposition *of f. A $(0, \infty)$-decomposition of f will also be referred to as a decomposition of f into* upper and lower parts *(respectively).*

(The reason for using the words "upper" and "lower" here is the existence of certain canonical such decompositions for L_q^p functions; see Lemma 2.1, below).

We will now prove the following theorem, which will be the basic tool in our present brief study of L^p arithmetic. The result explains our choice of notation for the L_q^p spaces.

Theorem 2.1. *The following are equivalent:*

(1) $f \in L_q^p$.
(2) *There exists a decomposition of f into upper and lower parts which is also a (p,q)-decomposition of f.*
(3) *f has at least one decomposition into upper and lower parts, and every decomposition of f into upper and lower parts is a (p, q)-decomposition.*

We give a (lemma, lemma, lemma)-decomposition of the proof.

Lemma 2.1 (Canonical decompositions for L_q^p functions). *If $f \in L_q^p$ for some $0 \leq p \leq q \leq \infty$, then there exists a real number $c > 0$ such that the condition $|f(x)| \leq c$ holds for all x outside some set of finite measure. For any such c, multiplication of f by the characteristic function of the set of all x such that $|f(x)| > c$ (respectively $|f(x)| \leq c$) leads to a $(0, \infty)$-decomposition of f.*

Proof: The truth of the second sentence in the statement follows from that of the first. To prove the first, one may consider the value $c = 1$ in the case $p \leq q < \infty$, $c = \mathit{esssup}|f|$ if $p = q = \infty$, and otherwise letting (g, h) be a (p, q)-decomposition of f, one may let $c = 1 + \mathit{esssup}|h|$. ■

Proving that (1) implies (2) in Theorem 2.1 will reduce to showing that if $f \in L_q^p$, then a certain quasi-canonical decomposition of f into upper and lower parts is in fact a (p, q)-decomposition. The following fact provides the key remaining ingredient for this argument.

Lemma 2.2 (Behavior of upper and lower parts under index changes).

(1) *Suppose f is in L^p and vanishes outside some set of finite measure (i.e. $f \in L^p \cap L^0$). Then $f \in L^q$ for every $q < p$.*
(2) *If f is in L^p and is bounded (i.e. $f \in L^p \cap L^\infty$), then $f \in L^q$ for every $q > p$.*

Proof: We content ourselves with claiming that part (1) follows from Hölder's inequality (unless $p = 0$ or $q = \infty$, which are even easier) and that (2) clearly reduces to the trivial case $|f| \leq 1$ a.e. ■

Corollary. *$L^0 \cap L^\infty \subset L^p$ for every $p \in [0, \infty]$.*

To complete the proof of the implication (1) $\Rightarrow$ (2) in Theorem 2.1, suppose that $f = g + h$, where $g \in L^p$ and $h \in L^q$, $p \leq q$. Let $(\bar{g}, \underline{g})$ and $(\bar{h}, \underline{h})$ be canonical decompositions of g and h, respectively, into upper and lower parts, as in Lemma 2.1. By Lemma 2.2, $\bar{h} \in L^p$ and $\underline{g} \in L^q$, since $p \leq q$. Therefore $(\bar{g} + \bar{h}, \underline{g} + \underline{h})$ is a (p, q)-decomposition of f, and of course it is also a $(0, \infty)$-decomposition. This proves (1) $\Rightarrow$ (2). The proof of the implication (2) $\Rightarrow$ (3) of the Theorem will follow from our next observation, which shows that any two decompositions into upper and lower parts for the same function differ precisely by a $L^0 \cap L^\infty$-decomposition of the zero function.

Lemma 2.3 (Rigidity of decompositions of a given function). *Suppose that $(\bar{f}, \underline{f})$ is a decomposition of f into upper and lower parts. Then a given pair of functions is also a decomposition of f into upper and lower parts iff it is of the form $(\bar{f} + \delta, \underline{f} - \delta)$ for some $\delta \in L^0 \cap L^\infty$.*

Proof: It is clear from the Corollary to Lemma 2.2 above that $(\bar{f} + \delta, \underline{f} - \delta)$ is a $(0, \infty)$-decomposition of f for any δ belonging to $L^0 \cap L^\infty$. To prove the other direction, suppose that (g, h) is any decomposition of f into upper and lower parts. Then $\bar{f} + \underline{f} = g + h$, so that $\bar{f} - g = h - \underline{f}$. The function on the left side of this equality belongs to L^0, while that on the right side is in L^∞; it follows that both sides are in $L^0 \cap L^\infty$, and that (g, h) is of the given form with $\delta = \underline{f} - h$. ■

The proof of Theorem 2.1 is now finished, as (3) $\Rightarrow$ (1) follows from the trivial observation that any function which admits a (p, q)-decomposition must belong to $L^p + L^q = L^p_q$.

3. GENERAL ARITHMETIC IDENTITIES. An important idea behind the proof of Theorem 2.1 is that, at least for the present L^p_q context, measurable functions may be thought of as having "parts", namely the upper and lower parts of the statement, and, most importantly, that when two such functions interact through addition, their upper parts combine to form the upper part of the result, and their lower parts combine to form the new lower part, without any "cross terms" arising from interaction of the upper part of one function with the lower part of the other (when we say "the" upper or lower part here, we of course mean modulo an $L^0 \cap L^\infty$ function as in the Rigidity Lemma 2.3). More briefly: *If $(\bar{f}, \underline{f})$ is a decomposition of f into upper and lower parts and if $(\bar{g}, \underline{g})$ is such a decomposition for g, then $(\bar{f} + \bar{g}, \underline{f} + \underline{g})$ is a decomposition of $f + g$ into upper and lower parts.*

Keeping this idea in mind, we may derive some nice consequences of Theorem 2.1 for L^p arithmetic, which yield answers to the questions stated in the Introduction. Letting $\vee$, $\wedge$ denote, respectively, the maximum and the minimum operators for extended real numbers, we obtain the following result (see **[1]**, **[2]**).

Theorem 3.1 (Sums of L^p_q spaces are L^p_q spaces).

$$L^p_q + L^r_s = L^{p \wedge r}_{q \vee s}.$$

Proof: We show that each of the above objects is contained in the other, heavily using Theorem 2.1 and Lemma 2.2 throughout. A sketch follows.

" $\subset$ ": By the observation preceding the statement of this Theorem.

" $\supset$ ": $p \wedge r$ equals either p or r, and $q \vee s$ equals either q or s; assume $f \in L_{q \vee s}^{p \wedge r}$ and consider a decomposition $(\bar{f}, \underline{f})$ of f into upper and lower parts; then consider the decompositions $(\bar{f}, 0)$ of $\bar{f}$ and $(0, \underline{f})$ of $\underline{f}$ to conclude that each of the functions $\bar{f}, \underline{f}$ belongs to either L_q^p or L_s^r. ■

This implies that the collection of L_q^p spaces, $0 \leq p \leq q \leq \infty$, is closed under addition, as claimed before. It suffices to observe that if $0 \leq p \leq q \leq \infty$ and $0 \leq r \leq s \leq \infty$, then we have e.g. $0 \leq p \wedge r \leq p \leq q \leq q \vee s \leq \infty$.

Well, what about the product? In light of the above arguments, one would expect to get a formula for the product of L_q^p and L_s^r by operating upper parts and lower parts independently, as for the sum. Using Proposition 1.1 concerning products of L^p spaces, we are lead to conjecture that $L_q^p \cdot L_s^r = L_{q\|s}^{p\|r}$. However, caution is necessary. In contrast to the case for the sum, it is *not* true that the upper (respectively, lower) part of a product is the product of the upper (respectively, lower) parts of the factors.

Example. On the positive real line, consider the functions

$$\bar{f}(x) = \begin{cases} x^{-1}, & \text{if } x < 1 \\ 0, & \text{otherwise.,} \end{cases} \qquad \underline{g}(x) = 1.$$

Then $(\bar{f}, 0)$ and $(0, \underline{g})$ are decompositions into upper and lower parts for the functions $f = \bar{f}$, $g = \underline{g} = 1$, respectively, which satisfy $f \cdot g = f = \bar{f}$. Thus, the upper part of the product $f \cdot g$ is just $\overline{f \cdot g} = \bar{f} \cdot \underline{g} = \bar{f}$. However, the product of the upper parts of f, g is identically zero, since the upper part $\bar{g}$ of g is zero. Notice that the difference between $\overline{f \cdot g}$ and $\bar{f} \cdot \bar{g}$ is not an $L^0 \cap L^\infty$ function, so we get essentially different answers in each case.

But not all is lost. Though "cross terms" appear in the product, we may show that, as far as our conjecture about products of L_q^p spaces is concerned, they are not too large. We *do* have in general: *if $(\bar{f}, \underline{f})$ and $(\bar{g}, \underline{g})$ are decompositions into upper and lower parts of f, g respectively, then $(\bar{f} \cdot \bar{g} + \bar{f} \cdot \underline{g} + \underline{f} \cdot \bar{g}, \underline{f} \cdot \underline{g})$ is a decomposition of $f \cdot g$ into upper and lower parts.*

We will show that if $f \in L_q^p$ and $g \in L_s^r$, then this decomposition is also a $(p\|r, q\|s)$-decomposition of $f \cdot g$, thus proving that $L_q^p \cdot L_s^r$ is contained in $L_{q\|s}^{p\|r}$.

So, suppose $f \in L_q^p$ and $g \in L_s^r$. By Theorem 2.1 and Lemma 2.2, and observing that the product of an L^p function and a bounded function is also an L^p function,

$$\bar{f} \cdot \underline{g} \in (L^p \cap L^0) \cdot (L^s \cap L^\infty) \subset (L^p \cap L^0) \cdot L^\infty \subset L^p \cap L^0$$

$$\underline{f} \cdot \bar{g} \in (L^q \cap L^\infty) \cdot (L^r \cap L^0) \subset L^\infty \cdot (L^r \cap L^0) \subset L^r \cap L^0.$$

The "canonical terms" in the product satisfy, using Proposition 1.1 regarding products of L^p spaces,

$$\bar{f} \cdot \bar{g} \in (L^p \cap L^0) \cdot (L^r \cap L^0) \subset (L^p \cdot L^r) \cap L^0 = L^{p\|r} \cap L^0$$

$$\underline{f} \cdot \underline{g} \in (L^q \cap L^\infty) \cdot (L^s \cap L^\infty) \subset (L^q \cdot L^s) \cap L^\infty = L^{q\|s} \cap L^\infty$$

as expected. By Lemma 2.2, it follows that $(\bar{f} \cdot \bar{g} + \bar{f} \cdot \underline{g} + \underline{f} \cdot \bar{g}, \underline{f} \cdot \underline{g})$ is in fact a $(p\|r, q\|s)$-decomposition of $f \cdot g$ into upper and lower parts as claimed, so that $f \cdot g \in L_{q\|s}^{p\|r}$.

The converse also holds. If $h \in L_{q\|s}^{p\|r}$, then assuming without loss of generality that h is non-negative, and choosing a decomposition $(\bar{h}, \underline{h})$ of h into upper and

lower parts *supported on disjoint sets* (e.g. any canonical decomposition as in Lemma 2.1), we have

$$h = \left((\bar{h})^{r/(p+r)} + (\underline{h})^{s/(q+s)}\right) \cdot \left((\bar{h})^{p/(p+r)} + (\underline{h})^{q/(q+s)}\right).$$

Here, we interpret the formal expressions $0/(0+0)$ and $\infty/(\infty+\infty)$ as $1/2$, and $\infty/(\infty+0)$ as 1, in order to cover all possible cases. The right-hand side of the equality is in $L_q^p \cdot L_s^r$ by Theorem 2.1, so $h \in L_q^p \cdot L_s^r$. We have now proven our conjecture:

Theorem 3.2 (Products of L_q^p spaces are L_q^p spaces).

$$L_q^p \cdot L_s^r = L_{q\|s}^{p\|r}.$$

Again, it is necessary to verify that if $p \leq q$ and $r \leq s$, then also $p\|r \leq q\|s$; however, this is straightforward.

By using Theorems 3.1 and 3.2, the fact that $\|$ distributes over $\vee$ and $\wedge$ (an easy computation) now immediately yields the distributivity of multiplication over addition for L_q^p spaces, which, as observed in the Introduction, is *not* a trivial consequence of the corresponding property for individual functions (the difficulty lies in proving that, in the equality given in the statement below, the space on the right-hand side is contained in the space on the left-hand side).

Corollary.

$$L_q^p \cdot \left(L_s^r + L_u^t\right) = \left(L_q^p \cdot L_s^r\right) + \left(L_q^p \cdot L_u^t\right).$$

In view of the identity $p\|\infty = p$, Theorem 3.2 also allows us to recover the easy fact, a special case of which was used in the proof of Theorem 3.2 itself, that L^∞ is a multiplicative identity for the collection of L_q^p spaces. Other consequences may be derived in a similar fashion. For example, the fact that we have $2p\|2p = p$ implies that L_q^p is the square of L_{2q}^{2p}; more generally, for any natural number n, L_q^p has an n-th root, namely L_{nq}^{np}. Given any finite collection $C = \{L^{p_1}, \dots, L^{p_k}\}$ of L^p spaces, the "substructure" of $(\{L_q^p\colon 0 \leq p \leq q \leq \infty\}, +, \cdot\,)$ which is generated by C may easily be shown to equal

$$\left\{L_{p_1/m_1\|\cdots\|p_k/m_k}^{p_1/n_1\|\cdots\|p_k/n_k}\colon \quad \text{each } m_j, n_j \text{ is a natural number}, \ m_j \leq n_j\right\}$$

where the value 0 is allowed for m_j, n_j so that the multiplicative identity L^∞ is included. Arbitrary collections of L^p spaces generate the union, over all finite subcollections, of the corresponding substructures.

Recalling the original Basic L^p Arithmetic Question (Question 2 in the Introduction), we may now confidently give a happy

Answer. *YES, assuming we enlarge our universe to include all the L_q^p spaces, the usual arithmetic operations are well-defined and satisfy simple identities in terms of the indices p, q. There* is *an L_q^p Arithmetic.*

CONCLUSIONS. The identities of Theorems 3.1 and 3.2 provide us with tools for systematically answering questions about sums and products of L^p spaces, allowing us to use for this purpose the properties of the operations $\wedge$, $\vee$, and $\|$ on the extended real interval $[0, \infty]$.

As is often the case, the clarity afforded by new knowledge raises new issues. If we try to use Theorem 3.1 to determine whether there exists an *additive* identity

for the collection of L_q^p spaces, $p \le q$, we are led to the problem of finding numbers $0 \le p \le q \le \infty$ such that whenever $0 \le r \le s \le \infty$, then $p \wedge r = r$ and $q \vee s = s$. This forces $p = \infty$ and $q = 0$, which don't satisfy $p \le q$. Interestingly enough, it may be shown (see **[1]**) that if for $p > q$ one uses Theorem 2.1 to *define* L_q^p, then the resulting space is not the sum $L^p + L^q$ but rather the *intersection* $L^p \cap L^q$; in particular, L_0^∞ should be interpreted as $L^0 \cap L^\infty$. One may proceed to study the behavior of the intersection and show that it gives rise to a formula which is "dual" to that for the sum:

$$L_q^p \cap L_s^r = L_{q \wedge s}^{p \vee r},$$

and from which it follows in particular that $L^0 \cap L^\infty$ is indeed an additive identity for the entire collection of L_q^p spaces, $p, q \in [0, \infty]$.

All of the arithmetic formulas given above hold for any measure μ. Which suggests the question: *how does $L^p(\mu)$ arithmetic depend on the underlying measure μ?*

We intend to discuss this matter in a forthcoming note.

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> No mathematician nowadays sets any store on the discovery of isolated theorems, except as affording hints of an unsuspected new sphere of thought, like meteorites detached from some undiscovered planetary orb of speculation. —*J. J. Sylvester*

A Vector Approach to Euler's Line of a Triangle

J. Ferrer

Among the many interesting properties that triangles possess there is one that quickly attracts our curiosity and stays easily in our mind: The centroid, circumcentre and orthocentre all lie in a common line (Euler's Line).

An elementary simple proof can be obtained using metric and affine properties of the points involved, [1]. Our aim here is to illustrate a proof using vectors.

We identify points in the plane with their position vectors. It is easy to see that the centroid G of the triangle ABC is given by the identity

$$G = \tfrac{1}{3}(A + B + C).$$

Similar formulae for the circumcentre O, the orthocentre H and the incentre I are not as immediate, but inner product relations such as the laws of sines and cosines can be used as to show

$$O = \frac{1}{2s}[(s - vw)A + (s - uw)B + (s - uv)C]$$

$$H = \frac{1}{s}(vwA + uwB + uvC)$$

$$I = \frac{1}{p}(aA + bB + cC)$$

with the obvious terminology:

$$a, b, c \text{ the lengths of sides } BC, AC, AB$$

$$u = b^2 + c^2 - a^2, \qquad v = a^2 + c^2 - b^2, \qquad w = a^2 + b^2 - c^2$$

$$s = uv + uw + vw, \qquad p = a + b + c.$$

Using these vectors, we have the straightforward relation:

$$2O + H = 3G,$$

that is, G, O, H are always colinear.

It also can be verified that the incentre I belongs to Euler's Line whenever the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ \sin A & \sin B & \sin C \\ \sin A \cos B \cos C & \cos A \sin B \cos C & \cos A \cos B \sin C \end{vmatrix}$$

vanishes. But the value of this determinant is

$$2\sin\left(\frac{B-A}{2}\right)\sin\left(\frac{C-A}{2}\right)\sin\left(\frac{C-B}{2}\right)$$
$$\times\left[\sin(A+B)+\sin(A+C)+\sin(B+C)\right].$$

Thus, since the terms $\sin(A+B), \sin(A+C), \sin(B+C)$ are all positive, it is clear that the triangle must be isosceles.

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Picture Puzzle
(from the collection of Paul Halmos)

Do these two men have anything in common?
(*See page* 687.)

> Angling may be said to be so like the mathematics, that it can never be fully learnt.
>
> —*Isaac Walton*

Underwood Dudley, Gerald A. Edgar, Michael A. Filaseta, Ira M. Gessel, Richard A. Gibbs, Douglas A. Hensley, John R. Isbell, Mourad E. H. Ismail, Murray Klamkin, Daniel J. Kleitman, Frederick W. Luttmann, Frank B. Miles, Richard Pfiefer, Stephen L. Portnoy, J. O. Shallit, John Henry Steelman, Kenneth B. Stolarsky, Douglas B. Tyler, Daniel Ullman, Edward T. H. Wang, and William E. Watkins.

Answer to Picture Puzzle

(*on page* 665)

Eberhard Hopf, the analyst, and Heinz Hopf, the topologist—no relation.

Call Archimedes from his buried tomb
Upon the plain of vanished Syracuse,
And feelingly the sage shall make report
How insecure, how baseless in itself,
Is the philosophy, whose sway depends
On mere material instruments—how weak
Those arts, and high inventions, if unpropped
By virtue

—*Wordsworth*

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FREEMAN DYSON grew up in England where he studied mathematics with Besicovitch and Davenport. In 1947 he came to Cornell University to learn physics, and since 1953 he has been a professor in the School of Natural Sciences of the Institute for Advanced Study in Princeton. He works in physics and number-theory, two branches of applied mathematics that both use concepts taken from pure mathematics to solve concrete problems.

HAROLD FALK grew up in Iowa where he ate sweet corn and played the violin badly. He studied physics in Seattle but never learned to ski. *Life Magazine* once paid him $125 for some photographs. He enjoys probability theory and statistical mechanics and is enthusiastically learning number-theory.

DAVID M. BRESSOUD graduated from Swarthmore College, spent two years with the Peace Corps in the Eastern Caribbean, returned to earn his doctorate under Emil Grosswald at Temple University, and has been at Penn State ever since except for visiting positions at the Institute for Advanced Study and the universities of Wisconsin, Minnesota, and Strasbourg. His research is in Partition Theory overlapping with Number Theory, Combinatorics, Special Functions, and Representation Theory.

RICHARD L. ROTH did his undergraduate work at Harvard and received his Ph.D. in mathematics in 1963 at the University of California at Berkeley. He has been a member of the mathematics department of the University of Colorado, Boulder, since 1963. He has also taught as a visiting professor in Central America (1965–66) and Colombia, South America (1969). His research has usually been related to group theory including character theory, and applications of groups to projective planes, color symmetry and tilings and patterns.

JONATHAN BORWEIN was an Ontario Rhodes Scholar (1971) and completed an Oxford D.Phil. (1974) with Michael Dempster. He has worked at Dalhousie University ever since, and is now Professor of Combinatorics and Optimization at Waterloo. He has also been on faculty at Carnegie-Mellon University (1980–82). He was the 1987 Coxeter-James lecturer of the Canadian Mathematical Society and received the 1988 Atlantic Provinces Gold Medal for Research in the Sciences. His research interests include optimization theory, functional and classical analysis.

PETER BORWEIN is a Professor of Mathematics at Dalhousie University where he has been for much of the last eleven years. His Ph.D. was under the supervision of David Boyd at the University of British Columbia. This was followed by post-doctoral training at Oxford. He has a fondness for classical analytic and number theoretic problems that lend themselves to extensive computational experimentation and computer assisted proofs.

STEVEN ROMAN is currently Professor of Mathematics at the California State University, Fullerton. He received his Ph.D. in 1975 at the University of Washington, under Branko Grünbaum, and has taught at MIT, the University of California at Santa Barbara, and the University of South Florida. Dr. Roman has written several research articles in the areas of combinatorics, graph theory and the umbral calculus. He has also written several books: *The Umbral Calculus* (Academic Press), *Discrete Mathematics* (HBJ), *Linear Algebra* (HBJ), and *College Algebra and Trigonometry* (HBJ). In addition, he has written a series of books entitled *Modules in Mathematics* (Innovative Textbooks), designed for a course in Liberal Arts Mathematics. Currently, he is working on a graduate text in Coding and Information Theory.

BART BRADEN wrote his Ph.D. thesis under the direction of Charles W. Curtis at the University of Oregon in 1966. He has been teaching at Northern Kentucky since 1971. This article is dedicated with gratitude and admiration to Professor Curtis on the occasion of his retirement in January 1992.

SERGIO A. ALVAREZ was born in Bogotá, Colombia, where he received his degree in engineering and his Master's in mathematics from the Universidad de los Andes. He is presently pursuing a Ph.D. in applied mathematics at the University of Maryland, College Park, supported by a Graduate School Fellowship. Mr. Alvarez has done work in computer systems for private industry, published research in mathematics and engineering, and is a contributing author to a textbook on mathematics for social science students. His main interests are in functional analysis, dynamical systems, and robotics.

JESÚS FERRER was born in Miramar, a small town near Valencia, Spain. In 1970 he came to the United States as an American Field Service scholar to graduate from High School in Long Prairie, Minnesota. He graduated and received his Ph.D. from the University of Valencia while enjoying a fellowship at IBM/Spain. Being a High School teacher for many years, he developed a special enthusiasm for all fields of elementary Mathematics. He teaches now at the University of Valencia where he is doing research in General Topology and in Banach Space Geometry. He enjoys soccer and doing chores as a central Minnesotan in Lake Wobegon.

STEVEN R. FINCH studied composition, piano and mathematics at Oberlin College. He received his MS in applied mathematics at the University of Illinois (in Urbana/Champaign) in 1985 and worked until recently as a statistician at TASC (outside Boston). This paper arose from his friendship with Jane A. Hale, author of a book surveying Raymond Queneau's literary work.

JOHN P. BURGESS received BA and MS degrees in mathematics from Princeton and Ohio State, respectively, and a Ph.D. degree in logic from Berkeley (1974). The following year he joined the philosophy department at Princeton, where he is now a professor. He is an editor of the Journal of Symbolic Logic and the Notre Dame Journal of Formal Logic, and a frequent contributor to logic journals and anthologies.

FRANK SWETZ is Professor of Mathematics and Education at the Pennsylvania State University at Harrisburg. His research interests have focused on the humanization of the learning and teaching of mathematics and have led him into studies on ethonomathematics and the history of mathematics. His most recent works in these fields are *The Sea Island Mathematical Manual: Mathematics and Surveying in Ancient China* and *The History of Mathematics: A Collection of Readings*. He holds a D.Ed. from Columbia University.

The moving power of mathematical invention is not reasoning but imagination.

—*A. De Morgan*

UNSOLVED PROBLEMS

Edited by: **Richard Guy**

In this department the MONTHLY presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial or related results. Typescripts should be sent to Richard Guy, Department of Mathematics & Statistics, The University of Calgary, Alberta, Canada T2N 1N4.

Are 0-Additive Sequences Always Regular?

Steven R. Finch

Starting with m positive integers $a_1 < a_2 < \cdots < a_m$, Queneau [**1**] defined the 0-*additive sequence* with *base* $\{a_1, a_2, \dots, a_m\}$ as the infinite sequence $a_1, a_2, a_3, \dots$ with a_{n+1}, for $n \geq m$, equal to the least integer exceeding a_n which is *not* of the form $a_i + a_j, i < j$. For example, when $m = 2$ and $a_1 = 1, a_2 = 6$, the first few terms of the sequence are

$$1, 6, 8; \quad 10, (12), 15, 17; \quad 19, 24, 26; \quad 28, 33, 35; \quad 37, 42, 44; \dots$$

which, apart from the extra term 12, breaks naturally into segments of three terms, with successive differences 5, 2, 2.

A 0-additive sequence is said to be *regular* if successive differences $a_{n+1} - a_n$ are eventually periodic; i.e., if there is a positive integer N such that $a_{N+n+1} - a_{N+n} = a_{n+1} - a_n$ for all sufficiently large n. Call the smallest such N the *period* of the sequence. The 0-additive sequence with base $\{1, 6\}$, hence, is regular with period $N = 3$. Queneau established the regularity of many families of 0-additive sequences and found, for instance, that

$$\left(\begin{array}{c}\text{the period of the}\\ \text{0-additive sequence}\\ \text{with base}\\ \{1, k\}, k > 1\end{array}\right) = \left\{\begin{array}{ll}1 & \text{if } k \text{ is odd or } k = 2\\ 4 & \text{if } k = 4\\ 3 & \text{if } k = 6\\ k + 3 & \text{if } k \geq 8 \text{ is even}\end{array}\right\}$$

and

$$\begin{pmatrix} \text{the period of the} \\ \text{0-additive sequence} \\ \text{with base} \\ \{2,k\},\ k>2 \end{pmatrix} = \begin{Bmatrix} 2 & \text{if } k = 3 \text{ or } 8 \\ 1 & \text{if } k = 4 \\ k & \text{if } 5 \le k \equiv 1 \bmod 4 \\ k+1 & \text{if } 6 \le k \equiv 2 \bmod 4 \\ \dfrac{3k+3}{4} & \text{if } 7 \le k \equiv 3 \bmod 4 \\ \dfrac{3k}{4} & \text{if } 12 \le k \equiv 0 \bmod 4 \end{Bmatrix}.$$

Given the sheer magnitude of Queneau's computations, one cannot help but conjecture that *all* 0-additive sequences are regular (no matter the choice of base). His formulas, however, do not suggest a proof of the conjecture.

Before continuing, we point out that sequences similar to 0-additive sequences were constructed by Dickson [**2**]. The only distinction is that Dickson's criterion for a_{n+1} to be a term is slightly more stringent: terms of the form $a_i + a_j, i \le j$, are prohibited (instead of merely $i < j$). Computational evidence suggests that Dickson's sequences are likewise always regular (Guy [**3**]). For the sake of definiteness, we shall focus on Queneau's sequences rather than Dickson's sequences in the remainder of this paper. Only minor adjustments are needed to shift formulation from one to the other.

There is an attractive way of rephrasing the conjecture that all 0-additive sequences are regular. It involves the simple Boolean algebra $B = \{0, 1\}$, where 0 and 1 are treated as the logical values *false* and *true*, respectively, and where the arithmetic operations + and $\cdot$ in the algebra are isomorphic to the logical connectives *or* and *and*, respectively (Dwinger [**4**]). Thus $0 + 0 = 0$, $0 + 1 = 1 + 0 = 1 + 1 = 1$, $0 \cdot 0 = 1 \cdot 0 = 0 \cdot 1 = 0$ and $1 \cdot 1 = 1$ in B. Let also $0' = 1$ and $1' = 0$, which correspond to logical negation.

For a specific 0-additive sequence $a_1, a_2, \ldots,$ define a sequence $b_1, b_2, \ldots$ in B by $b_n = 1$ if $n = a_j$ for some j and $b_n = 0$ otherwise. Recall that m denotes the number of base elements in $a_1, a_2, \ldots$ and define $p = a_m$. For $n > p$, clearly $b_n = 0$ if and only if $b_k \cdot b_{n-k} = 1$ for some $k \ne n/2$. By properties of summation in B, the regularity conjecture may be expressed as follows.

Conjecture. *For any integer $p \ge 2$ and any choice of initial conditions $(b_1, b_2, \ldots, b_p) \in B^p$, the* convolution sequence *in B defined by*

$$b'_n = \sum_{k=1}^{\left[\frac{n-1}{2}\right]} b_k \cdot b_{n-k} \quad \text{for } n > p$$

is eventually periodic.

Here $[x]$ denotes the greatest integer $\le x$.

To prove the Conjecture, it will suffice to replace the upper terminal $[(n-1)/2]$ by some constant c (independent of n) so large that convolution sequence terms remain unchanged. The constant c may depend on p. If c_p denotes the smallest constant c that works for any choice of initial conditions $(b_1, b_2, \ldots, b_p)$, then the values of c_p for $2 \le p \le 8$ have been computed to be 6, 8, 10, 14, 16, 51 and 156.

It's interesting to compare the form of this Conjecture with parallel analysis by Finch [**5**] of what are known as 1-additive sequences. In [**5**], it was useful to regard

the sequence indicator variables b_{2n+1} as elements of the binary field $\mathbb{Z}_2 = \{0, 1\}$ (arithmetic modulo 2) in certain cases. A linear recursive formula for b_{2n+1} gave rise to an approximation for the period N of such sequences. This contrasts with the use of the Boolean algebra $B = \{0, 1\}$ and a nonlinear recursion in the present paper.

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4. Philip Dwinger, *Introduction to Boolean Algebras*, Physica-Verlag, Wurzburg, 1971.
5. Steven R. Finch, On the regularity of certain 1-additive sequences, *J. Combinatorial Theory Ser. A*, 60 (1992), 123–130.

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A Computer's View

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PROBLEMS AND SOLUTIONS

Edited by:
Richard T. Bumby, Fred Kochman and Douglas B. West

Proposed problems should be sent to the MONTHLY PROBLEMS address given on the inside front cover. Please include solutions, relevant references, etc. Three copies are requested.

Solutions of published problems should arrive before February 28, 1993 at the MONTHLY PROBLEMS address given on the inside front cover. Solutions should be typed with double spacing, including the problem number and the solver's name and mailing address. Two copies suffice. A self-addressed postcard or label should be included if an acknowledgement is desired.

*An asterisk (*) after the number of a problem, or part of a problem, indicates that no solution is currently available. Partial solutions will be useful in such cases. Otherwise, the published solution is likely to be based on a solution which is complete and correct. Of course, an elegant partial solution or a method leading to a more general result is always useful and welcome. In addition, references to other appearances of MONTHLY problems or to solutions of these problems in the literature are also solicited.*

PROBLEMS

10238. *Proposed by David M. Bloom, Brooklyn College of CUNY, Brooklyn, NY.*

(a) Show that there exist infinitely many positive integers a such that both $a + 1$ and $3a + 1$ are perfect squares.

(b) Let $a_1 < a_2 < \ldots$ be the sequence of all solutions of (a). Show that $a_n a_{n+1} + 1$ is also a perfect square.

10239. *Proposed by Ismor Fischer, Naval Postgraduate School, Monterey, CA.*

A continuous vector field $\vec{F}$ (in $\mathbb{R}^2$ or $\mathbb{R}^3$) and a simple closed curve Γ are given. Show that, for every point $x \in \Gamma$, there exists a point $y \in \Gamma$ and a path γ from x to y (nontrivial if $x = y$) such that the work $W = \int_\gamma \vec{F} \cdot d\vec{r}$ is zero.

10240. *Proposed by Michael Golomb, Purdue University, West Lafayette, IN.*

Fix an integer n. For each integer m with $0 \le m \le n$, let p_m be a polynomial of degree n for which $\int_0^1 p_m(x) x^l \, dx = 0$ for $0 \le l \le n$ with $l \ne m$, while $\int_0^1 p_m(x) x^m \, dx = 1$.

(a) Determine the value of $\int_0^1 p_m^2(x)\,dx$.

(b) Find an explicit expression for p_m and prove that the coefficient of x^l in p_m is the same as the coefficient of x^m in p_l for $0 \le l < m \le n$.

10241. *Proposed by Roger W. Johnson, Carleton College, Northfield, MN.*

Let m and n be positive integers with $m \ge n$. Show that

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^n \left(\frac{\sin(mx)}{x}\right) dx = \frac{\pi}{2}.$$

10242. *Proposed by S. Brocco, Brandeis University, Waltham, MA, and F. Mignosi, Institut Blaise Pascal, Paris, France and Università di Palermo, Palermo, Italy.*

Let α be a fixed irrational number.

(a) For fixed integer n with $n > 1$, show that it is possible to find a constant $c(n)$ such that there are infinitely many rationals p/q with q relatively prime to n and $|\alpha - p/q| < c(n)/q^2$.

(b) If the continued fraction of α has unbounded partial quotients and $\varepsilon > 0$ is given, can one find $c(n) < \varepsilon$ satisfying the above condition?

10243. *Proposed by Michel Balazard, Université Bordeaux I, Talence, France.*

Define a sequence of functions $f_k(t)$ for $t > k$ recursively by

$$f_1(t) = 1$$

$$f_{k+1}(t) = \int_k^{t-1} f_k(u)\frac{du}{u}.$$

Prove that, for every real number $t > 1$, the sequence $\langle f_k(t) : 1 \le k < t \rangle$ is unimodal.

10244. *Proposed by Ken Bromberg (student), Brown University and Stan Wagon, The Geometry Center, Minneapolis, MN and Macalester College, St. Paul, MN.*

A classical construction of Miquel starts with an n-vertex polygon and a point P in the plane (not a vertex of the n-gon), and forms another n-gon as follows:

1. draw the perpendiculars from P to the (extended) sides of the polygon;
2. connect the feet to obtain another n-gon.

These steps are then repeated n times (provided that none of the polygons has P as a vertex). The resulting polygon, denoted $M(P)$ is similar to the initial n-gon.

(a) Given a triangle, construct the point P for which $M(P)$ is largest.

(b)* Given a quadrilateral, is there a Euclidean construction of the point P for which $M(P)$ is largest?

10245. *Proposed by M. A. Bezem, Utrecht University, Utrecht, The Netherlands and A. J. C. Hurkens, Catholic University, Nijmegen, The Netherlands.*

Let $\mathcal{S}$ be a set of finite, non-empty sets. A *transversal* of $\mathcal{S}$ is a set which has a non-empty intersection with every element of $\mathcal{S}$. The Principle of Minimal Transversal states that every such $\mathcal{S}$ has a transversal which is minimal with respect to set inclusion. Prove that the Axiom of Choice is equivalent to the Principle of Minimal Transversal.

10246. *Proposed by B. C. Carlson, Iowa State University, Ames, IA.*

For integers m and n with $n \geq 0$ and $-n \leq m \leq n$, find values of A, N, and λ such that

$$\frac{1}{2\pi(1-\rho^2)^{1/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^{n+m}y^{n-m}\exp\left(\frac{-x^2+2\rho xy-y^2}{2(1-\rho^2)}\right)dx\,dy = AC_N^{\lambda}(-\rho)$$

for $-1 < \rho < 1$, where C_N^{λ} is a Gegenbauer polynomial.

NOTES

(10242) It is known that, without the condition on relative primality, there are infinitely many p/q with $|\alpha - p/q| < 1/q^2$. Furthermore, if the continued fraction of α has unbounded partial quotients, then $|\alpha - p/q| < \varepsilon/q^2$ has infinitely many solutions. **(10243)** A sequence $\langle s_k \rangle$ is called "unimodal" if there is an index k_0 such that $s_k < s_{k+1}$ for $k < k_0$ and $s_k > s_{k+1}$ for $k > k_0$. **(10244)** Further details of this construction can be found in B. M. Stewart, "Cyclic properties of Miquel polygons", this MONTHLY, 47 (1940), 462–466, and in H. S. M. Coxeter, *Introduction to Geometry*, p. 16. **(10245)** The book, H. Rubin & J. E. Rubin, *Equivalents of the Axiom of Choice, II*, North-Holland, 1985 gives a description of some recent work on the Axiom of Choice and its relatives. **(10246)** The Gegenbauer (or ultraspherical) polynomials are defined by the generating function $(1 - 2tz + t^2)^{-\lambda} = \sum_{N=0}^{\infty} t^N C_N^{\lambda}(z)$. Details may be found in A. Erdélyi, et al., *Higher Transcendental Functions*, vol. 2, Sect. 10.9.

SOLUTIONS

The Product of Two Sides of a Triangle

E 3417 [1991, 54]. *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, WI.*

Suppose ABC is a triangle with $AB \neq AC$, and let D, E, F, G be points on the line through B and C defined as follows: D is the midpoint of BC, AE is the bisector of the angle BAC, F is the foot of the perpendicular from A to BC, and AG is perpendicular to AE (i.e., AG bisects one of the exterior angles at A).

Prove that $AB \cdot AC = DF \cdot EG$.

Solution I by S. Belbas, University of Alabama, Tuscaloosa, AL. All symbols refer to non-oriented line segment lengths. We may assume $AB < AC$. We set $a = BC$, $b = AC$, $c = AB$, $h = AF$, $x = BE$, $y = BG$, $z = DF$.

From the well-known theorems about internal and external bisectors, we have

$$\frac{BE}{CE} = \frac{BG}{CG} = \frac{AB}{AC}, \quad \text{or} \quad \frac{x}{a-x} = \frac{y}{a+y} = \frac{c}{b}.$$

Solving for x and y, we find $x = ac/(b+c)$ and $y = ac/(b-c)$. Thus $EG = x + y = 2abc/(b^2 - c^2)$.

Applying the Pythagorean Theorem to the right triangles AFC and AFB, we obtain $b^2 = (a/2 + z)^2 + h^2$ and $c^2 = (a/2 - z)^2 + h^2$. The difference between these is $b^2 - c^2 = 2az$; hence $DF = z = (b^2 - c^2)/2a$.

Therefore, $DF \cdot EG = ((b^2 - c^2)/2a)(2abc/(b^2 - c^2)) = bc = AB \cdot AC$.

Solution II by Jiro Fukuta, Shinsei-Cho, Motosu-Gun, Gifu-Ken, Japan. Let H be the point of intersection between the line AE and the circumcircle of the triangle ABC. From the similarity of triangles ABH and ACE, we have $AB \cdot AC = AE \cdot AH = AE^2 + AE \cdot EH$. Because AEG is a right triangle, and because A, D, H, G are concyclic (from $HD \perp BC$ and $HA \perp AG$), we have $AB \cdot AC = EF \cdot EG + AE \cdot EH = EF \cdot EG + DE \cdot EG = DF \cdot EG$.

Solved also by 44 readers and the proposer.

A Consequence of Classical Inequalities

6646 [1991, 63]. *Proposed by Daniel Goffinet, St. Etienne, France.*

Suppose f is a continuous function from $[0,1] \times [0,1]$ to $\mathbb{R}$ such that

$$\left(\int_0^1 \left\{\int_0^1 f(x,y)\,dx\right\}^2 dy\right)^{1/2} = \int_0^1 \left(\int_0^1 \{f(x,y)\}^2\,dy\right)^{1/2} dx.$$

Prove that there exist continuous functions u and r from $[0,1]$ to $\mathbb{R}$ such that $\int_0^1 \{u(y)\}^2\,dy = 1$, $r(x) \geq 0$ for all x in $[0,1]$, and $f(x,y) = r(x)u(y)$ for all x and y in $[0,1]$.

Solution by Kiran S. Kedlaya (student), Georgetown Day High School, Washington, DC. Ignoring the trivial case $f = 0$ we can replace f by cf for an appropriate $c > 0$ to assure that

$$\int_0^1 \{u(y)\}^2\,dy = \int_0^1 h(x)\,dx = 1$$

for $u(y) = \int_0^1 f(x,y)\,dx$ and $h(x) = (\int_0^1 \{f(x,y)\}^2\,dy)^{1/2}$.

Let $r(x) = \int_0^1 f(x,y)u(y)\,dy$ and $o(x,y) = f(x,y) - r(x)u(y)$.

All these functions are continuous and

$$\int_0^1 o(x,y)u(y)\,dy = \int_0^1 \left[f(x,y)u(y) - r(x)\{u(y)\}^2\right] dy = r(x) - r(x) = 0.$$

So

$$h(x) = \left\{\int_0^1 \left[\{o(x,y)\}^2 + 2o(x,y)r(x)u(y) + \{r(x)u(y)\}^2\right] dy\right\}^{1/2}$$
$$= \left\{\int_0^1 \{o(x,y)\}^2\,dy + \{r(x)\}^2\right\}^{1/2} \geq r(x).$$

But

$$\int_0^1 r(x)\,dx = \int_0^1 \int_0^1 f(x, y) u(y)\,dy\,dx = \int_0^1 u(y) \int_0^1 f(x, y)\,dx\,dy$$

$$= \int_0^1 \{u(y)\}^2\,dy = \int_0^1 h(x)\,dx.$$

Therefore $r = h \geq 0$ since $h \geq r$ and both r and h are continuous. This implies $\int_0^1 \{o(x, y)\}^2\,dy = 0$. Hence $o(x, y) = 0$.

Editorial comment. A number of respondents noted that the result is essentially a special case of Theorem 202, pgs. 148–150 of Hardy, Littlewood, and Pólya, *Inequalities* (Cambridge University Press, 1952), dealing with a continuous variant of Minkowski's inequality for sums of L_p norms. Frédéric Brulois discussed conditions for equality in an inequality

$$\left\| \int_X f \right\| \leq \int_X \|f\|$$

where $f\colon X \to B$ is defined on a measure space X, with values in a Banach space B. In the problem at hand X is the interval $[0, 1]$ and B is $L^2[0, 1]$.

Solved also by K. F. Anderson (Canada), F. Brulois, C. P. Grant, G. L. Isaacs, R. B. Israel (Canada), T. Kunkle, O. P. Lossers (The Netherlands), J. M. Monier (France), K. Schilling, R. Stong and the proposer. One incorrect solution was received.

An Inequality for Specially Selected Real Numbers

E 3421 [1991, 158]. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that $n \geq 1$ and that $w_1, w_2, \ldots, w_n$ are positive real numbers with

$$\sum_{i=1}^{n} \frac{1}{1 + w_i} = 1.$$

Prove that

$$\sum_{i=1}^{n} w_i^{1/2} \geq (n - 1) \sum_{i=1}^{n} w_i^{-1/2}.$$

Solution by Fumio Kubo, Toyama University, Gofuku, Toyama, Japan. Summing the identity

$$\frac{w_j}{1 + w_j} = 1 - \frac{1}{1 + w_j}$$

over all j yields

$$\sum_{j=1}^{n} \frac{w_j}{1 + w_j} = n - 1.$$

Hence

$$\sum_{i=1}^{n} w_i^{1/2} - (n-1)\sum_{i=1}^{n} w_i^{-1/2}$$
$$= \left(\sum_{j=1}^{n} \frac{1}{1+w_j}\right)\cdot\left(\sum_{i=1}^{n} w_i^{1/2}\right) - \left(\sum_{j=1}^{n} \frac{w_j}{1+w_j}\right)\cdot\left(\sum_{i=1}^{n} w_i^{-1/2}\right)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{w_i - w_j}{w_i^{1/2}(1+w_j)}\right)$$
$$= \sum_{i>j} \frac{\left(w_i^{1/2}w_j^{1/2}-1\right)\left(w_i^{1/2}-w_j^{1/2}\right)^2\left(w_i^{1/2}+w_j^{1/2}\right)}{w_i^{1/2}w_j^{1/2}(1+w_i)(1+w_j)}.$$

Here the final equality follows from

$$\frac{1}{w_i^{1/2}(1+w_j)} - \frac{1}{w_j^{1/2}(1+w_i)} = \frac{\left(w_i^{1/2}w_j^{1/2}-1\right)\left(w_i^{1/2}-w_j^{1/2}\right)}{w_i^{1/2}w_j^{1/2}(1+w_i)(1+w_j)}.$$

To complete the proof, it suffices to show $w_i w_j \geq 1$ for each distinct pair, for then the difference computed above is nonnegative. This follows from

$$1 \geq \frac{1}{1+w_i} + \frac{1}{1+w_j} = \frac{2+w_i+w_j}{1+w_i+w_j+w_iw_j}.$$

From this proof, it is also evident that equality holds if and only if $n = 2$ or $w_1 = w_2 = \cdots = w_n$.

Editorial comment. Jean-Charles Leccia noted that the problem is reduced to problem E 2874 [1981, 208; 1982, 601] by the substitution $w_i = \tan^2 A_i$.

Solved also by R. Betts (student), K. David, J. S. Frame, G. Greybeard, E. A. Herman, J. M. Huntley and D. E. Tepper, M. E. Kuczma (Poland), J.-C. Leccia (France), O. P. Lossers (The Netherlands), I. A. Sakmar (Turkey), K. Schilling, H.-J. Seiffert (Germany), R. Stong, G. W. Teck (student, England), J. Tòth (Czechoslovakia), M. Vowe (Switzerland), and the proposer. One incomplete solution was received.

Tangents Intersect on the Axis of Involution

E 3422 [1991, 158]. *Proposed by H. Demir and C. Tezer, Middle East Technical University, Ankara, Turkey.*

Suppose F and F' are points situated symmetrically with respect to the center of a given circle, and suppose S is a point on the circle not on the line FF'. Let P and P' be the second points of intersection of SF and SF' respectively with the circle. If the tangents to the circle at P and P' intersect at T, prove that the perpendicular bisector of FF' passes through the midpoint of the line segment ST.

Solution I by Jean-Pierre Grivaux, Paris, France. We work in the complex plane, with lower-case letters denoting the complex representations of points designated by the corresponding upper-case letters. We may assume that the circle is $\mathbf{U} = \{Z\colon |z| = 1\}$ and that the points F and F' are on the real axis.

If $A, B \in \mathbf{U}$, then Z is on the line through A and B if and only if $z + ab\bar{z} = a + b$, which we shall refer to as equation $\mathscr{E}_{ab}$. To derive this equation, note that

the line is the set of Z whose numerical representation satisfies $z = a + r(b - a)$, where r is real. Conjugating this and using $\bar{a} = 1/a$ and $\bar{b} = 1/b$ yields $\bar{z} = \bar{a} + r(\bar{b} - \bar{a})$, which when multiplied by ab and added to the first equation yields $\mathscr{E}_{ab}$. This form of $\mathscr{E}_{ab}$ remains valid when $a = b$.

Since $\mathscr{E}_{pp}$ and $\mathscr{E}_{p'p'}$ are the equations of the tangents to $\mathbf{U}$ at P and P', we have $t + p^2\bar{t} = 2p$ and $t + (p')^2\bar{t} = 2p'$. Solving for t by eliminating $\bar{t}$ (when $p \neq p'$) yields $t = 2/(\bar{p} + \bar{p'})$. Note that $p + p' \neq 0$ because s is not real. The midpoint of ST is Z, where

$$z = \frac{1}{2}(s + t) = \frac{1}{2}\left(s + \frac{2}{\bar{p} + \bar{p'}}\right),$$

and the result we want to prove is $z + \bar{z} = 0$, which by the above is

$$\left(s + \frac{2}{1/p + 1/p'}\right) + \left(\frac{1}{s} + \frac{2}{p + p'}\right) = 0.$$

This is equivalent by algebraic manipulation to

$$\left(-\frac{2s}{1 + s^2}\right)(1 + pp') = p + p'. \qquad (*)$$

Sine F and F' belong to the lines PS and $P'S$ respectively, f and $f'(= -f)$ satisfy the equations $\mathscr{E}_{ps}$ and $\mathscr{E}_{p's}$ respectively, namely $(f)(1 + ps) = p + s$ and $(-f)(1 + p') = p' + s$, where we use the fact that $\bar{f} = f$. Elimination of f from these two equations produces the desired equality $(*)$.

Solution II by the proposers. We exclude the case in which F and F' coincide. Let K be the point diametrically opposite S. Let S' be the additional point where the line through S parallel to FF' intersects the circle (S' may coincide with S). The lines SS', SP, SK, SP' form a harmonic pencil, as the center of the circle bisects FF'. Consequently, for any point X on the circle, the lines XS', XP, XK, XP' form a harmonic pencil. Choosing $X = P$ or $X = P'$ in particular, we find that the pencils PS', PT, PK, PP' and $P'S', P'P, P'K, P'T$ are harmonic. Since the line PP' is common to both pencils, the points S', K, T lie on a line which is clearly perpendicular to FF'. Hence the perpendicular bisector of FF' bisects ST.

Editorial comment. Most solvers used straightforward analytic geometry and brute force calculation to prove the result. Several used synthetic Euclidean geometry. H. Kappus gave another proof using complex numbers. O. P. Lossers gave another proof using projective geometry. A nice approach by J. Dou uses a classical property of projective involutions of a conic (involutions sending a conic to itself and preserving cross ratios). We briefly describe this and its relationship to Grivaux's solution, using the notational conventions of that solution.

The mapping $\sigma: \mathbb{R} \to \mathbb{R}$ given by $\sigma(x) = -x$ yields a projective involution of the real line which extends to a projective involution of the real projective line $\mathbf{P}$ by defining $\sigma(\infty) = \infty$. With point S given on the unit circle $\mathbf{U}$, we define $\pi: \mathbf{P} \to \mathbf{U}$ by letting $\pi(X)$ be the point where the line joining S to $X \in \mathbf{P}$ again intersects $\mathbf{U}$. In particular, π applied to the point at infinity is the other point of intersection with $\mathbf{U}$ of the line through S parallel to the real axis. It then follows that the mapping $g: \mathbf{U} \to \mathbf{U}$ given by $g = \pi \circ \sigma \circ \pi^{-1}$ is a projective involution of $\mathbf{U}$.

The numbers corresponding to the fixed points of g are $-s$ and $-\bar{s}$, since σ fixes 0 and ∞. Now a classical result of projective geometry implies that for each $P \in \mathbf{U}$, the tangent lines of $\mathbf{U}$ at P and $g(P)$ intersect on the line $\mathbf{l}$ through the fixed points of g. Since T is the intersection of the tangent lines at P and $P' = g(P)$, we see that T is on the line $\mathbf{l}$, and it is then obvious that the midpoint of ST is on the pure-imaginary axis, as was to be proved.

It is easy to calculate that $\pi(X)$ is represented by $(x - s)/(1 - xs)$ for $x \in \mathbb{R}$, and $g(P)$ is represented by $(\lambda - p)/(1 - \lambda p)$ for $p \in \mathbf{U}$, where $\lambda = -2s/(1 + s^2)$. In fact, λ is real (or ∞) and it represents the intersection of the tangent lines at the fixed points of g. Indeed, the relation $(*)$ in Grivaux's solution expresses the fact that λ satisfies the equation $\mathscr{E}_{pp'}$; thus the line through P and P' always passes through Λ. This shows that the involution g is obtained by sending each point $P \in \mathbf{U}$ to the other point where $\mathbf{U}$ intersects the line through Λ and P. The line $\mathbf{l}$ (the "axis" of the involution g) is the polar of Λ with respect to $\mathbf{U}$.

For a detailed discussion of involutions of conics, see H. F. Baker, *An Introduction to Plane Geometry* (Cambridge University Press, 1943), Chapter IX, or M. Berger, *Geometry II* (Springer, 1987), Section 16.3. In Berger's book the above point λ is called the "Frégier point" of the involution g.

Solved by 26 readers (including those cited) and the proposers.

Conditions for Solving a Matrix Equation

E 3425 [1991, 159]. *Proposed by Shmuel Rosset and Tamir Shalom, Tel Aviv University, Tel Aviv, Israel.*

Suppose A and B are n matrices over a field k.

(i) Prove that there is an n by n matrix X over k with $AX + XA = B$ if and only if $\mathrm{tr}(BC) = 0$ for every n by n C with $AC = -CA$.

(ii) Assume $\mathrm{char}(k) = 0$. If $AB = -BA$ and if the matrix equation $AX + XA = B$ has a solution over k, prove that B is nilpotent; for each positive integer n give an example in which $B^n = 0$ but $B^{n-1} \neq 0$.

Solution by Robin J. Chapman, University of Exeter, United Kingdom. Let $V = M_n(k)$ be the vector space of n by n matrices over k. Since $\mathrm{tr}(XY) = \mathrm{tr}(YX)$, the function $\langle X, Y\rangle = \mathrm{tr}(XY)$ is a symmetric bilinear form on V. It is non-singular, because $\langle X, E_{ji}\rangle = x_{ij}$, where E_{ji} is the matrix with 0 entries except for a 1 in position (j, i). The function $f_A(X) = AX + XA$ is a linear transformation from V to itself. Denote its kernel and image by K and I, respectively. Assertion (i) states that $I = K^{\perp} = \{B \in V : \langle B, K\rangle = 0\}$. To see that $I \subseteq K^{\perp}$, note that if $B = AX + XA \in I$ and $C \in K$, then $AC = -CA$ and $BC = AXC + XAC = A(XC) - (XC)A$, which has $\mathrm{tr}\, 0$. Since $\langle\, ,\rangle$ is nonsingular, $\dim_k(K^{\perp}) = n^2 - \dim_k(K) = \dim_k(I)$, and so I and $K^{\perp}$ must be equal.

Suppose k has characteristic zero and $AB = -BA$, where $B = AX + XA$. For every integer $m \geq 1$, $AB^{2m-1} = -B^{2m-1}A$, and so $B^{2m} = (AX + XA)B^{2m-1} = A(XB^{2m-1}) - (XB^{2m-1})A$ has trace zero. Thus B^2 is nilpotent (see I. N. Herstein, *Topics in Algebra*, Wiley, 1975, Lemma 6.8.1), and hence B is nilpotent.

Finally, let $k = \mathbb{Q}$ and define $A = (a_{ij})$, $B = (b_{ij})$, $X = (x_{ij}) \in V$ as follows. Let $a_{ij} = b_{ij} = 0$ if $j \neq i + 1$, otherwise $a_{ij} = 1$ and $b_{ij} = (-1)^{i+1}$. Let $x_{ij} = 0$ for $i \neq j$ and $x_{ii} = (-1)^i i$. We may easily verify that $AB = -BA$, $B = AX + XA$, $B^{n-1} \neq 0$, and $B^n = 0$.

Editorial comment. E. D. Dixon noted that it suffices to have $\text{char}(k) > n$ in part (ii). F. J. Flanigan points out that part (ii) is equivalent to the fact that every (necessarily isotropic) vector B in $K \cap K^{\perp}$ is nilpotent.

Solved also by D. Callan, F. J. Flanigan, W. H. Gustafson, C. Lanski, J.-C. Leccia (France), R. Stong, and the proposer. Partially solved by J.-P. Grivaux (France), E. Cohen (France), and E. D. Dixon.

Convergence of a Subseries

E 3426 [1991, 159]. *Proposed by J. Michael Steele, Princeton University, Princeton, NJ.*

Suppose that $\{a_k\}_{k=1}^{\infty}$ is a sequence of non-negative real numbers such that $\Sigma a_k^2 < \infty$. Show that, for any positive constant c, there exists an increasing sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ such that $n_k = ck^2 + O(k)$ and such that

$$\sum_{k=1}^{\infty} a_{n_k}^2 \sum_{j=1}^{n_k} a_j < \infty.$$

Note: J. Michael Steele is now at the University of Pennsylvania.

Solution by Grażyna Bartoszek and Wojciech Bartoszek, Potchefstroom University, Potchefstroom, South Africa. We establish the stronger result that for every $p > 1$, if $a_k \geq 0$ and $\Sigma_{k=1}^{\infty} a_k^p < \infty$, then for every positive constant C there exists an increasing sequence n_k of natural numbers such that

$$\limsup_{k\to\infty} \frac{|n_k - Ck^p|}{Cpk^{p-1}} \leq 1 \quad \text{and} \quad \sum_{k=1}^{\infty} a_{n_k}^p \sum_{j=1}^{n_k} a_j < \infty.$$

Letting $p = 2$ solves the proposed problem.

We may assume that $a_n \neq 0$ for some n. For a fixed p and C, let K be a natural number strictly greater than 1, such that $\lfloor C(k-1)^p \rfloor < \lfloor Ck^p \rfloor$ for all $k \geq K$. Define n_k to be the smallest natural number from the interval $I_k = [\lfloor C(k-1)^p \rfloor + 1, \lfloor Ck^p \rfloor]$ such that $a_{n_k} = \min\{a_j\colon\ j \in I_k\}$. Let $c(I_k) = \lfloor Ck^p \rfloor - \lfloor C(k-1)^p \rfloor$. From the fact that $f'(k-1) \leq f(k) - f(k-1) \leq f'(k)$ when $f(x) = ax^p$, we have $Cpk^{p-1} + 1 \geq c(I_k) \geq Cp(k-1)^{p-1} - 1$. Now we compute

$$\begin{aligned}\sum_{k=1}^{\infty} a_k^p &\geq \sum_{k\geq K} \sum_{j\in I_k} a_j^p \geq \sum_{k\geq K} a_{n_k}^p c(I_k) \geq \sum_{k\geq K} a_{n_k}^p\left(Cp(k-1)^{p-1} - 1\right)\\ &= \sum_{k\geq K} Cp(k-1)^{p-1} a_{n_k}^p - \sum_{k\geq K} a_{n_k}^p.\end{aligned}$$

By Hölder's Inequality, we also have

$$\begin{aligned}\sum_{j=1}^{n_k} a_j &\leq \left(\sum_{j=1}^{n_k} a_j^p\right)^{1/p} \cdot n_k^{1/q} \leq \left(\sum_{j=1}^{\infty} a_j^p\right)^{1/p} \cdot (Ck^p)^{1/q}\\ &= C^{1/q}k^{p/q}\left(\sum_{j=1}^{\infty} a_j^p\right)^{1/p} = C^{1/q}k^{p-1}\left(\sum_{j=1}^{\infty} a_j^p\right)^{1/p},\end{aligned}$$

where $1/p + 1/q = 1$. Hence

$$\sum_{k \geq K} a_{n_k}^p \sum_{j=1}^{n_k} a_j \leq C^{1/q} \left(\sum_{j=1}^{\infty} a_j^p \right)^{1/p} \sum_{k \geq K} k^{p-1} a_{n_k}^p.$$

For $k \geq 2$, we have $(k-1)^{p-1} \geq (1/2)^{p-1} k^{p-1}$, so

$$+\infty > \sum_{k \geq K} Cpa_{n_k}^p (k-1)^{p-1} - \sum_{k \geq K} a_{n_k}^p \geq \sum_{k \geq k} Cp(1/2)^{p-1} k^{p-1} a_{n_k}^p - \sum_{k \geq K} a_{n_k}^p$$

$$\geq pC^{1/p}(1/2)^{p-1} \left(\sum_{j=1}^{\infty} a_j^p \right)^{-1/p} \sum_{k \geq K} a_{n_k}^p \sum_{j=1}^{n_k} a_j - \sum_{k \geq K} a_{n_k}^p.$$

In particular, $\sum_{k=1}^{\infty} a_{n_k}^p \sum_{j=1}^{n_k} a_j < \infty$. Since

$$\limsup_{k \to \infty} \frac{|n_k - Ck^p|}{Cpk^{p-1}} \leq \lim_{k \to \infty} \frac{c(I_k)}{Cpk^{p-1}} = 1,$$

the proof is complete.

Solved also by G. Bennett, R. High, O. P. Lossers (The Netherlands), R. Martin, A. Riese, K. Schilling, R. Stong, University of South Alabama Problem Group and the proposer.

No Even Break

6650 [91, 168]. *Proposed by Keith Ball, Trinity College, Cambridge, England, and Herman J. Tiersma, University of Technology, Eindhoven, The Netherlands.*

Let S be the set of natural numbers k such that every matrix of zeros and ones containing exactly $2k$ ones must have a submatrix containing exactly k ones. (In the language of graph theory S is the set of natural numbers k such that every bipartite graph with $2k$ edges has an induced subgraph with k edges.)

(a) Show that if p is a prime congruent to 1 modulo 5 such that $2p + 1$ is also prime, then $2p$ and $2p + 1$ are not in S.

(b)* Does S contain infinitely many natural numbers? Are the powers of 2 all in S?

Solution of (a) *by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands.* Suppose $p \equiv 1 \bmod 5$ and let both p and $2p + 1$ be primes. We write $p = 5t + 1$ $(t \geq 2)$.

First, $2(2p) = 5(4t + 1) - 1$. We form a matrix M of size $5 \times (4t + 1)$ consisting of ones only except for a single place. So M has $2(2p)$ ones. A submatrix $\widetilde{M}$, comprising a rows and b columns $(1 \leq a \leq 5,\ 1 \leq b \leq 4t + 1)$ contains ab or $ab - 1$ ones.

(i) $ab = 2p$ implies $a \geq p$ or $b \geq p$, because p is prime.

(ii) $ab - 1 = 2p$ implies $ab = 2p + 1$, so $a = 2p + 1$ or $b = 2p + 1$ because $2p + 1$ is prime.

Both cases contradict the size of M. So $2p \notin S$.

Second, $2(2p + 1) = 5(4t + 1) + 1$. We form a matrix N of size $5 \times (4t + 2)$ consisting of ones only except for one column that contains only a single one. So N has $2(2p + 1)$ ones. A submatrix $\widetilde{N}$, comprising a rows and b *all-one* columns $(1 \leq a \leq 5,\ 0 \leq b \leq 4t + 1)$ contains ab or $ab + 1$ ones.

(i) $ab = 2p + 1$ implies $a = 2p + 1$ or $b = 2p + 1$ because $2p + 1$ is prime.

(ii) $ab + 1 = 2p + 1$ implies $a \geq p$ or $b \geq p$ since p is prime.

Again, this contradicts the size of N. So $2p + 1 \notin S$.

Editorial comment. No solutions of part (b) were received.

Solved also by the proposers (part (a) only).

Big Pills and Little Pills

E 3429 [1991, 264]. *Proposed by Donald E. Knuth and John McCarthy, Stanford University, Stanford, CA.*

A certain pill bottle contains m large pills and n small pills initially, where each large pill is equivalent to two small ones. Each day the patient chooses a pill at random, if a small pill is selected, (s)he eats it; otherwise (s)he breaks the selected pill and eats one half, replacing the other half, which thenceforth is considered to be a small pill.

(a) What is the expected number of small pills remaining when the last large pill is selected?

(b) On which day can be expect the last large pill to be selected?

Composite solution by Walter Stromquist, Daniel H. Wagner, Associates, Paoli, PA and Tim Hesterberg, Franklin & Marshall College, Lancaster, PA. The answers are (a) $n/(m+1) + \Sigma_{k=1}^{m}(1/k)$, and (b) $2m + n - (n/(m+1)) - \Sigma_{k=1}^{m}(1/k)$. The answer to (a) assumes that the small pill created by breaking the last large pill is to be counted.

A small pill present initially remains when the last large pill is selected if and only if it is chosen last from among the $m+1$ element set consisting of itself and the large pills—an event of probability $1/(m+1)$. Thus the expected number of survivors from the original small pills is $n/(m+1)$. Similarly, when the kth large pill is selected ($k = 1, 2, \ldots, m$), the resulting small pill will outlast the remaining large pills with probability $1/(m-k+1)$, so the expected number of created small pills remaining at the end is $\Sigma_{k=1}^{m}(1/k)$. Hence the answer to (a) is as above. The bottle will last $2m+n$ days, so the answer to (b) is just $2m+n$ minus the answer to (a), as above.

Editorial comment. Most solvers derived a recurrence relation, guessed the answer, and verified it by induction. Several commented on the origins of the problem. Robert High saw a version of it in the MIT Technology Review of April, 1990. Helmut Prodinger reports that he proposed it in the Canary Islands in 1982. Daniel Moran attributes the problem to Charles MacCluer of Michigan State University, where it has been known for some time.

Solved by 38 readers (including those cited) and the proposer. One incorrect solution was received.

Asymptotics of the Harmonic Sum

E 3432 [1991, 264]. *Proposed by Lászlo Tóth, Satu Mare, Romania.*

(i) Prove that for every positive integer n we have

$$\frac{1}{2n + 2/5} < 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n - \gamma < \frac{1}{2n + 1/3},$$

where γ is Euler's constant.

(ii) Show that $2/5$ can be replaced by a slightly smaller number, but that $1/3$ cannot be replaced by a slightly larger number.

Solution by R. High, New York City, NY. Both D. E. Knuth, *The Art of Computer Programming*, Vol. I, Addison-Wesley, 1973, sect. 1.2.7, formula 3, p. 74 and R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics, A Foundation for Computer Science*, Addison-Wesley, 1989, p. 466 present the result that

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{\varepsilon_n}{120n^4},$$

where $0 < \varepsilon_n < 1$. It thus suffices to show that

$$\frac{1}{2n + 2/5} < \frac{1}{2n} - \frac{1}{12n^2} + \frac{\varepsilon_n}{120n^4} < \frac{1}{2n + 1/3}.$$

We obtain the first inequality from

$$\frac{1}{2n + 2/5} = \frac{1}{2n}\left(1 - \frac{1}{5n + 1}\right) = \frac{1}{2n} - \frac{1}{10n^2 + 2n},$$

since $12n^2 \geq 10n^2 + 2n$ and $\varepsilon_n > 0$. On the other hand,

$$\frac{1}{2n + 1/3} = \frac{1}{2n}\left(1 - \frac{1}{6n + 1}\right) = \frac{1}{2n} - \frac{1}{12n^2}\left(1 - \frac{1}{6n + 1}\right),$$

so the second inequality follows from the fact that $120n^4 > 12n^2(6n + 1)$ for all $n \in \mathbb{N}$.

For (ii), suppose we subtract $1/k$ from $2/5$. Since $2/5 - 1/k = (2k - 5)/(5k)$, we are asking whether sufficiently large k guarantees for all n that

$$\frac{1}{2n + (2k - 5)/(5k)} < \frac{1}{2n} - \frac{1}{12n^2} + \frac{\varepsilon_n}{120n^4}.$$

The left side can be written as $1/(2n) - (2k - 5)/(2n(10kn + 2k - 5))$. Comparison of the latter term with $1/(12n^2)$ shows that the inequality is satisfied without help from ε_n whenever $4k > 60 + (4k - 10)/n$. For $n > 1$, $k \geq 28$ suffices. For $n = 1$, direct calculation shows that $(2k - 5)/(24k - 10) > 1/12 = \varepsilon_1/120$ when $k \geq 30$. Hence $2/5$ can be replaced by $11/30$.

On the other hand, if we add $1/k$ to $1/3$ in the other inequality, we are asking whether sufficiently large k guarantees

$$\frac{1}{2n} - \frac{1}{12n^2} + \frac{\varepsilon_n}{120n^4} < \frac{1}{2n + (k + 3)/(3k)} = \frac{1}{2n} - \frac{k + 3}{2n(6nk + k + 3)}.$$

For each fixed k, sufficiently large n will violate the inequality, so $1/3$ cannot be replaced by a larger number.

Editorial comment. Solvers Jean Anglesio, David M. Bloom, Douglas B. Tyler, and Michael Vowe showed that $2/5$ can be replaced by $(2\gamma - 1)/(1 - \gamma) = 0.36527\ldots$, and equality holds only when $n = 1$.

Solved also by J. Anglesio (France), R. Betts, D. M. Bloom, P. Bracken (Canada), M. Dindos (Czechoslovakia), J. S. Frame, M. E. Kuczma (Poland), L. E. Mattics, H. Morris, R. E. Shafer, R. Stong, D. B. Tyler, M. Vowe (Switzerland), E. A. Weinstein, and the proposer.

A Fast Runner and a Slow One

6654 [1991, 273]. *Proposed by W. O. Egerland and C. E. Hansen, Aberdeen Proving Ground, Aberdeen, MD.*

Suppose ω is real, n is a positive integer greater than 1, and $a_a, a_2, \dots, a_n$ are complex numbers with $|a_k| < 1$ for $k = 1, 2, \dots, n$. Prove that the equation

$$e^{i\omega}(z-a_1)(z-a_2)\cdots(z-a_n) = z(1-\bar{a}_1 z)(1-\bar{a}_2 z)\cdots(1-\bar{a}_n z)$$

has at least $n-1$ roots on the unit circle.

Solution by Richard Holzsager, American University, Washington, D.C. If a is a complex number that is not on the unit circle, then the linear fractional transformation $g(z) = (z-a)/(1-\bar{a}z)$ carries the unit circle onto itself, winding once around. If a is inside the circle, the winding direction is positive, while outside it is negative.

Multiplying self-maps of the circle adds winding numbers, so

$$\frac{e^{i\omega}(z-a_1)\cdots(z-a_n)}{z(1-\bar{a}_1 z)\cdots(1-\bar{a}_n z)}$$

winds the circle around itself $n-1$ times (where the -1 comes from the z in the denominator). It must therefore take the value 1 at least $n-1$ times.

A few comments:

1. More generally, this reasoning shows that

$$\begin{aligned} e^{i\omega}(z-a_1)\cdots(z-a_n)(1-\bar{b}_1 z)\cdots(1-\bar{b}_m z) \\ = (1-\bar{a}_1 z)\cdots(1-\bar{a}_n z)(z-b_1)\cdots(z-b_m) \end{aligned}$$

has at least $|m-n|$ roots. The problem is the special case $m = 1$, $b_1 = 0$.

2. Moving any of the a's outside the unit circle reverses the corresponding rotation, and makes that a act like one of the b's.

3. The transformations $g(z)$ defined above commute with the map $z \to 1/\bar{z}$, so roots off the circle occur in inverse-conjugate pairs.

Editorial comment. The generalization in Comment 1 was also given by O. P. Lossers, S. G. Merzlyakov, and W. F. Trench.

C. C. Rousseau and J. Warren said that the variables z and $f(z)$, related by the formula

$$f(z) = e^{i\omega}\frac{(z-a_1)(z-a_2)\cdots(z-a_n)}{(1-\overline{a_1}z)(1-\overline{a_2}z)\cdots(1-\overline{a_n}z)},$$

are analogous to two runners on a circular track: If $f(z)$ makes it around the track n times while z does so just once, then the faster runner has to pass the slower one at least $n-1$ times.

Solved also by J. Angelsio (France), S.-J. Bang (Korea), D. Borwein, R. J. Chapman (U.K.), D. Cruz-Uribe, T. N. Delmer, F. Flanigan, A. Horwitz, O. P. Lossers (The Netherlands), S. G. Merzlyakov (Russia), R. Mortini (Germany), Y. Nievergelt, R. Richberg (Germany), C. C. Rousseau and J. Warren, R. Stong, W. F. Trench, C. Vanden Eynden, Y. Wang, M. Winter (Germany), National Security Agency Problems Group, Western Maryland College Problems Group and the proposer.

Underwood Dudley, Gerald A. Edgar, Michael A. Filaseta, Ira M. Gessel, Richard A. Gibbs, Douglas A. Hensley, John R. Isbell, Mourad E. H. Ismail, Murray Klamkin, Daniel J. Kleitman, Frederick W. Luttmann, Frank B. Miles, Richard Pfiefer, Stephen L. Portnoy, J. O. Shallit, John Henry Steelman, Kenneth B. Stolarsky, Douglas B. Tyler, Daniel Ullman, Edward T. H. Wang, and William E. Watkins.

Answer to Picture Puzzle

(*on page* 665)

Eberhard Hopf, the analyst, and Heinz Hopf, the topologist—no relation.

Call Archimedes from his buried tomb
Upon the plain of vanished Syracuse,
And feelingly the sage shall make report
How insecure, how baseless in itself,
Is the philosophy, whose sway depends
On mere material instruments—how weak
Those arts, and high inventions, if unpropped
By virtue

—*Wordsworth*

REVIEWS

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Mathematics and the Image of Reason. By Mary Tiles, Routledge: London and New York, 1991, xii + 188 pp.

Reviewed by **John P. Burgess**

Few scientifically oriented readers spend much time following the shenanigans of those *hauts couturiers* of the mind, the grand theorists of contemporary culture criticism; though like everyone else who works in a university environment, they will have heard the buzz-words "critical theory," "de-centering," "discourse analysis," "deconstruction," "new historicism," and "post-this," "post-that," and "post-the-other." Much of the prose of the trend-setters in these movements seems designed to repel not merely scientists who might be curious about them, but also anyone else, other than the most dedicated and determined graduate students of comparative literature. Much of it is as full of unexplained if not inexplicable polysyllabic jargon as a memo from a dean of a school of education.

Yet some of these works—or "texts" to use the preferred term—can be enjoyable if one reads them like symbolist poetry, not hoping to understand everything, being carried along more by the sound than by the sense; or like a stream-of-consciousness novel, where one looks not for logical inferences between propositions, but rather for free associations among fleeting ideas. In some of these works/texts the non-stop punning and word-play can be highly amusing, if approached in the right humor.

It is only if one asks for reasoned arguments rather than rhetorical tropes that one may become exasperated, and this to the point of literally gasping for breath: Certainly the atmosphere is very far from that of pure mathematics, where one expects rigorous arguments, and above all rigorous definitions; it is, if anything, even farther from that of applied mathematics, where in order to build mathematical models of empirical phenomena one strives to replace fuzzy notions by sharper ones. Instead, the most fashionable authors—or "writers" to use the preferred term—are fond of taking words that do have precise, technical definitions (such as "undecidability" or "incommensurability") and using them in wildly metaphoric senses.

Yet scientists perhaps ought to take more notice of such authors/writers than they generally do: For among them are purveyors of science-bashing arguments (or trains of thought) that have been immensely influential outside the scientific culture, arguments (or trains of thought) whose influence threatens to become a major obstacle—as if there were not enough such obstacles already—to the spread of scientific literacy and of scientific ways of thought, especially to thought about important public issues. Notably, the influence of such writers and texts is inimical to the spread of scientific skepticism, of the attitude that demands extraordinary evidence for extraordinary claims, to thinking about human affairs (and above all to thinking about grand theories in culture criticism). The skeptical demand for

supporting argument and evidence ends up being denounced as "onto-Euro-theo-phono-phallogocentrism," and blamed for everything bad that has happened in the world in the last half-millenium (at least).

The train of thought that seems to have been the most influential, and about whose influence scientists most need to be forewarned, runs—or drifts—along roughly the following lines (with the reviewer's commentary as logician in parentheses): Either, it is argued, the procedures for evaluating scientific claims and hypotheses can be reduced to definite rules, or they cannot. (So far, so good.) But if it is so reducible, then scientific reasoning is mechanical, and being mechanical is good only for designing machines and other technological gadgets, and not for enlarging our understanding. (This looks like a fallacy of equivocation on "mechanical".) While if it is not a matter of definite rules, then "Science stands unmasked; its authority does not lie in the rationality of its methods but in the politics of power relations." (This looks like a fallacy of false dichotomy.)

In her *Mathematics and the Image of Reason*, Mary Tiles aims to meet the proponents of the foregoing sort of dilemma on their own ground, or rather—since "ground" is too suggestive of solidity—she proposes to wade after them into their own bog. Seizing the second horn of the dilemma, she makes it her project to develop an "image of reason," especially of mathematical reason, as something that is neither reducible to mechanical rules, nor yet a matter of arbitrary power dictatorially imposing a decision when mechanical application of rules leaves a question undecided.

Unfortunately, in pursuing her quarry into the morass that is its native habitat, she seems to have allowed a certain lack of clarity and distinctness to affect and infect some of her own formulations. Time and again the reviewer found himself tearing his hair and asking, "What can she possibly mean by this?" An example may help readers of this review to judge whether this reaction is due to a lack of perspicacity on the part of the reviewer or to a lack of perspicuousness on the part of the author. Early on (pp. 5–6) she briefly sounds a theme to whose development she will return at length later (pp. 170–171), writing:

> In the late twentieth century intuition is becoming mathematically respectable once more as mathematicians use computers to help them develop methods of studying nonlinear functions in ways which have never before been available to them (the theory of chaos).

Here a distinction seems urgently needed, between (a) intuition as a means of justifying the acceptance of mathematical propositions by professional mathematicians, sufficient in itself and making rigorous proof superfluous, and (b) intuition as a means of discovery of mathematical conjectures and of strategies of proof for converting those conjectures into theorems (and as a source of understanding of theorems and their proofs, once these have been discovered, by students and professionals alike). It seems to the reviewer that intuition in sense (b), *heuristic* (and *pedagogical*) appeal to intuition, has always been respectable—indeed, the question in the contexts of discovery and teaching is not whether it is "respectable" to use intuition, but rather whether it would be even the least bit possible to try to make do without it—and has not just become respectable *again* in the last few years. And it seems again to the reviewer that intuition in sense (a), intuition as a substitute for proof, has *still* not become respectable as the twenty-first century approaches—less than anywhere in connection with uses of computers in mathematics, and least of all in connection with nonlinear dynamics—and that it would

be an insult to the many fine mathematicians involved in these developments to suggest that it has. Let me enlarge on both points.

Throughout the period (a couple of decades on either side of the turn of the century) when rigor was being instilled into mathematics, one heard from mathematicians of the period similes comparing rigor either to a court needed to give legal sanction to claims staked by intuition, or else to a hygienic regimen to which intuition must submit if it is to stay healthy and fit. The mathematicians most prominent in rigorizing the work of their predecessors were generally quite eloquent in praise of intuition—in its proper place. No one did more than Hilbert, for example, to rigorize geometry, and no one was more vociferous in insisting on the indispensable role of intuition in that splendid branch of mathematics: The very same Hilbert who wrote the *Foundations of Geometry* also wrote (with Cohn-Vossen) *Geometry and the Imagination*. Nor is there any inconsistency between what he says in the one work and what he says in the other, provided the two roles (a) and (b) of intuition are distinguished.

Computers provide a very considerable extension, whose full scope cannot yet be taken in, of the mathematician's ability to experiment and explore. Far more numerical cases can be checked than ever could with paper and pencil or slate and chalk. Graphics on a screen can be manipulated far more freely than models made of plaster, cardboard, or pipe-cleaners. If one reads, however, what has been written by mathematicians at the forefront of involvement in such developments—say, Robert D. Silverman, "A Perspective on Computational Number Theory," *Notices of the American Mathematical Society*, volume 38, number 6 (July/August 1991), pages 562–568; or David Hoffman, "The Computer-Aided Discovery of New Embedded Minimal Surfaces," *The Mathematical Intelligencer*, volume 9, number 3 (1987), pages 8–21—one finds it clearly stated, and even emphasized, that the end product produced by such researchers consists of theorems as rigorously proved as anyone else's. No significant trend back towards acceptance of pictures in place of logical proofs, or calculations up to 10^9 places in lieu of rigorous deductions has as yet emerged among mathematicians.

In connection with dynamical systems, and specifically with the sensitive dependence on initial conditions of solutions to systems of nonlinear ordinary differential equations, there is a huge body of mathematics from Poincaré to Smale and beyond—see, for instance, the research-expository article of D. S. Ornstein and B. Weiss, "Statistical Properties of Chaotic Systems," *Bulletin of the American Mathematical Society*, volume 24, number 1 (January 1991), pages 11–129—much of it dating from before the computer era, most of it making little or no use of computers, and all of it as rigorous as you please. There also exist a number of computer simulations by meteorologists, zoologist, etc. of particular systems of equations thought to be descriptive of various kinds of natural phenomena, producing apparently "chaotic" results. Such simulations seem to have convinced many not just—what the mathematicians already knew—that chaotic behavior is *in principle* the rule and not the exception, but that it is *in practice* the rule and not the exception in systems arising in the description of nature. Such work by empirical scientists has furnished mathematicians with examples on which to test their techniques, with problems to test their ingenuity—the meteorological example of Lorenz seems to have been particularly tough challenge—just as good work in empirical science has always done. But there has been less than no trend among mathematicians towards accepting something's looking "chaotic" on the tube as any substitute for a rigorous deduction that it is "chaotic" in a rigorously defined sense. (This issue has been thoroughly aired in letters and columns in *The*

Mathematical Intelligencer, volume 11, numbers 1 and 3 (Winter and Summer 1989) —see especially the remarks of Morris Hirsch, "Chaos, Rigor, and Hype," in the latter number, pages 6–8.)

This complaint lodged, let me hasten to add that much of the book fully merits the praise for "accuracy" and "lucidity" it receives in the publisher's blurb on the dust-jacket. The core of the book, so to speak, the middle suite of three chapters on Frege, Russell, and Hilbert, consists of very scholarly work, carefully reconstructing, in the manner of a true historian, the issues of the day as they appeared to those actively engaged in debating them. Though much of what Tiles has to say will be not unfamiliar to specialists (in part from the books she lists as "Further Reading"), there are also many novel insights. To any mathematician or user or teacher of mathematics whose understanding about just what was going on in the debates of the Three Schools of Russell's Logicism, Brouwer's Intuitionism, and Hilbert's Formalism is hazy, Tiles' account in these three core chapters can be recommended as a fine, mathematically-informed and historically-sensitive guide, at least to the perspectives of the first and third schools.

The trouble all comes in the more fluid mantle and unstable crust in which this solid core is wrapped. Assertions that are difficult to interpret—or at least, difficult to interpret as meaning anything *true*—like that about intuition, computers, and chaos quoted above, cluster in the first and last chapters and sections of the book, where Frege meets Nietzsche and Hilbert confronts Derrida. Even here the author deserves praise for her courage in attempting to handle such explosive combinations. But her own positive project, her projection of a new "image of reason," suffers, remaining somewhat obscured by gasses emanating from the swamp of contemporary literary theory.

Her evocation of the old Kantian conception of a rational agent as "an agent who is able to act not only according to a rule but also according to his conception of a rule" and "to reflect on his own rule following" remains undeveloped, or at least, is never brought down to earth by considering concrete cases (such as attempts to escape from the Gödel incompleteness phenomenon by "reflection principles"), and so never has its sense clarified by spelling out just how it is supposed to apply to, and just what it is supposed to amount to in, such concrete cases. Perhaps a spelling out—with the same degree of disciplined thought as one finds in the more purely historical parts of the present book—of the concrete implications of her philosophical position can be expected in a future work. The present work illustrates, by its successes and its failures, how history and philosophy of mathematics benefit when they possess, and suffer when they lack, their own "informal rigor."

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The Crest of the Peacock: Non-European Roots of Mathematics. By George Cheverghese Joseph, London/New York (I. B. Tauris & Co.) [Distributed in North America by St. Martin's Press, New York], 1991, xv + 368 pp.

Reviewed by **Frank J. Swetz**

In the mid-nineteenth century, the British scholar Alexander Wylie, while working as a scientific translator for the Manchu court, chided his Chinese colleagues for the inaccurate value of π then in use. They replied that of course their value for π was merely an approximation and that the Chinese had obtained very accurate values for π at an early date. Wylie dismissed their claims as mere "face-saving" bravado and never traced their factual origins. If he could have, and had, he would have discovered that the traditional mathematician Zu Chongzhi (429–500 A.D.) had obtained a value of π accurate to seven decimal places, an accuracy that would not be achieved in Europe for another thousand years. Wylie lived in a climate where it was easy and convenient to dismiss claims of non-occidental achievements in the field of science and mathematics. European colonial domination of the non-Western world was at its height. In a sense, the domination had been driven by Western science and technology. The "superior" ruled the "inferior" and while there were a few isolated efforts to understand certain aspects of non-Western cultures, on the whole most achievements of these cultures were either ignored or devalued. This was also the era that gave rise to a modern interest in the history of mathematics. Unfortunately the proclivity to ignore non-Western accomplishments also extended into the field of historical scholarship involving mathematics. In 1888 W. W. Rouse Ball published his *A Short Account of the History of Mathematics*. He noted in the Preface that "The history of mathematics begins with that of the Ionian Greeks" and thereby established a theory that would be conveyed and expanded upon in mathematical histories for the next century—mathematics originated in Greece and was developed in the European milieu. Thus the history of mathematics most of us learned was Eurocentric. Quite simply, it is a biased view conceived within the illusions of racial and ethnic superiority and supported by the limitations of available scholarship.

In the last two decades, rapid progress has been made in correcting this situation. An appreciation of alternate world-views, including views on science and mathematics, is growing in the West. One way this appreciation is being expressed is through the rise and study of new disciplines such as ethnomathematics, a subject within which the existence of varied people-centered mathematics is openly acknowledged. Non-Western mathematical accomplishments are being recognized. The most recent revisions of standard texts in the history of mathematics, e.g. Boyer, Burton, and Eves [**1**, **2**, **3**], include increased, albeit still token, coverage of non-Western achievements. A more global view of the development of mathematics is slowly emerging.

George Cheverghese Joseph's *The Crest of the Peacock*: *Non-European Roots of Mathematics* is a welcome stimulus to this reexamination and reconsideration of non-Western mathematical accomplishments. In this work Joseph clearly identifies the existence of a Eurocentric bias in the history of mathematics and goes on to survey the non-Western development of mathematics from ancient times until the seventeenth century. His survey touches on the mathematical accomplishments of pre-Columbian America, Africa and the Arab world and explores in further depth the mathematics of Mesopotamia, ancient Egypt, traditional China and India. The

most extensive discussion focuses on Indian accomplishments, followed next by those of the Chinese.

In his initial chapter, "The History of Mathematics: Alternate Perspectives," Joseph strikes his case for the existence of a Eurocentric bias in the history of mathematics. He grounds his contentions primarily on two factors: colonial imperialism and the lingering belief of cultural/racial superiority. This argument is rather simplistic. Although these factors contributed greatly to the existence of historical bias, they are neither the sole contributing nor supporting factors in this issue. In several instances, for example in China and India, there was a distinct lack of indigenous knowledge on and documentation of traditional mathematics upon which western scholarship could build. Phlogistic and archeological research focused on broad aspects of culture and tended to ignore the mathematical and scientific accomplishments of the societies under investigation. In particular, language barriers were formidable! The detailing of specific Babylonian mathematical accomplishments would have to wait for the pioneering work of François Thureau-Dangin and Otto Neugebauer in the 1930s. A similar appreciation of Egyptian mathematics appeared at about the same time. Besides linguistic barriers, political and social obstacles to research had to be overcome. During the past hundred years, there have been relatively few "windows" open for a foreign researcher to pursue studies on traditional mathematics within China. Thus the neglect of non-Western mathematical accomplishments may have resulted as much from physical and temporal limitations as from psycho-social preconceptions.

The second chapter, "Mathematics from Bones, Strings and Standing Stones," surveys numeration techniques and systems from several ancient and traditional societies. The testimony of artifacts such as tally bones, quipus, the Inca abacus and Mayan glyphs is examined and commented upon. Successive chapters then discuss traditional mathematical accomplishments in several principal non-Western societies: Egypt, Babylonia, China and India. The book closes with a consideration of "Arab Contributions." Chapter contents are organized in a chronological manner and, in each instance, ample cultural and societal perspectives are provided. Charts, diagrams and illustrative mathematical problems enhance the presentations. Joseph's writing style is pleasant and the work is highly readable. Occasional footnotes amplify or explain obscure points—I would have preferred to see more such footnotes. All the accomplishments described were previously known and discussed and documented in a scattered literature. This is the first attempt I know of that has collected such material into a unified and coherent survey of non-Western mathematics and, as such, this work serves as a valuable reference. Despite the lack of new material, *per se*, even a knowledgeable reader will be gratified by personal discoveries and insights provided by the contents of *The Crest of the Peacock*. My own personal discovery was the existence of a Jainist conception of transfinite numbers (p. 251) a millennium before the appearance of Georg Cantor's work (1872).

In any history intended to be comprehensive in scope, omissions of specific events or persons are bound to occur. While the judgments as to what to include in a book are relative and rest with the author, there are several omissions I found disturbing. For example, no mention is made of the discovery or existence of clay tally tokens found in the mideast and their association with early concrete counting. Eventually, such tokens were impressed on wet clay (ca. 3500 B.C.) resulting in the appearance of the first known numerals. These theories have been explored and developed in the work of the French researchers A. Lebrun and F. Vallat [6] and more recently in the writings of the American Denise Schmandt-

Besserat [7]. A mention of such tokens could have appeared in the primary chapter on numeration, or, at least, during the examination of Babylonian accomplishments. I felt that the discussion of Chinese mathematics undertaken in Chapters 6 and 7 was particularly well done, especially the review of the contents of the mathematical classic *Chiu chang suanshu* (ca. 200 B.C.). (Joseph chooses to use the Wade-Giles system for transcribing Chinese names rather than the currently popular Pinyin system.) However, such noteworthy Chinese mathematical achievements as a use of decimal fractions (A.D. 300) or the Mohist attempts to formalize geometry (300 B.C.) are ignored. A similar dissatisfaction also extends to the bibliography. Several standard references in ethnomathematics and non-Western mathematics are ignored. Two such references that easily come to mind are: Georges Ifrah, *From One to Zero: A Universal History of Numbers* [4] and David Lancy's, *Cross-Cultural Studies in Cognition and Mathematics* [5].

Despite these limitations, this is a most worthwhile book and is highly recommended for library acquisition and personal use. It confronts us with the fact that the historical reporting of mathematics is often clouded by bias and offers a partial solution to this situation by documenting and discussing non-Western mathematical accomplishments. Its contents resurrect some issues that are frequently forgotten but which are fundamental in understanding the history of mathematics and the development of mathematical ideas. For example, 'How does a society's world-view affect its perception and use of mathematics?' 'How are mathematical discoveries communicated across cultural boundaries?' and even the most fundamental question of all, 'What is mathematics?' George Joseph's *The Crest of the Peacock* can leave its reader with many questions. Can one ask more of a book?

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Joe Konhauser died on February 28, 1992 of complications following heart surgery. Joe served as editor of this column from 1987 until 1991, and for many years served as editor of the Pi Mu Epsilon Journal. He recently retired from Macalester College where he had taught since 1968. He received his degrees from Pennsylvania State University and taught there as well as at the University of Minnesota before going to Macalester. In 1988 he was awarded a Distinguished Service Award for his work with the North Central Section of the MAA. Joe was a gentle man with an enthusiasm for all things geometric that engaged both his students and his friends. We will miss him.

TELEGRAPHIC REVIEWS

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General, S*, P*, L.** *Chance and Chaos.* David Ruelle. Princeton Univ Pr, 1991, xi + 195 pp, $24.95. [ISBN: 0-691-08574-9] A superb, concise exposition of twentieth-century science woven around the red thread of chance. From turbulence to quanta, from strange attractors to "the true meaning of sex," this brief monograph explains for lay persons the profound shift of natural philosophy from phenomena ruled by determinism to events determined by chaos. LAS

General, P*, L. *Selected Works of A.N. Kolmogorov, Volume I: Mathematics and Mechanics.* Ed: V.M. Tikhomirov. Math. & Its Applic., V. 25. Kluwer Academic, 1991, xix + 551 pp, $199. [ISBN: 90-277-2796-1] "Includes the most important papers by Kolmogorov on mathematics and natural science" (not including probability theory or mathematical statistics). Also contains a short biographical sketch of Kolmogorov, extensive, insightful section of commentaries on his works and a complete list of his works. Everything in English. Note price! BH

General, T*(12-13: 1), S, L.** *Mathematics Meets Technology.* Brian Bolt. Cambridge Univ Pr, 1991, x + 203 pp, $24.95 (P). [ISBN: 0-521-37692-0] A marvelous potpourri of mechanisms (pulleys, gear trains, cams and ratchets, trapezium linkages, rollers and wheels, scissor action, robotics) with associated mathematics (geometry, measurement, motion). Profusely illustrated; numerous exercises (with answers in back); fascinating ideas for practical projects. A superb "hands-on" approach to kinematics and the geometry of space. LAS

General, S*, P*, L.** *The Art of Mathematics.* Jerry P. King. Plenum Pr, 1992, vi + 313 pp, $24.50. [ISBN: 0-306-44129-2] An articulate exploration of the role of aesthetics of mathematics as art revealing to lay audiences not only the aesthetic of mathematical thought and the power of applicable mathematics, but also the sociology and psychology academic mathematicians. Compelling, poetic, engaging: enriched with telling anecdotes and acerbic commentary. LAS

Reference, P, L. *Mathematical Book Review Index: 1800–1940.* Louise S. Grinstein. Garland Pub, 1992, xxxvi + 448 pp, $72. [ISBN: 0-8240-4114-3] An index to published reviews of 3,200 English-language books on mathematics and mathematics education that appeared between 1800 and 1940 (the year *Math Reviews* began). Arranged alphabetically by book, with review references and topic words listed as annotations. Indices provide access via topic words and lists of periodicals that were checked for reviews. LAS

Mathematics Appreciation, S*, P*, L. *From Zero to Infinity: What Makes Numbers Interesting, Fourth Edition.* Constance

Reid. Spectrum. MAA, 1992, xiv + 186 pp, $19 (P). [ISBN: 88385-505-4] Reprint of a 1955 classic, a "small book on numbers" that offers a mixture of mathematics, history, and folklore about each digit $0 \ldots 9$ plus e and $\aleph_0$. Reid's first book; still one of the best on this subject at this elementary level. LAS

Finite Mathematics, T(13), S. *Finite Mathematics for Business and the Social and Life Sciences: A Problem-Solving Approach.* Ruric E. Wheeler. Saunders College, 1991, xviii + 545 pp, $38 net. [ISBN: 0-03-046939-2] Covers standard material for a finite mathematics course—linear equations, matrices, linear programming, and probability and statistics. Nothing exceptional, but a solid text for a business-oriented course. MPR

Education, P. *Educating Mathematical Scientists: Doctoral Study and the Postdoctoral Experience in the United States.* National Research Council. National Academy Press, 1992, xii + 64 pp, (P). [ISBN: 0-309-04690-4] Report of a study intended to determine characteristics of doctoral and postdoctoral programs in mathematics that are successful in meeting national needs for quantity, quality, diversity, and breadth. Based on visits to ten representative campuses. Stresses the importance of a focused and realistic mission (standard or specialized), positive learning environment, and relevant professional development. "Action, if it starts at all, will start from the faculty." LAS

Education, P, L. *Testing in American Schools: Asking the Right Questions.* John H. Gibbons. Office of Technology Assessment (US Government Printing Office, Washington, DC 20510-8025), 1992, ix + 39 pp, (P). A balanced report on the role of testing in schools, focusing especially on new approaches, current controversies, and policy options. Prepared at the request of Congress, it illustrates various uses of tests, documents common misuses, explores new testing technologies, and analyzes pros and cons of national assessment. A good primer for one of today's most important educational policy issues. LAS

Education, P*, L. *A Core Curriculum: Making Mathematics Count for Everyone.* Steven P. Meiring, *et al.* NCTM, 1992, viii + 150 pp, $17 (P). [ISBN: 0-87353-328-3] Three options for curricula—called "crossover," "enrichment," and "differentiated"—that meet the objectives of the 1989 NCTM *Standards* for a three-year high school core curriculum for all students. Also includes a special chapter on "matrices for all" adapted from an innovative Dutch curriculum. Extensive examples illustrate approaches to instruction and assessment that are adaptable to students at different levels. Concludes with a chapter on the process of change—suggestions for how a district can develop and implement a common core curriculum. LAS

Education, L*, P. *Statistical Abstract of Undergraduate Programs in the Mathematical Sciences and Computer Science in the United States: 1990–91 CBMS Survey.* Donald J. Albers, *et al.* MAA Notes No. 23. MAA, 1992, xx + 173 pp, $20 (P). [ISBN: 0-88385-080-X] Sixth in a series of data studies published every five years since 1965. In contrast to prior volumes, this report consists primarily of data and charts with minimal interpretive narrative. Some highlights: 38% of total enrollments are in two-year colleges, where there has been a "staggering increase" in part-time faculty, and where over half the courses are remedial. Advanced (post-calculus) enrollments still constitute only 6% of the total. LAS

Education, P*, L*. *Advanced Mathematical Thinking.* Ed: David Tall. Math. Educ. Lib., V. 11. Kluwer Academic, 1991, xvii + 289 pp, $89. [ISBN: 0-7923-1456-5] Fourteen individually authored chapters carefully edited into a coherent monograph with unified bibliography and index. Surveys contemporary views on mathematical thinking, creativity, and proof from the perspective of cognitive theory, then explores empirical research about learning various parts of college-level mathematics. Seeks to explore and document the immense and surprising cognitive hurdles faced by university mathematics students. An excellent foundation for anyone interested in learning about undergraduate educational research. LAS

History, P, L. *The Apprenticeship of a Mathematician.* André Weil. Transl: Jennifer Gage. Birkhäuser, 1992, 197 pp, $29.50. [ISBN: 0-8176-2650-6] Sketchy but fascinating memoirs of Weil's early life (up until Hiroshima), including a sojourn in India where he met with Gandhi and Nehru; a detour as a prisoner (for draft dodging) in Finland, England, and France; life in England during the Battle of Lon-

don; and hardships of refugee status in the United States. No mathematics, but many sketches of mathematical people. LAS

History, S*, P*, L.** *The Crest of the Peacock: Non-European Roots of Mathematics.* George Gheverghese Joseph. Penguin Books, 1991, xv + 371 pp, $12 (P). [ISBN: 0-14-012529-9] An exploration of the global nature of mathematical creativity, motivated by the example of Ramanujan, emphasizing the influence of culture, the diversity of methods, and, fundamentally, the nature of mathematics. Author Joseph, whose own roots are embedded in four widely scattered world cultures, employs a convincing array of evidence to puncture the comfortable uncritical "fertile soil" myth of the Eurocentric evolution of mathematics. Joseph documents the crucial importance of transmission of diverse mathematics across cultures. Well-written, engaging, and unrelenting in its assault on hazy age-old stories. LAS

Number Theory, P*. *Quaternary Quadratic Forms: Computer Generated Tables.* Gordon L. Nipp. Springer-Verlag, 1991, vii + 155 pp, $59. [ISBN: 0-387-97601-9] Computer-age successor to Brandt-Intrau tables of reduced positive ternary forms. Forms are grouped by discriminant (up through 500), then by genus. Entire table also available on 3.5 inch disc. Should be of interest to researchers interested in classification problems for quadratic forms. MPR

Algebra, P. *Lecture Notes in Mathematics-1437: K-theory and Homological Algebra.* Ed: H. Inassaridze. Springer-Verlag, 1990, 313 pp, $32 (P). [ISBN: 0-387-52836-9] A selection of nine articles on K-theory and homological algebra presented at the 1987–1988 Seminar on Algebra at Razmadze Mathematical Institute of the Georgian Academy of Sciences, Tbilisi. These are the first publications of works from that long-standing seminar. RB

Complex Analysis, T(17-18), S, P, L. *Classical Complex Analysis.* Mario O. González. Pure & Appl. Math., V. 151. Marcel Dekker, 1992, xiv + 767 pp, $150. [ISBN: 0-8247-8415-4] Topics from classical theory of analytic functions usually taught in first course. Also covers non-analytic and generalized analytic functions. Comprehensive and dense; will most likely be considered as a reference from which many courses can be drawn. Many examples and exercises. KS

Complex Analysis, S(17-18), P, L. *Complex Analysis: Selected Topics.* Mario O. González. Pure & Appl. Math., V. 152. Marcel Dekker, 1992, xi + 518 pp, $115. [ISBN: 0-8247-8416-2] Sequel to author's *Classical Complex Analysis* (see TR above). In-depth study of analytic continuation, conformal mappings, entire functions of finite order, meromorphic functions, and an alternative approach to elliptic functions. KS

Complex Analysis, P. *Complex Analysis.* Ed: Klas Diederich. Aspects of Math., V. E17. Friedr Vieweg, 1991, ix + 341 pp, DM 89. [ISBN: 3-528-06413-7] Forty-nine papers on many aspects of complex analysis, mainly in several variables, comprise proceedings of a 1990 International Workshop, Wuppertal, held in honor of H. Grauert. Subjects span Grauert's own wide-ranging areas of interest. PZ

Differential Equations, P. *Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems.* Ser. in Computat. Math., V. 14. E. Hairer, G. Wanner. Springer-Verlag, 1991, xv + 601 pp, $79. [ISBN: 0-387-53775-9] Contains three chapters: Runge–Kutta methods for stiff problems, multi-step methods for stiff problems, and singular perturbation and differential-algebraic equations. MLR

Partial Differential Equations, P*. *Some Applications of Functional Analysis in Mathematical Physics, Third Edition.* S.L. Sobolev. Transl. of Math. Mono., V. 90. AMS, 1991, vii + 286 pp, $161. [ISBN: 0-8218-4549-7] A unified treatment via functional analysis of variational methods (with applications to the Dirichlet problem and polyharmonic equations), and of the Cauchy problem for linear equations. Includes a chapter on the necessary results from functional analysis and, as an appendix, the author's 1936 classic paper "Méthode nouvelle à résoudre le problème de Cauchy pour les équations linéaires hyperboliques normales." Note price. MPR

Numerical Analysis, P. *Computer Arithmetic and Self-Validating Numerical Methods.* Ed: Christian Ullrich. Notes & Reports in Math. in Sci. & Eng., V. 7. Academic Pr, 1990, xi + 302 pp, $39.95. [ISBN: 0-12-708245-X] As computer simulation becomes more common, automatic verification of computed results becomes important so that computational inaccuracies can be distinguished from the effects of the

mathematical model underlying a simulation. This volume presents thirteen invited papers from the first international conference on the topic, Basel, Switzerland, October 1989, together with recommendations for computer manufacturers. RB

Functional Analysis, T(18), S, P. *Rearrangements of Series in Banach Spaces.* V.M. Kadets, M.I. Kadets. Transl. of Math. Mono., V. 86. AMS, 1991, iv + 123 pp, $72. [ISBN: 0-8218-4546-2] A series $\Sigma_{i=1}^{\infty} x_i$ converges unconditionally if it converges for any rearrangement of its terms. Text explores relationship between unconditionally and absolutely convergent series in Banach spaces. Initial chapters treat primarily L_p-space setting. Later chapters cover general Banach spaces. Large number of exercises of varying difficulty. Translated from Russian. Clear, well-motivated presentation. BH

Functional Analysis, P. *Lecture Notes in Mathematics-1466: Additive Subgroups of Topological Vector Spaces.* Wojciech Banaszczyk. Springer-Verlag, 1991, vii + 178 pp, $19 (P). [ISBN: 0-387-53917-4] Several important theorems of commutative harmonic analysis apply more generally than in the traditional setting of locally compact groups. In this monograph the author extends several such results to the more general setting of "nuclear groups." Basic definitions are provided in early chapters. PZ

Geometry, S*, L. *The Fractal Explorer.* Linda Garcia. Dynamic Pr (POB 7534, Santa Cruz, CA 95061), 1991, 108 pp, (P). [ISBN: 0-9628659-0-7] A spritely personal exploration of fractals by a member of the "Designer Fractal" team that created *FractaSketch* (TR, May 1991), *MandelMovie*, and *Chaos* (TR, May 1991). Includes numerous illustrations, virtually no equations, many quotations from the expository literature, and an extensive bibliography of articles and books about fractals. No exercises. LAS

Geometry, S*, L. *Fractals: Endlessly Repeated Geometrical Figures.* Hans Lauwerier. Transl: Sophia Gill-Hoffstädt. Penguin Books, 1991, xiv + 209 pp, £9.99 (P). [ISBN: 0-14-014411-0] British edition of the Princeton paperback *Fractals: Endlessly Repeated Geometrical Figures* (TR, Februry 1992). A careful exposition for beginners with many full color illustrations. Translated from a 1987 Dutch monograph. LAS

Algebraic Topology, S(17-18), P. *Lecture Notes in Mathematics-1443: Equivariant Surgery Theories and Their Periodicity Properties.* Karl Heinz Dovermann, Reinhard Schultz. Springer-Verlag, 1990, vi + 227 pp, $24 (P). [ISBN: 0-387-53042-8] A monograph consisting of an introductory survey of equivariant surgery theory (that incorporates the approach of Lück/Madsen), and an exposition of the authors' periodicity results. RB

Topology, P, L. *Hassler Whitney: Collected Papers, Volumes I–II.* Eds: James Eells, Domingo Toledo. Birkhäuser, 1992, $115 each. *Volume I,* xiv + 590 pp; *Volume II,* xv + 596 pp. [ISBN: 0-8176-3560-2] Virtually all of Whitney's influential papers arranged by subject (graphs and combinatorics; singularities, analytic spaces, manifolds, bundles and characteristic classes, topology, geometric integration theory) and introduced by his own recent retrospective "Moscow 1935: Topology Moves Toward America." Strangely, neither volume contains any hint of Whitney's extensive work in school education. LAS

Topology, P. *Lecture Notes in Mathematics-1440: Topology and Combinatorial Group Theory.* Ed: P. Latiolais. Springer-Verlag, 1990, vi + 207 pp, $22 (P). [ISBN: 0-387-52990-X] Proceedings of the Fall Foliage Topology Seminar held in New Hampshire (1986–1988) where lively interaction took place in the areas of one- and two-dimensional topology, algebraic topology, and combinatorial group theory amidst an informal, rustic atmosphere. Nineteen papers. RB

Control Theory, P. *Lecture Notes in Control and Information Sciences-164: Differential and Algebraic Riccati Equations with Application to Boundary/Point Control Problems: Continuous Theory and Approximation Theory.* I. Lasiecka, R. Triggiani. Springer-Verlag, 1991, xi + 160 pp, $29 (P). [ISBN: 0-387-54339-2] From the Preface: "These notes collect, in a unified framework, an updated and rather comprehensive account of results centered on the theory of optimal control with quadratic cost functionals for abstract (linear) equations in a Hilbert space." AWR

Elementary Statistics, T(13-14: 1), S, L. *Quick Answers to Quantitative Problems: A Pocket Primer.* G. William Page, Carl V. Patton. Academic Pr, 1991, xiii

+ 277 pp, $32.95 (P). [ISBN: 0-12-543570-3] A concise introduction to back-of-the-envelope methods of data analysis for describing, comparing, predicting, validating, and analyzing data required for making decisions. Many examples but no exercises; methods suitable for calculator or spreadsheet use, but also for paper and pencil. Extensive appendices are filled with miscellaneous data often required in business analyses (e.g., international distances, measurement units, city populations, consumer price indices). LAS

Statistical Methods, T(14-18: 1), L. *Exploratory and Multivariate Data Analysis.* Michel Jambu. Stat. Modeling & Decision Sci. Academic Pr, 1991, xiii + 474 pp, $79. [ISBN: 0-12-380090-0] Thoughtful introduction to data-analytic methods. Written for people with limited experience of data analysis, but covers a wide variety of topics from one-, two-, and n-dimensional statistical analysis to factor analysis, principal components analysis, correspondence analysis, classification methods, and computing issues. Appendix contains data sets from a variety of disciplines. Text is nice mix of examples and theory. MK

Statistics. *Aspects of Nonparametric Density Estimation.* A.J. van Es. CWI Tract, V. 77. Centrum voor Wiskunde en Informatica, 1991, 137 pp, Dfl. 39 (P). [ISBN: 90-6196-397-4] Revision of the author's 1988 dissertation. Treats non-smooth densities, bandwidth, and deconvolution. LAS

Programming, T(14-15: 1), S*, P, L.** *The Standard C Library.* P.J. Plauger. Prentice Hall, 1992, xiv + 498 pp, $28 (P). [ISBN: 0-13-131509-9] A users' and implementers' guide to the ANSI and ISO standard library functions for the C programming language, by an experienced master of programming style. Each chapter (per header file) includes background, Standards excerpts, how to use, implement, and test the library functions, and graded exercises. Unspoken subtleties exposed, software design principles emphasized, 9000 lines of exemplary code included. RB

Languages, T(18: 1), P. *Logic of Domains.* Guo-Qiang Zhang. Progress in Theoret. Comp. Sci. Birkhäuser, 1991, 259 pp, $49.50. [ISBN: 0-8176-3570-X] In denotational semantics, one assigns meaning to a program written in a programming language by mapping the elements of that language into a mathematical construct called a domain. This monograph explores mathematical logical aspects of (SFP and stable) domains, with applications to proof systems for reasoning about programs. Uses denotational semantics, mathematical logic, general topology, category theory. RB

Computer Systems, P, L. *Digital Control Systems, Volume 2: Stochastic Control, Multivariable Control, Adaptive Control, Applications, Second Revised Edition.* Rolf Isermann. Springer-Verlag, 1991, xxi + 325 pp, $79. [ISBN: 0-387-50997-6] A complete revision of the *First Edition* split into two volumes. This volume includes thorough discussions of control systems for stochastic disturbances; interconnected, multivariable, and adaptive control systems; digital control with process computers and microcomputers. Aimed at students and engineers in industry looking for an introduction to theory and application of digital control systems. MK

Computer Systems, S, C. *Mathematica Help Stack.* Robert Campbell. Variable Symbols (2161 Shattuck Ave., Suite 202, Berkeley, CA 94704-1313), 1990, iii + 16 pp, $99 (P). *Second Edition* (for *Mathematica Version 2*) of a Macintosh hypercard application that provides an on-line tree-structured reference manual for *Mathematica* (*First Edition,* TR, December 1991). Requires at least 8MB to run since the stack itself is 4.1MB; to use simultaneously with *Mathematica* requires either more than 8MB or the program HyperDA. LAS

Computer Systems, S, P*, L.** *Learning GNU Emacs.* Debra Cameron, Bill Rosenblatt. O'Reilly & Assoc, 1991, xxvii + 411 pp, $24.95 (P). [ISBN: 0-937175-84-6] GNU (for "GNU's Not Unix") Emacs (for Editing Macros), a product of Richard Stallman's Free Software Foundation (FSF), is a powerful, flexible, and very popular "copyleft" UNIX editor (users are authorized to share copies) that includes special features for editing `troff`, TEX, and Scribe documents and C, Fortran, and (especially) Lisp programs. This highly readable guide provides a comprehensive survey of standard (Version 18) Emacs, including special features for Lisp and for text formatters. Appendices provide detail on customization, information on FSF philosophy and licenses, and a quick reference guide. LAS

Computer Systems, P. *A Performance Monitor for Parallel Programs.* Matthew

H. Reilly. Academic Pr, 1990, xv + 178 pp, $32.95. [ISBN: 0-12-586330-6] To measure the performance of parallel programs on a multiprocessor system, the author designed and directed construction of a hardware component for the M31 VAX system for monitoring that system in action. The resulting general-purpose monitoring device is an event collector that recognizes and records each action on the multiple processors, with limited effect on the computation being monitored. This dissertation focuses on design tradeoffs. RB

Computer Systems, S*, L. *TEX by Example: A Beginner's Guide.* Arvind Borde. Academic Pr, 1992, xiv + 169 pp, (P). [ISBN: 0-12-117650-9] An innovative approach to explaining TEX: narrative pages on the right illustrating various typographical features, with the corresponding TEX code on the top of the facing left page, accompanied by explanatory footnotes on the bottom of this page. Forty of these two-page examples are followed by a sixty-page glossary with expansive explanations of both commands and features. An epilog gives TEX code used in production of the book, illustrating yet more advanced features. LAS

Computer Systems, S*. *Mathematica Quick Reference, Version 2.* Nancy Blachman. Variable Symbols (Distr: Addison-Wesley), 1992, 304 pp, $18.95 (P). [ISBN: 0-201-62880-5] *Second Edition* (*First Edition*, TR, December 1991) of a slim, spiral-bound guide to *Mathematica* commands for Version 2. Includes commands in standard distribution packages, sources of electronic information, and other helpful tidbits. A valuable aid for both novice and expert *Mathematica* users. LAS

Theory of Computation, T(17-18: 1), S, P. *Lecture Notes in Computer Science-454: Combinatorics on Traces.* Volker Diekert. Springer-Verlag, 1990, xii + 165 pp, $20 (P). [ISBN: 0-387-53031-2] In theoretical computer science, Mazurkiewicz's trace theory concerns free partially-commutative monoids used for the semantics of nonsequential systems, including distributed computing systems, multiprocessor configurations, and communication networks. The trace approach distinguishes concurrency from nondeterminism (cf. Petri nets) while incorporating well-understood sequential theory (cf. Hoare's CSP). Self-contained, no exercises. RB

Computer Science, P, L. *Advances in Computers, Volume 33.* Ed: Marshall C. Yovits. Academic Pr, 1991, xi + 336 pp, $79.95. [ISBN: 0-12-012133-6] Extended retrospective and prospective articles on computing: a look towards a reusable software-component industry; a review of object-oriented modelling and discrete event simulation; human factors principles for design of dialog with computers; neural networks applied to artificial intelligence; use of computer-assisted visualization in scientific fields. RB

Computer Science, P. *Advances in Computers, Volume 31.* Ed: Marshall C. Yovits. Academic Pr, 1990, x + 405 pp, $69.95. [ISBN: 0-12-012131-X] Five articles: a multi-disciplinary system design and development methodology based on a military model; modelling human perception in speaker-independent automated speech recognition; analyzing reliability, maintainability, and availability of computer systems as a product selection criterion; molecular computers; the nature of information science. RB

Computer Science, S*(13-18), P*, L*. *Computer Security Basics.* Deborah Russell, G.T. Gangemi, Sr. O'Reilly & Assoc, 1991, xx + 441 pp, $29.95 (P). [ISBN: 0-937175-71-4] A handbook on computer security that provides both "the big picture and quite a few helpful details." Introductory and historical overview; definitions of viruses, worms, etc.; secure systems administration; encryption; network security; legislation and standards such as U.S. government "Orange Book." A readable source that could be used profitably by professors to enhance courses throughout the curriculum. RB

Applications (Engineering), P. *Lecture Notes in Control and Information Sciences-155: High-Resolution Methods in Underwater Acoustics.* Eds: M. Bouvet, G. Bienvenu. Springer-Verlag, 1991, v + 249 pp, $35 (P). [ISBN: 0-387-53716-3] Six separately authored chapters in a text on finding "targets" underwater. Slightly more technical than Tom Clancy's *Hunt for Red October.* BC

Applications (Engineering), P. *An Introduction to Direct Access Storage Devices.* Hugh M. Sierra. Academic Pr, 1990, xviii + 260 pp, $44.95. [ISBN: 0-12-642580-9] Everything you ever wanted to know about the technology of magnetic disk drives, which

are known in the IBM community as direct access storage devices (DASDs), by a long-time designer of those devices. Historical perspective combined with mountains of technical information. Assumes knowledge of magnetic recording, servomechanism design, coding. RB

Applications (Fluid Dynamics), P. *Invariant Manifold Theory for Hydrodynamic Transition.* S.S. Sritharan. Pitman Res. Notes in Math. Ser., V. 241. Longman Scientific & Technical (US Distr: Wiley), 1990, 163 pp, $32 (P). [ISBN: 0-582-06781-2] Rigorous treatment provides link between hydrodynamic transition and finite dimensional dynamical systems. Results include spectral theorems and smoothness theorems. SP

Applications (Physical Science), P, L. *Cellular Automata: Theory and Experiment.* Ed: Howard Gutowitz. MIT Pr, 1991, xvii + 483 pp, $37.50 (P). [ISBN: 0-262-57086-6] Thirty-four articles on cellular automata with titles like "cellular automata and multifractals," "criticality in cellular automata," "simulation of HIV-infection in artificial immune systems," "knot invariants and cellular automata," and "cellular automata and discrete neural networks." BC

Applications (Physics), T(18: 1, 2), S, P. *Quantum Signatures of Chaos.* Fritz Haake. Ser. in Synergetics, V. 54. Springer-Verlag, 1991, xv + 242 pp, $59. [ISBN: 0-387-53144-0] Written with unusual clarity and sensitivity for the fine points of mathematics, this text emphasizes random-matrix theory rather than periodic orbit theory. MU

Applications (Physics), T(18: 1, 2), S, P. *Renormalization and Asymptotic Expansions.* V.A. Smirnov. Progress in Physics, V. 14. Birkhäuser, 1991, x + 380 pp, $85. [ISBN: 0-8176-2640-9] Organized into three parts: Part I, Regularized Feynman Amplitudes describes divergences and singularities of Feynman amplitudes; Part II, Removal of Divergences characterizes standard renormalization schemes; and Part III, Asymptotic Expansions provides explicitly finite formulae for coefficient functions of operator and diagrammatic expansions in the limits of large momenta and masses. Each part begins with an introduction and ends with a detailed bibliography. Analytic proofs are usually included. MU

Applications (Physics), S(18), P. *Dynamical Systems and Statistical Mechanics.* Ed: Ya. G. Sinaĭ. Advances in Soviet Math., V. 3. AMS, 1991, viii + 254 pp, $127. [ISBN: 0-8218-4102-5] Collection of papers presented at the Seminar on Statistical Physics held at Moscow State University covering such topics as the renormalization group method in the theory of dynamical systems, the hyperbolic theory of dynamical systems, and the theory of random media. MU

Applications (Physics), P, L. *Twistors in Mathematics and Physics.* Eds: T.N. Bailey, R.J. Baston. London Math. Soc. Lect. Note Ser., V. 156. Cambridge Univ Pr, 1990, 384 pp, $34.50 (P). [ISBN: 0-521-39783-9] The "Twistor Program" is the search for a theory which unites Einstein's General Relativity with quantum physics. Herein are eighteen review articles covering twistors from both the mathematical and physical perspectives. Includes an introductory article by Penrose surveying the history of twistors and its future. MPR

Applications (Physics), T(18: 1-3), S, P, L. *Quantum Physics, Relativity, and Complex Spacetime: Towards a New Synthesis.* Gerald Kaiser. Math. Stud., V. 163. North-Holland (US Distr: Elsevier Science), 1990, xvi + 359 pp, $85.75. [ISBN: 0-444-88465-3] Complex differential geometry facilitates the synthesis of quantum mechanics and relativity. Wave functions and fields can be extended to complex spacetime and these extensions form a relativistic generalization of the coherent-state representation. A lucid text filled with carefully developed mathematics. MU

Applications (Physics), S(18), P. *Symmetries in Science V: Algebraic Systems, Their Representations, Realizations, and Physical Applications.* Eds: Bruno Gruber, L.C. Biedenharn, H.D. Doebner. Plenum Pr, 1991, ix + 613 pp, $135. [ISBN: 0-306-43895-X] The proceedings of a symposium (of the same name) held at the Landesbildungszentrum Schloss Hofen, Vorarlberg, Austria during the summer of 1990. MU

Applications (Physics), S(18), P. *Geometry and Theoretical Physics.* Eds: J. Debrus, A.C. Hirshfeld. Springer-Verlag, 1991, x + 323 pp, $59. [ISBN: 0-387-53570-5] Focuses on the applications of twistor geometry to problems arising from theoretical physics. Divided into three parts: Part I, Geometry (the Klein Correspon-

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Applications (Physics), S(18), P. *Many Particle Hamiltonians: Spectra and Scattering.* Ed: R.A. Minlos. Adv. in Soviet Math., V. 5. AMS, 1991, vi + 194 pp, $75. [ISBN: 0-8218-4104-1] Collection of six papers covering the following topics: the spectral properties of the matrix-valued Friedrichs Model, asymptotic completeness for an infinite number of Fermions, the pointlike interaction of three different particles, Meson states in lattice QCD, and Hamiltonians in solid-state physics as multiparticle discrete Schrödinger operators. MU

Applications (Physics), T(17). *Chaos in Classical and Quantum Mechanics.* Martin C. Gutzwiller. Interdiscip. Appl. Math., V. 1. Springer-Verlag, 1990, xiii + 432 pp, $39.95. [ISBN: 0-387-97173-4] This text, at a first-year graduate level in physics, explores the open question of whether there are chaotic features in quantum mechanics as in classical mechanics. Ideas and examples are offered, appealing to geometric intuition rather than to general concepts, mathematical theorems and algebraic "manipulations." Includes cultural and historical background. RB

Applications (Physics), P, L. *Lattice Gas Methods: Theory, Applications, and Hardware.* Ed: Gary D. Doolen. MIT Pr, 1991, ix + 339 pp, $37.50 (P). [ISBN: 0-262-54063-0] Twenty-seven papers on one of the hot new approaches to statistical mechanics and related fields. Includes an article on the intriguing concept of "programmable matter." BC

Applications, P. *Robotics.* R.W. Brockett, *et al.* Proc. of Symp. in Appl. Math., V. 41. AMS, 1990, x + 196 pp, $51. [ISBN: 0-8218-0163-5] Lecture notes for an AMS short course held in Louisville, Kentucky, January 1990. Techniques from diverse mathematical fields including differential geometry, multivariate polynomials, homotopy theory, and formal languages are applied to robot motion problems described in terms of kinematic chains; mathematical frameworks are presented for constrained motion (e.g., grasping) and for motion planning with uncertainty. RB

Applications, P. *Mathematical and Computer Modelling in Science and Technology.* Ed: Xavier J.R. Avula. Math. & Comp. Modelling, V. 14. Pergamon Pr, 1990, xxi + 1191 pp, (P). Proceedings of the seventh international conference of that title, August 1989. Six plenary lectures and 218 papers: methodology; optimization; neural networks; circuits, networks, and power systems; dynamical systems and control; artificial intelligence and robotics; biomedical systems and biological sciences; fluid mechanics; heat transfer; structures and materials; structural dynamics; industrial problems; etc. RB

Applications, P. *SOLSTICE: An Electronic Journal of Geography and Mathematics.* Ed: Sandra L. Arlinghaus. Institute of Mathematical Geography, (2790 Briarcliff, Ann Arbor, MI 48105), 1990. *V. I, No. 1*, 49 pp; *V. I, No. 2*, 67 pp, (P). [ISBN: 1-877751-44-8]; *V. II, No. 1*, 56 pp, (P). [ISBN: 1-877751-52-9] One of the world's first electronic journals, in TEX, distributed both on paper (for a fee—$15.95 per year) and electronically (for free). Contents are quite eclectic, including reprints, puzzles, mathematical articles, and miscellany. LAS

Applications, S(13-14), L*. *New Applications of Mathematics.* Ed: Christine Bondi. Penguin Books, 1991, x + 289 pp, £12.99 (P). [ISBN: 0-14-012491-8] A lucid account of diverse applications of mathematics in Great Britain. Sponsored by the other IMA, the (British) Institute of Mathematics and Its Applications; intended for A-level students and teachers. Samples: graphs and derivatives in oil wells; vibrations in violin strings and squealing brakes; biological models; parallel computers. Uses mathematical techniques from elementary calculus and linear algebra. LAS

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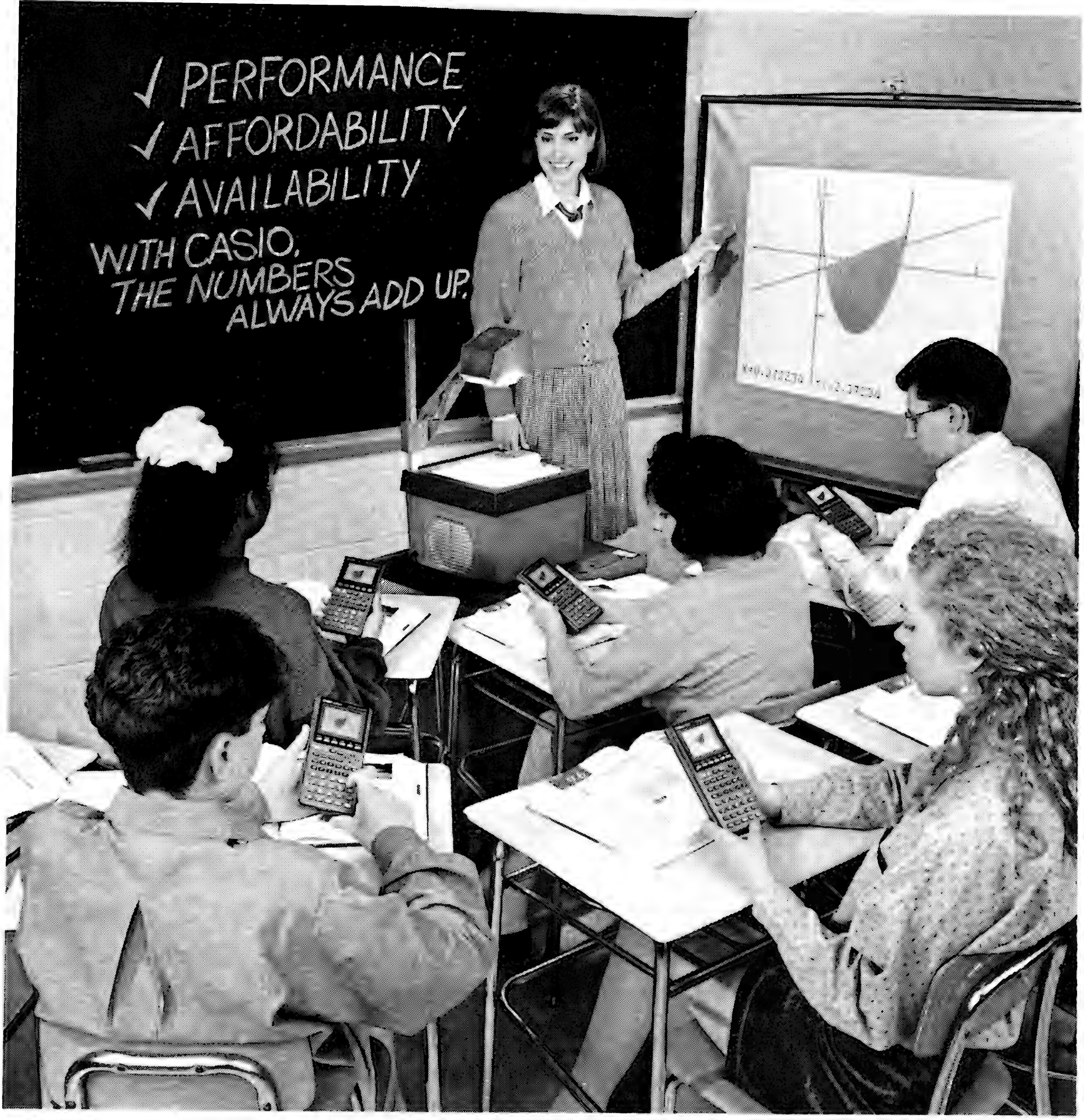

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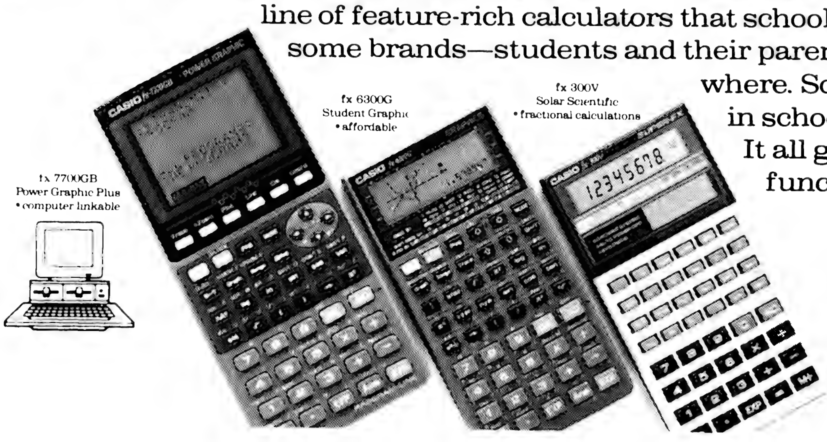

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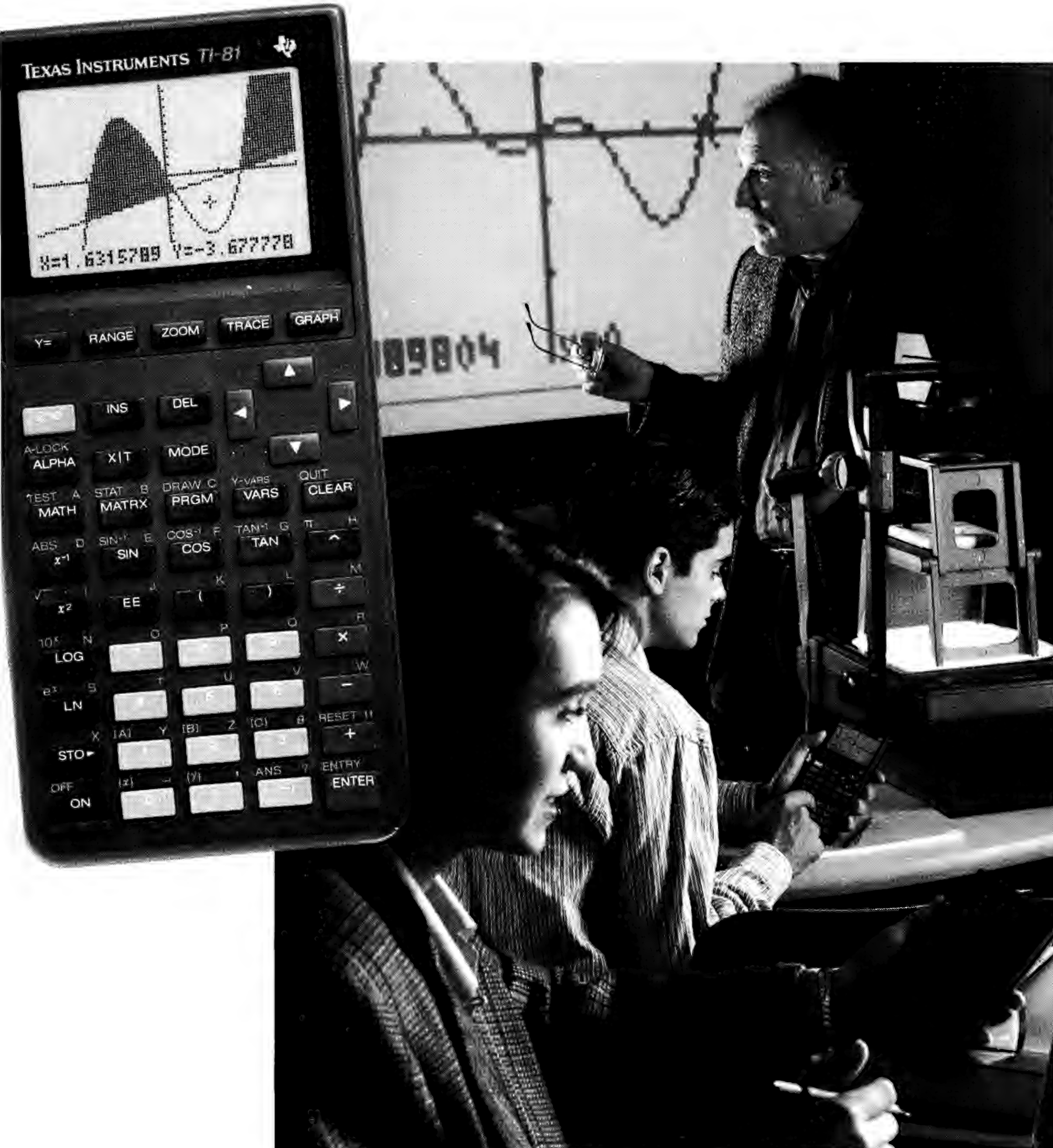
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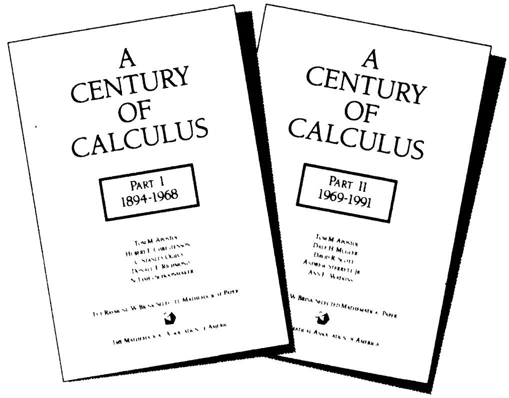

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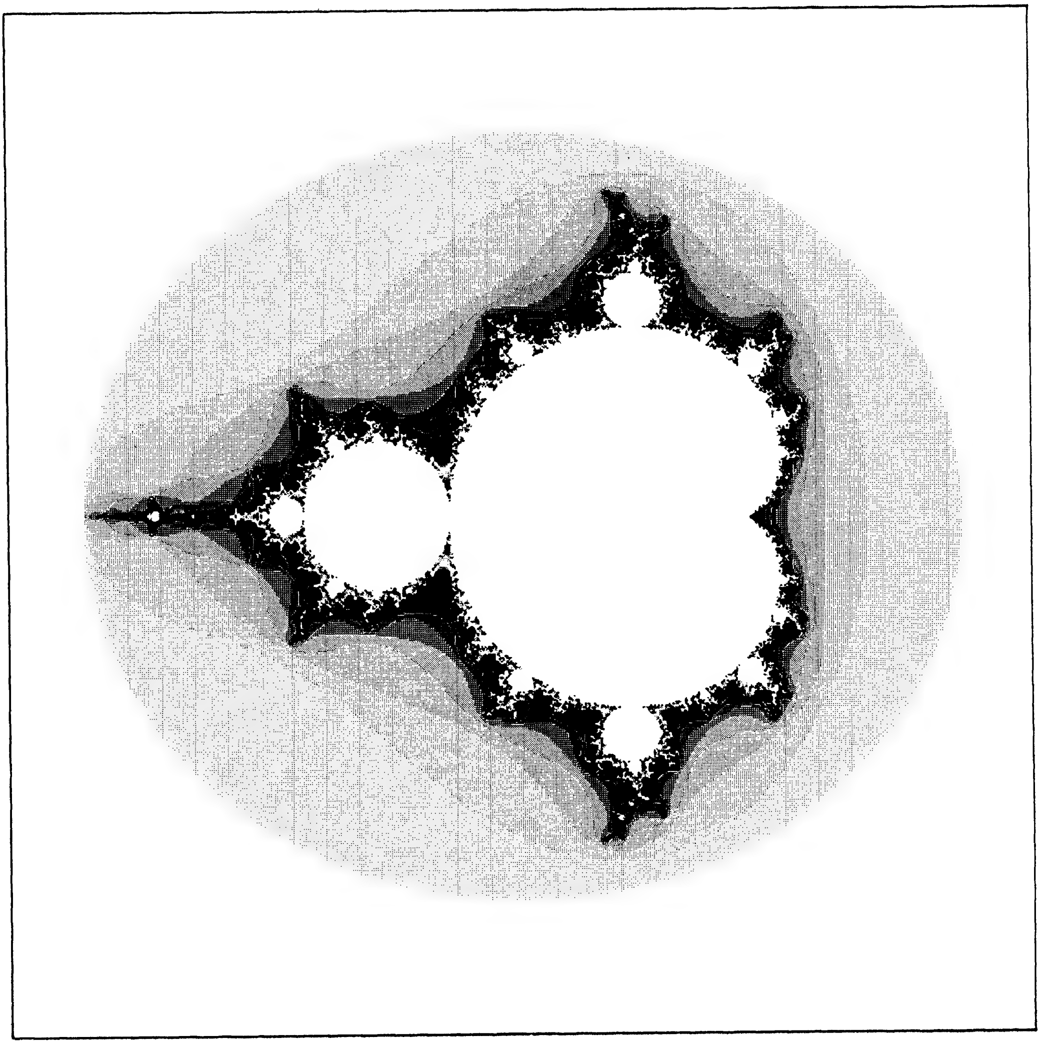

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Cover: The Mandelbrot set. Neither it nor the word fractal is mentioned anywhere in this issue of the *Monthly*, except on the inside front cover.

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COMMENTS

Mathematics libraries in this country are in crisis. Journal costs have steadily risen during the past decade, outpacing increases in available funds year after year. Libraries have exhausted their funds—they cancel subscriptions, order fewer (or no) books, and make desperate plans for a bleak future.

Who's the culprit? Everyone wants to find a simple cause for an institutional crisis. Find it, eliminate it, and the problem goes away. But like all institutional crises, this one has more than one villain. To be sure, *some* publishers of journals have increased prices dramatically and (presumably) profits as well. On the other hand, many journals have more pages, larger editorial costs, fancier production. Someone has to pay, and it's not the publisher. There are more journals too, both for general mathematics and for specialties. And even *more* journals are on their way. Who is to blame? Well, a weaker dollar is partially to blame, and greedy publishers, and European Societies that charge American libraries more than their own, but mainly *we* are the ones responsible—we publish too many papers in too many journals.

The problem of escalating publication costs is not new. In 1931, the American Mathematical Society faced serious financial problems caused largely (but not wholly) by publication costs. A joint committee of the Society and the Association sought a solution, and some members suggested a simple one: publish fewer pages. A tremendous uproar ensued, and one irate mathematician wrote to denounce the "Jehovah complex", which meant "arrogating to oneself the prescience and wisdom which some of us still like to think belong only to Almighty God." Could anyone tell what would be looked at in ten or fifty years? Impossible, nearly everyone agreed, we cannot judge which research is important, and hence we can only ask which research is correct. No recommendation came from the committee (mainly due to the untimely death of its Chairman, John Wesley Young), but the debate had a profound effect on publication policy for the next 50 years.

Today, the problem is worse than ever. All journals make *some* decisions about the importance of papers, of course, but they do so quietly, almost surreptitiously (without guidelines or public debate). Libraries cancel subscriptions year after year, making hard choices based on the needs of the present. The legacy we leave for the future is shameful: Mathematicians will find few libraries capable of fully supporting research.

Should we publish so much? Are there better ways to disseminate research? Can we find a method to finance journal publication so that the mathematical community, present *and* future, is better served? Those are tough questions, but they deserve honest and open discussion. Before we create one more new journal, before we add 100 pages to an old one, before we raise our costs or prices, we ought to step back to a time before 1931 and open the debate once again. Our present policy merely transfers the Jehovah complex from editors to librarians; they now make the decisions about the future importance of mathematics, choosing journals rather than papers.

-John Ewing

The Fifty-Second William Lowell Putnam Mathematical Competition

Leonard F. Klosinski
Gerald L. Alexanderson
Loren C. Larson

The following results of the fifty-second William Lowell Putnam Mathematical Competition, held on December 7, 1991, have been determined in accordance with the governing regulations. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship, left by Mrs. Putnam in memory of her husband, and is held under the auspices of the Mathematical Association of America.

The first prize, $5,000, was awarded to the Department of Mathematics of Harvard University. The members of the winning team were: Jordan S. Ellenberg, Samuel A. Kutin, and Eric K. Wepsic; each was awarded a prize of $250.

The second prize, $2,500, was awarded to the Department of Mathematics of the University of Waterloo. The members of the winning team were Daniel R. L. Brown, Ian A. Goldberg, and Colin M. Springer; each was awarded a prize of $200.

The third prize, $1,500, was awarded to the Department of Mathematics of Harvey Mudd College. The members of the winning team were Timothy P. Kokesh, Jon H. Leonard, and Guy D. Moore; each was awarded a prize of $150.

The fourth prize, $1,000, was awarded to the Department of Mathematics of Stanford University. The members of the winning team were Gregory G. Martin, Garrett R. Vargas, and András Vasy; each was awarded a prize of $100.

The fifth prize, $500, was awarded to the Department of Mathematics of Yale University. The members of the winning team were Zuwei Thomas Feng, Evan M. Gilbert, and Andrew H. Kresch; each was awarded a prize of $50.

The five highest ranking individual contestants, in alphabetical order, were Xi Chen, University of Missouri, Rolla; Joshua B. Fischman, Princeton University; Samuel A. Kutin, Harvard University; Ravi D. Vakil, University of Toronto; and Eric K. Wepsic, Harvard University. Each of these was designated a Putnam Fellow by the Mathematical Association of America and awarded a prize of $500 by the Putnam Prize Fund.

The next five highest ranking individuals, in alphabetical order, were Daniel R. L. Brown, University of Waterloo; Gregory G. Martin, Stanford University; David M. Patrick, Carnegie Mellon University; Jun Teng, California Institute of

Technology; and Jeffrey M. Vanderkam, Duke University. Each was awarded a prize of $250.

The following teams, named in alphabetical order, received honorable mention: the University of British Columbia, with team members Rob M. Deary, Malik M. Kalfane, and Mark A. Van Raamsdonk; the Massachusetts Institute of Technology, with team members Christos Athanasiadis, Henry L. Cohn, and Mikhail Grinberg; Oberlin College, with team members Gary N. Felder, Susan J. Patterson, and Ian B. Robertson; Princeton University, with team members Joshua B. Fischman, Peter R. Kramer, and Gregory D. Landweber; and the University of Toronto, with team members Nima Arkani-Hamed, Jeff T. Higham, and Ravi D. Vakil.

Honorable mention was achieved by the following thirty-four individuals named in alphabetical order: Christos Athanasiadis, Massachusetts Institute of Technology; Radu Bacioiu, Dartmouth College; David S. Bigham, Duke University; Hubert L. Bray, Rice University; Daniel P. Cory, Stanford University; Graham C. Denham, University of Alberta; Jordan S. Ellenberg, Harvard University; Ian A. Goldberg, University of Waterloo; Steven S. Gubser, Princeton University; F. Dean Hildebrandt, Harvard University; Daniel C. Isaksen, University of California, Berkeley; Dmitry A. Ivanov, Georgia Institute of Technology; Timothy P. Kokesh, Harvey Mudd College; Andrew H. Kresch, Yale University; Gregory D. Landweber, Princeton University; Roger W. Lee, Harvard University; Andrew P. Lewis, Harvard University; Jacob R. Lorch, Michigan State University; Samuel J. Maltby, University of Calgary; David K. McKinnon, Harvard University; Peter L. Milley, University of Waterloo; Guy D. Moore, Harvey Mudd College; Demetrio A. Muñoz, Cornell University; Lev Novik, University of Maryland, College Park; Joel E. Rosenberg, Princeton University; Colin M. Springer, University of Waterloo; Andrej Šuch, Queen's University; Dylan P. Thurston, Harvard University; Samuel K. Vandervelde, Swarthmore College; Garrett R. Vargas, Stanford University; Kevin M. Wald, Harvard University; Erick Wong, Simon Fraser University; John H. Woo, Harvard University; and Michael E. Zieve, Harvard University.

The other individuals who achieved ranks among the top 100, in alphabetical order of their schools, were: Boston University, Michael G. Szydlo; University of British Columbia, Rob M. Deary, Mark A. Van Raamsdonk; Brown University, Kenneth W. Bromberg; California Institute of Technology, William M. Watson; University of California, Berkeley, Benjamin J. Davis; University of California, Los Angeles, Christopher B. Baker; Carleton College, Mark J. Logan; Dartmouth College, Paul B. Larson, Dan O. Popa; Duke University, David M. Jones; Harvard University, David B. Carlton, Tal N. Kubo, Lawren M. Smithline; Harvey Mudd College, Jon H. Leonard; Hope College, Alexey G. Stepanov; University of Illinois, Champaign-Urbana, David E. Beckman; Kalamazoo College, Kenneth P. Mulder; Le Tourneau University, Bryan D. Greer; Massachusetts Institute of Technology, Thomas C. Chou, Henry L. Cohn, Michael J. Lawlor, Patrick J. LoPresti, Todd W. Rowland, Jason M. Sachs, David E. Tang; Michigan State University, Thomas P. Hayes; University of Michigan, Ann Arbor, Soundararajan Kannan; New York University, David P. Gamarnik; Northwestern University, Ashvin M. Sangoram; Oberlin College, Ian B. Robertson; University of Pennsylvania, Frosti Petursson; Princeton University, Ze-Yu Chen, Jonathan T. Higa, Adam M. Logan, Mark W. Lucianovic; Rice University, Clark B. Bray; University of Rochester, Daniel B. Finn; Rose Hulman Institute of Technology, Jonathan E. Atkins; Stanford University, James M. Mailhot; Swarthmore College, David A. Packer; University of Texas, Austin, Douglas S. Hauge; University of Toronto, Jeff T. Higham, Colin J.

Rust, Hugh A. Thomas; Trinity College, Hartford, Marshall A. Whittlesey; University of Victoria, Benjamin J. Tilly; Washington State University, Julie B. Kerr; Washington University, St. Louis, Scott P. Nudelman, Jeremy T. Strzynski; University of Waterloo, Paul L. Check, James H. Coleman, Jie J. Lou; Yale University, Zuwei Thomas Feng, Matthew Frank, Zhaohui Zhang.

There were 2325 individual contestants from 383 colleges and universities in Canada and the United States in the competition of December 7, 1991. Teams were entered by 291 institutions.

The Questions Committee for the fifty-second competition consisted of George E. Andrews, George T. Gilbert, and Kenneth A. Stolarsky (Chair); they composed the problems listed below and were most prominent among those suggesting solutions.

PROBLEMS

Problem A-1.

A 2×3 rectangle has vertices at $(0,0)$, $(2,0)$, $(0,3)$, and $(2,3)$. It rotates $90°$ clockwise about the point $(2,0)$. It then rotates $90°$ clockwise about the point $(5,0)$, then $90°$ clockwise about the point $(7,0)$, and finally, $90°$ clockwise about the point $(10,0)$. (The side originally on the x-axis is now back on the x-axis.) Find the area of the region above the x-axis and below the curve traced out by the point whose initial position is $(1,1)$.

Problem A-2.

Let $\mathbf{A}$ and $\mathbf{B}$ be different $n \times n$ matrices with real entries. If $\mathbf{A}^3 = \mathbf{B}^3$ and $\mathbf{A}^2\mathbf{B} = \mathbf{B}^2\mathbf{A}$, can $\mathbf{A}^2 + \mathbf{B}^2$ be invertible?

Problem A-3.

Find all real polynomials $p(x)$ of degree $n \geq 2$ for which there exist real numbers $r_1 < r_2 < \cdots < r_n$ such that

$$\text{(i)} \quad p(r_i) = 0, \qquad i = 1, 2, \ldots, n,$$

and

$$\text{(ii)} \quad p'\left(\frac{r_i + r_{i+1}}{2}\right) = 0, \qquad i = 1, 2, \ldots, n-1,$$

where $p'(x)$ denotes the derivative of $p(x)$.

Problem A-4.

Does there exist an infinite sequence of closed discs $D_1, D_2, D_3, \ldots$ in the plane, with centers $c_1, c_2, c_3, \ldots$, respectively, such that

(i) the c_i have no limit point in the finite plane,
(ii) the sum of the areas of the D_i is finite, and
(iii) every line in the plane intersects at least one of the D_i?

Problem A-5.

Find the maximum value of

$$\int_0^y \sqrt{x^4 + (y - y^2)^2}\, dx$$

for $0 \le y \le 1$.

Problem A-6.

Let $A(n)$ denote the number of sums of positive integers $a_1 + a_2 + \cdots + a_r$ that add up to n with $a_1 > a_2 + a_3$, $a_2 > a_3 + a_4, \ldots, a_{r-2} > a_{r-1} + a_r$, $a_{r-1} > a_r$. Let $B(n)$ denote the number of $b_1 + b_2 + \cdots + b_s$ that add up to n, with

(i) $b_1 \ge b_2 \ge \cdots \ge b_s$,
(ii) each b_i is in the sequence $1, 2, 4, \ldots, g_j, \ldots$ defined by $g_1 = 1$, $g_2 = 2$, and $g_j = g_{j-1} + g_{j-2} + 1$, and
(iii) if $b_1 = g_k$ then every element in $\{1, 2, 4, \ldots, g_k\}$ appears at least once as a b_i.

Prove that $A(n) = B(n)$ for each $n \ge 1$.
(For example, $A(7) = 5$ because the relevant sums are 7, $6 + 1$, $5 + 2$, $4 + 3$, $4 + 2 + 1$, and $B(7) = 5$ because the relevant sums are $4 + 2 + 1$, $2 + 2 + 2 + 1$, $2 + 2 + 1 + 1 + 1$, $2 + 1 + 1 + 1 + 1 + 1$, $1 + 1 + 1 + 1 + 1 + 1 + 1$.)

Problem B-1.

For each integer $n \ge 0$, let $S(n) = n - m^2$, where m is the greatest integer with $m^2 \le n$. Define a sequence $(a_k)_{k=0}^{\infty}$ by $a_0 = A$ and $a_{k+1} = a_k + S(a_k)$ for $k \ge 0$. For what positive integers A is this sequence eventually constant?

Problem B-2.

Suppose f and g are nonconstant, differentiable, real-valued functions on $\mathbf{R}$. Furthermore, suppose that for each pair of real numbers x and y,

$$f(x + y) = f(x)f(y) - g(x)g(y),$$

$$g(x + y) = f(x)g(y) + g(x)f(y).$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

Problem B-3.

Does there exist a real number L such that, if m and n are integers greater than L, then an $m \times n$ rectangle may be expressed as a union of 4×6 and 5×7 rectangles, any two of which intersect at most along their boundaries?

Problem B-4.

Suppose p is an odd prime. Prove that

$$\sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

Problem B-5.

Let p be an odd prime and let $\mathbf{Z}_p$ denote (the field of) the integers modulo p. How many elements are in the set

$$\{x^2 : x \in \mathbf{Z}_p\} \cap \{y^2 + 1 : y \in \mathbf{Z}_p\}?$$

Problem B-6.

Let a and b be positive numbers. Find the largest number c, in terms of a and b, such that

$$a^x b^{1-x} \le a \frac{\sinh ux}{\sinh u} + b \frac{\sinh u(1-x)}{\sinh u}$$

for all u with $0 < |u| \le c$ and for all x, $0 < x < 1$. (Note: $\sinh u = (e^u - e^{-u})/2$.)

SOLUTIONS

In the 12-tuples $(n_{10}, n_9, \ldots, n_0, n_{-1})$ following each problem number below, n_i for $10 \ge i \ge 0$ is the number of students among the top 213 contestants achieving i points for the problem and n_{-1} is the number of those not submitting solution.

A-1 (189, 0, 3, 0, 0, 0, 0, 0, 0, 1, 20, 0)

Solution. The point $(1,1)$ rotates around $(2,0)$ to $(3,1)$, then around $(5,0)$ to $(6,2)$, then around $(7,0)$ to $(9,1)$, then around $(10,0)$ to $(11,1)$. The area of concern consists of four 1×1 right triangles of area $1/2$, four 1×2 right triangles of area 1, two quarter circles of area $(\pi/4)(\sqrt{2})^2 = \pi/2$, and two quarter circles of area $(\pi/4)(\sqrt{5})^2 = 5\pi/4$. Hence the total area is $7\pi/2 + 6$.

A-2 (150, 17, 2, 1, 0, 0, 0, 0, 3, 5, 15, 20)

Solution. No. If so, then $\mathbf{A} - \mathbf{B} = (\mathbf{A}^2 + \mathbf{B}^2)^{-1}(\mathbf{A}^2 + \mathbf{B}^2)(\mathbf{A} - \mathbf{B}) = (\mathbf{A}^2 + \mathbf{B}^2)^{-1}(\mathbf{A}^3 + \mathbf{B}^2\mathbf{A} - \mathbf{A}^2\mathbf{B} - \mathbf{B}^3) = (\mathbf{A}^2 + \mathbf{B}^2)^{-1}\mathbf{0} = \mathbf{0}$, so $\mathbf{A} = \mathbf{B}$, a contradiction.

A-3 (42, 35, 29, 0, 0, 0, 0, 0, 6, 5, 63, 33)

Solution. The set of polynomials is $\{ax^2 + bx + c : a \ne 0,\ b^2 - 4ac > 0\}$.

First, if $p(x)$ is such a polynomial, it must have two distinct real roots, say r_1, r_2, with $r_1 < r_2$. It is easy to check that such polynomials meet the condition. To show nothing else does, write

$$p(x) = a(x - r_1)(x - r_2) \cdots (x - r_n)$$

where $r_1 < r_2 < \cdots < r_n$ and $n \geq 3$. Then

$$p'(x) = a(2x - (r_1 + r_2))q(x) + a(x - r_1)(x - r_2)q'(x),$$

where $q(x) = (x - r_3) \cdots (x - r_n)$. By Rolle's Theorem, all the zeros of $q'(x)$ lie between r_3 and r_n. Hence $(r_1 + r_2)/2$ is not a zero of $q'(x)$, showing that $p(x)$ does not meet the condition.

A-4 (86, 33, 43, 0, 0, 0, 0, 0, 12, 3, 21, 15)

Solution. Let a_i be a decreasing sequence of positive numbers $a_1 \leq 1$, $\Sigma a_i = \infty$, and $\Sigma a_i^2 < \infty$ (for example, $a_i = 1/i$). Let D_i be a disc of radius a_i. Cover $x^2 + y^2 = 1$ by translates (each of which shall intersect $x^2 + y^2 = 1$) of $D_1, D_2, \ldots, D_{m_1}$ with $m_1 < \infty$. This can be done since $\Sigma \operatorname{diam}(D_i) = 2\Sigma a_i = \infty$.

Now cover $x^2 + y^2 = 2$ similarly by translates of $D_{m_1+1}, \ldots, D_{m_2}$ where $m_2 < \infty$ (same justification), ..., $x^2 + y^2 = k$ by $D_{m_{k-1}+1}, \ldots, D_{m_k}$, etc.

Clearly, every line intersects $x^2 + y^2 = k$ for some integer k; moreover, $\Sigma a_i^2 < \infty$ implies $\Sigma \operatorname{area}(D_i) = \pi \Sigma a_i^2$ is finite.

Finally, any disc is inside of a disc $x^2 + y^2 = k_0$, and the discs covering $x^2 + y^2 \leq h$ for $h > k_0 + 4$ cannot intersect $x^2 + y^2 \leq k_0$ (recall (a_i) is decreasing, $a_i \leq 1$). Hence the c_i have no limit point, since no disc may contain infinitely many of them.

A-5 (23, 4, 5, 0, 0, 0, 0, 0, 3, 6, 82, 90)

Solution. For $0 \leq y \leq 1$ let $I(y) = \int_0^y \sqrt{x^4 + (y - y^2)^2}\, dx$. Claim: $I'(y) \geq 0$ with equality only in the (clearly non-optimal) case $y = 0$.

To see this, observe that

$$I'(y) = \sqrt{y^4 + (y - y^2)^2} + \int_0^y \frac{(y - y^2)(1 - 2y)}{\sqrt{x^4 + (y - y^2)^2}}\, dx.$$

If $0 < y \leq 1/2$ clearly $I'(y)$ is positive. So suppose $y > 1/2$. Then $I'(y) > 0$ is equivalent to

$$\sqrt{y^4 + (y - y^2)^2} > (y - y^2)(2y - 1)\int_0^y \frac{dx}{\sqrt{x^4 + (y - y^2)^2}}.$$

Since

$$\int_0^y \frac{dx}{\sqrt{x^4 + (y - y^2)^2}} \leq \int_0^y \frac{dx}{\sqrt{(y - y^2)^2}} = \frac{y}{y - y^2},$$

it suffices to show $\sqrt{y^4 + (y - y^2)^2} \geq (2y - 1)y$, $1/2 \leq y \leq 1$. This is the same as

$$\begin{aligned} y^4 + (y - y^2)^2 &\geq (2y - 1)^2 y^2 \\ &\Leftrightarrow y^2 + (1 - y)^2 \geq (2y - 1)^2 \\ &\Leftrightarrow 2y^2 - 2y + 1 \geq 4y^2 - 4y + 1 \\ &\Leftrightarrow 2y \geq 2y^2, \end{aligned}$$

the last of which is clearly true.

Now, for $y < 1$, $I(y) < I(1) = \int_0^1 x^2\,dx = 1/3$, so $1/3$ is the maximum.

Note: If $y_0 < 1/2$ it is easy to see that $I(y_0) < I(1 - y_0)$ since the integrand is nonnegative and $(y(1-y))^2$ is invariant under $y \to 1 - y$. Hence one may restrict attention to $y \geq 1/2$ from the very beginning.

A-6 (8, 21, 8, 1, 0, 0, 0, 0, 6, 7, 40, 122)

Solution. The sums represented by $A(n)$ may be given an "array" representation using Fibonacci numbers.

Start with a_{r-1} and a_r using two rows of 1's, the lower row with a_r ones and the upper with a_{r-1} ones:

$$
\begin{aligned}
a_{r-1}&: 1\ 1\ 1\ 1\ 1\ 1\ 1\\
a_r&: 1\ 1\ 1\ 1\ 1
\end{aligned}
$$

The top row exceeds the bottom row since $a_{r-1} > a_r$.

Now $a_{r-2} > a_{r-1} + a_r$, hence we can uniquely write

$$
\begin{aligned}
a_{r-2}&: 2\ 2\ 2\ 2\ 2\ 1\ 1\ 1\ 1\ 1\\
a_{r-1}&: 1\ 1\ 1\ 1\ 1\ 1\ 1\\
a_r&: 1\ 1\ 1\ 1\ 1
\end{aligned}
$$

Next, $a_{r-3} > a_{r-2} + a_{r-1}$, so

$$
\begin{aligned}
a_{r-3}&: 3\ 3\ 3\ 3\ 3\ 2\ 2\ 1\ 1\ 1\ 1\\
a_{r-2}&: 2\ 2\ 2\ 2\ 2\ 1\ 1\ 1\ 1\ 1\\
a_{r-1}&: 1\ 1\ 1\ 1\ 1\ 1\ 1\\
a_r&: 1\ 1\ 1\ 1\ 1
\end{aligned}
$$

The total array of the representation will involve columns of the form $F_1 + F_2 + F_3 + \cdots + F_s$ and it is easy to see that this is just g_s. That is, by reading columns we see that we have a one-to-one correspondence between the partitions enumerated by $A(n)$ and those enumerated by $B(n)$.

Hence $A(n) = B(n)$ for all n.

B-1 (192, 6, 2, 0, 6, 0, 0, 0, 0, 5, 0, 2)

Solution. If A is a perfect square, the sequence is eventually constant, since it is identically A. Clearly the sequence diverges to infinity if it never contains a perfect square. So, say a_n is not a perfect square, but $a_{n+1} = (r+1)^2$. If $a_n \geq r^2$ then

$$
\begin{aligned}
a_{n+1} &= a_n + S(a_n),\\
(r+1)^2 &= a_n + \left(a_n - r^2\right),\\
r^2 + (r+1)^2 &= 2a_n,
\end{aligned}
$$

a contradiction because the left side is odd but the right side is even. On the other hand, if $a_n < r^2$ we have

$$(r+1)^2 = a_n + S(a_n) < r^2 + \left(r^2 - 1 - (r-1)^2\right) = r^2 + 2r - 2,$$

again a contradiction. Hence if A is not a perfect square, no a_n is a perfect square.

B-2 (93, 30, 8, 0, 0, 0, 0, 0, 7, 1, 57, 17)

Solution. Differentiate both sides of the two equations with respect to y, obtaining

$$f'(x+y) = f(x)f'(y) - g(x)g'(y),$$
$$g'(x+y) = f(x)g'(y) + g(x)f'(y).$$

Setting $y = 0$ yields

$$f'(x) = -g'(0)g(x) \quad \text{and} \quad g'(x) = g'(0)f(x).$$

Thus

$$2f(x)f'(x) + 2g(x)g'(x) = 0,$$

and therefore

$$(f(x))^2 + (g(x))^2 = C$$

for some constant C. Since f and g are nonconstant, $C \neq 0$. From the identity

$$[f(x+y)]^2 + [g(x+y)]^2 = \left[(f(x))^2 + (g(x))^2\right]\left[(f(y))^2 + (g(y))^2\right],$$

we see that $C = C^2$. Since $C \neq 0$, we have $C = 1$.

B-3 (38, 11, 4, 0, 0, 0, 0, 0, 5, 7, 49, 99)

Solution. Yes.

Claim: If a and b are positive integers, then there exists a number L_0 so that every multiple of (a, b) (the greatest common divisor of a and b) greater than L_0 may be written in the form $ra + sb$, where r and s are nonnegative integers.

Proof of Claim: Suppose first that $(a, b) = 1$. Then $0, a, 2a, \ldots, (b-1)a$ is a complete set of residues modulo b. Thus, for any integer k greater than $(b-1)a - 1$, $k - qb = ja$ for some $q \geq 0$, $j = 0, 1, 2, \ldots, b-1$, hence the claim for this special case.

In general, since $a/(a, b)$ and $b/(a, b)$ are relatively prime, we make use of the above to see that for some L_1, every integer greater than L_1 can be written in the form $ra/(a, b) + sb/(a, b)$. Multiplying through by (a, b) yields the claim.

To answer the question, we begin by forming 20×6 and 20×7 rectangles. From the claim, we may form $20 \times n$ rectangles for n sufficiently large. We may also form 35×5 and 35×7 rectangles, hence $35 \times n$ rectangles for n sufficiently large. We may further form 42×4 and 42×5 rectangles, hence $42 \times n$ rectangles for n sufficiently large.

Since $(20, 35) = 5$, there exists a multiple m_0 of 5, relatively prime to 42 and independent of sufficiently large n, for which we may form an $m_0 \times n$ rectangle. Finally, since $(m_0, 42) = 1$, we may form all $m \times n$ rectangles for m and n sufficiently large.

B-4 (21, 1, 7, 0, 0, 0, 0, 0, 23, 1, 37, 123)

Solution 1. The left side is equal to $\sum_{j=0}^{p}\binom{p}{j}\binom{p+j}{p}$. This is equal to the coefficient of x^p in $((1+x)+1)^p(1+x)^p$. To see this, note that for each j, $\binom{p}{j}$ is the coefficient of $(1+x)^j$ from the first factor, and therefore $\binom{p}{j}\binom{p+j}{p}$ is the coefficient of x^p in $(1+x)^{p+j}$. Summing over j establishes the claim.

On the other hand, the coefficient of x^p in $(2+x)^p(1+x)^p$ is $\sum_{k=0}^{p}\binom{p}{k}\binom{p}{p-k}2^k$. But p divides $\binom{p}{k}$ for $k \neq 0, p$. Thus,

$$\sum_{j=0}^{p}\binom{p}{j}\binom{p+j}{j} = \sum_{j=0}^{p}\binom{p}{j}\binom{p+j}{p} \equiv \binom{p}{0}\binom{p}{p}2^0 + \binom{p}{p}\binom{p}{0}2^p$$
$$\equiv 1 + 2^p \pmod{p^2}.$$

Solution 2. By the Vandermonde convolution,

$$\begin{aligned}\sum_{j=0}^{p}\binom{p}{j}\binom{p+j}{j} &= \sum_{j\geq 0}\binom{p}{j}\sum_{h\geq 0}\binom{j}{h}\binom{p}{p-h}\\ &= \sum_{h\geq 0}\binom{p}{p-h}\frac{p!}{h!(p-h)!}\sum_{j\geq 0}\frac{(p-h)!}{(p-j)!(j-h)!}\\ &= \sum_{h\geq 0}\binom{p}{h}^2 2^{p-h}\\ &\equiv 2^p + 1 \pmod{p^2}\end{aligned}$$

since the prime p divides $\binom{p}{h}$ for $0 < h < p$.

B-5 (38, 4, 3, 0, 3, 0, 1, 0, 9, 3, 50, 102)

Solution. There are $\lfloor (p+3)/4 \rfloor$ elements in the intersection.

Consider first the set of solutions to

$$x^2 = y^2 + 1. \qquad (*)$$

Rewriting this as $(x+y)(x-y) = 1$, we see that for each nonzero element r of $\mathbf{Z}_p$, there is exactly one solution to the above, namely, $x + y = r$, $x - y = r^{-1}$, or

$$x = \left(\frac{p+1}{2}\right)(r + r^{-1}), \qquad y = \left(\frac{p+1}{2}\right)(r - r^{-1}).$$

Thus, there are $p - 1$ solutions to $(*)$.

On the other hand, the element $x^2 = y^2 + 1$ in the intersection also arises from the pairs $(x, -y)$, $(-x, y)$, and $(-x, -y)$ as well as (x, y). These four pairs are distinct unless $x = 0$ or $y = 0$, in which case there are just two distinct pairs. Note that 1 arises from $(1, 0)$ and from $(-1, 0)$. Let $c = 1$ if there is a solution with $x = 0$ and let $c = 0$ if not. Then the intersection has $1 + c + d$ elements, where, from the above, $p - 1 = 2 + 2c + 4d$.

We see that $c = 1$ if and only if $p - 1$ is divisible by 4. Solving for d in each case, we find that $1 + c + d = \lfloor (p+3)/4 \rfloor$.

Note: *Ian Richards, University of Minnesota*, points out that this problem is a special case $(k = 1)$ of the following: If χ is the quadratic character mod p, then

$\sum_{n=0}^{p-1}\chi(n)\chi(n+k) = -1$, independent of k. This follows from the theory of Jacobi or Gauss sums.

B-6 (2, 0, 0, 0, 1, 0, 1, 2, 0, 4, 30, 173)

Solution. The inequality is satisfied if and only if $0 < |u| \leq |\ln(a/b)|$.

The right-hand side is an even function of u; hence it suffices to consider $u > 0$. Replacing x by $1 - x$ and interchanging a and b preserves the inequality, hence we may assume $a \geq b$. Set

$$F(u) = a\frac{\sinh ux}{\sinh u} + b\frac{\sinh u(1-x)}{\sinh u} - a^{x}b^{1-}$$

By differentiating

$$f(u) = \frac{\sinh ux}{\sinh u}$$

we find that $f'(u) < 0$ if and only if $g(u) = x\tanh u - \tanh xu < 0$. This latter inequality holds because $g(0) = 0$ and $g'(u) < 0$ for $u > 0$. Thus $f(u)$ is strictly decreasing in u, and therefore, so is $F(u)$. If $a > b$ then $F(\ln(a/b)) = 0$, whereas if $a = b$ then $\lim_{u \to 0^+} F(u) = 0$, and the proof is complete.

Note: By taking the limit as $u \to 0$, we obtain a proof of the weighted version of the arithmetic-mean–geometric-mean inequality.

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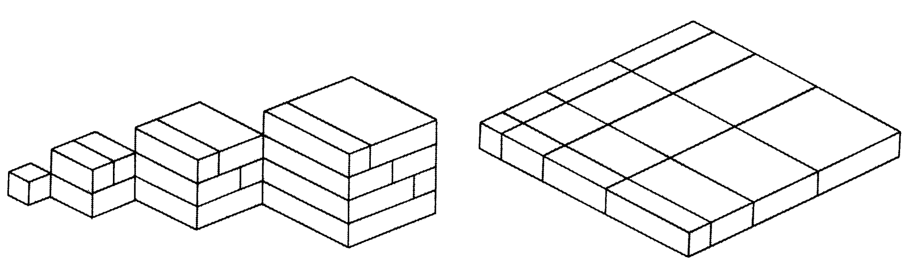

AUTHOR: Michael Hirschhorn

Dedekind's Theorem: $\sqrt{2} \times \sqrt{3} = \sqrt{6}$

David Fowler

1. DEDEKIND'S THEOREM. When the young Richard Dedekind, newly arrived at the Zurich Polytechnik (now the ETH), had to give for the first time the introductory calculus course, it had repercussions that were eventually to spread far beyond his class of students. He tells us in the introduction to his *Stetigkeit und irrationale Zahlen* [**3**] how his search for a satisfactory foundation for the calculus led him, on November 24, 1858, to his construction of the real numbers. (It was a Wednesday.) His immediate objective was to make precise and therefore, he argued, arithmetical the previously vague geometrical appeals to what we now call completeness, and every modern treatment develops lovingly and in detail this crucial property of the real numbers. But in the body of his essay, which is still the most lucid available account of his construction, he points to an equally fundamental achievement. After having described how to define addition, he goes on to say:

> Just as addition is defined, so can the other operations of the so-called elementary arithmetic be defined, viz., the formation of differences, products, quotients, powers, roots, logarithms, and in this way we arrive at real proofs of theorems (as, e.g. $\sqrt{2} \times \sqrt{3} = \sqrt{6}$), which to the best of my knowledge have never been established before (p. 22).

He then elaborates this opinion in letters to Lipschitz of 1876 [**3**], and repeats and emphasises it at the end of the introduction to his later *Was sind und was sollen die Zahlen*? (*What Are Numbers and What Should They Be*? is a better translation of this than the pusillanimous *The Nature and Meaning of Numbers* of [**3**]; and Dedekind throughout used the word *stetigkeit*, continuity, to denote our completeness.)

This contribution is often slighted or even completely overlooked in descriptions of the reals, so my objective here is to celebrate his achievement by illustrating some of the problems that lie in the way of some alternative interpretations of what I shall call Dedekind's theorem, $\sqrt{2} \times \sqrt{3} = \sqrt{6}$, and then discuss briefly the wider historical issue of the evolving idea of the real numbers since antiquity.

The illustrations will be of two sharply contrasted types. The first group will be arithmetical, in the spirit of Dedekind's approach. For simplicity, consider the non-negative numbers. Take a half-infinite line, with left end-point labelled 0 and another distinguished point labelled 1, and somehow describe a labelling system for the points of the line. Abstracting from this, the set of real numbers will then *be* the set of all possible labels, so the labels will determine what we then *conceive* as the points of the line, and the properties of these labels will *determine* the geometrical properties of the line. Throughout, we suppose that we have available the integers and their arithmetic and order, but nothing more; and we want to try to extend this arithmetic and order structure to the set of all labels. Dedekind's

insight was to use cuts in the rational numbers as labels, and to see that the features of arithmetic, order, and completeness are easy to define on these cuts.

Dedekind's cogent objections to referring other than like this to the 'points' of the line, or, in his terms, to 'extensive magnitudes', are that it is 'not scientific' ([**3**], p. 1), since these points or magnitudes are 'nowhere carefully defined' (p. 9; also see pp. 36-8); he calls his labels 'numbers' and describes his procedure as 'arithmetical'. My underlying historical point in this first group of examples is to emphasise that carefully defined arithmetic—addition, multiplication, etc.—for models of the number line was far from obvious before Dedekind, and it would have been difficult to satisfy his requirement that 'I demand that arithmetic should be developed out of itself' (p. 10). Behind these examples is also my dissent from an opinion that is almost universally held but is rarely articulated, and it is another indication of Dedekind's insight and lucidity that he brings it into the open:

> According to my view, the notion of the ratio between two magnitudes of the same kind can be clearly defined only after the introduction of irrational numbers (p. 10 footnote; the translation has 'two numbers', but 'two magnitudes' is clearly meant by the German original).

The first examples (in Sections 2, 3, & 5) show that there is no difficulty in defining the idea of the ratio of, for example, the diagonal and side of a square, and describing the order structure on these kinds of ratios; the problems arise precisely in trying to do, and prove results about, their arithmetic. And the final example (in Section 6) will abandon Dedekind's programme and will formulate his theorem in a geometrical model, Euclidean style, bypassing the need to talk about ratios.

The article [**2**] is highly recommended for details of Dedekind's intellectual and personal life.

2. THE CONTINUED FRACTION REPRESENTATION. We use the so-called Euclidean algorithm (or anthyphairesis) to generate a labelling system. Let X_0 be any point on the half-infinite line, and write x_0 for any line congruent to $0X_0$, x_1 for 01. Now express

$$\begin{aligned} x_0 &= n_0 x_1 + x_2 \quad \text{with} \quad x_2 < x_1, \\ x_1 &= n_1 x_2 + x_3 \quad \text{with} \quad x_3 < x_2, \text{etc.,} \end{aligned}$$

where if at any stage there is no remainder, the process terminates. (In some circumstances, it is more illuminating to conceive of the terminating case as finishing with an additional infinite term.) Thus far, this represents a process of decomposing $0X_0$ and 01 into subintervals, and we can use the n_i's to label the point X_0; let us write $X_0 = [n_0, n_1, n_2, \dots]$. Purely geometrical arguments show us that if $\sqrt{n}$ denotes the side of the square equal to n times the square on 01, then

$$\sqrt{2} = [1, \bar{2}], \qquad \sqrt{3} = \left[1, \overline{1, 2}\right], \qquad \text{and} \qquad \sqrt{6} = \left[2, \overline{2, 4}\right],$$

where the bar denotes an indefinitely repeating period. (Details of three different kinds of such proofs are given in [**6**], Chapter 3.)

We now reverse this way of looking at things, and define the set of all real numbers to be the set of all such terminating or non-terminating sequences of integers $[n_0, n_1, n_2, \dots]$, with $n_i \in \mathbb{Z}$, $n_i \geqslant 1$ if $i \geqslant 1$ and, if the sequence terminates with an n_K for which $K \geqslant 1$, then $n_K \geqslant 2$. Relaxing the description of the

process to

$$x_0 = n_0 x_1 + x_2 \qquad \text{with} \qquad x_2 \leqslant x_1, \text{etc.}$$

eliminates this last condition on terminating expansions but introduces an innocent ambiguity,

$$[n_0, n_1, n_2, \dots n_K] = [n_0, n_1, n_2, \dots n_K - 1, 1]$$
$$\text{where } K \geqslant 0, \text{ and if } K \geqslant 1 \text{ then } n_K \geqslant 2.$$

The order structure is easily described: lexicographic in the even-indexed terms, and reverse lexicographic in the odd-indexed terms, when terminating expansions have been put in standard notion ($n_K \geqslant 2$) with an infinite term adjoined. The final ingredient for the statement of Dedekind's theorem is a description of multiplication, but here a classic account of continued fractions describes how the situation is generally perceived:

> For continued fractions there are no practically applicable rules for arithmetical operations; even the problem of finding the continued fraction for a sum from the continued fractions representing the addends is exceedingly complicated, and unworkable in computational practice ([**8**], p. 20).

In fact there is a simple algorithm for arithmetic, an elaboration of the procedure for evaluating the convergents, discovered in the 1970's by R. W. Gosper but never published conventionally by him! It is described in [**6**], pp. 114–6 & 354–60, where it is illustrated for the evaluation of $[1, 2, 2, 2, 2, 2, 2, 2, 2, \dots] \times [1, 2, 1, 2, 1, 2, 1, 2, 1, \dots] = [2, 2, 4, 2, 4, \dots]$. But I cannot see how to go on to construct a direct proof of Dedekind's theorem using it.

I have argued (see [**6**]) that anthyphairesis may have been one of the early Greek ways of defining ratio, but anybody who knows anything about continued fractions would suspect that this approach to the real numbers and Dedekind's theorem might be unfruitful. Let us now try a much more promising and more familiar approach.

3. THE DECIMAL REPRESENTATION. Again, write x_0 for any line congruent to $0X_0$, x_1 for 01, and

$$\begin{aligned} x_0 &= n_0 x_1 + x_2 \quad \text{with } x_2 < x_1 \\ 10x_2 &= n_1 x_1 + x_3 \quad \text{with } x_3 < x_1 \\ 10x_3 &= n_2 x_1 + x_4 \quad \text{with } x_4 < x_1 \text{ etc.} \end{aligned}$$

Clearly $0 \leqslant n_i \leqslant 9$ if $i \geqslant 1$; and we have constructed a decimal expansion of x, traditionally written $x = n_0 \cdot n_1 n_2 n_3 \cdots$.

There is again no difficulty in describing the order structure, and so no difficulty in using decimal expansions to describe the ordered set of real numbers. But once again we have problems with arithmetic. Under pressure from calculations like $4/9 + 5/9$ and $3 \times 1/3$, expressed decimally, we are led to consider decimal expansions ending in strings of nines and allow identities like $0 \cdot 999 \dots = 1 \cdot 000 \dots$. (These do not arise from the algorithm as described above, but can occur if we modify it to

$$x_0 = n_0 x_1 + x_2 \qquad \text{with} \qquad x_2 \leqslant x_1, \text{etc.,}$$

and we now must decide whether to allow $n_i = 10$ or not.) This opens Pandora's box, letting out the confusion of non-unique representations and indefinitely long

carry, but it is difficult to conceive of any kind of decimal manipulation without these complicating features in some form or another.

Many mathematicians have a touching and naive belief that arithmetical operations on decimals pose no problems; or they pretend to believe this, as in some circumstances the most scrupulously honest among us may sometimes pretend to believe in Father Christmas (see, e.g. [**1**], pp. 26 & 47); or perhaps they have never considered the question to be problematic. Of course arithmetic with terminating decimal expansions is straightforward, since it is only arithmetic in $\mathbb{Z}$, represented decimally and slightly modified to accommodate the notation of the decimal point. But an example first shown to me by Christopher Zeeman shows how we again encounter problems long before we reach Dedekind's theorem: Let those who believe that an algorithm for decimal multiplication exists use it to evaluate the first non-zero digit of the expansion of the product of non-terminating periodic numbers $1 \cdot 222\ldots \times 0 \cdot 818181\ldots$. Is the answer 9, or 1, or 9 or 1? (For more discussion and another surprising example of Zeeman's, see [**5**].) The analogous problem with Gosper's algorithm for continued fraction arithmetic also concerns terminating expansions: in evaluating expressions like $\sqrt{2} \times \sqrt{2}$, the output from the algorithm will be of the form $[2, n_1]$ or $[1, 1, n_2]$, where n_1 and n_2 increase indefinitely as the algorithm struggles to evaluate them and move on to the next term. In fact, n_1 and n_2 here are infinite.

There is some evidence that mathematicians of the early nineteenth century and before, and especially those who were moving towards the developing ε–δ arithmetised analysis, were aware of the fundamental and messy problems with decimal arithmetic. For example, Cauchy's celebrated *Cour d'Analyse* has a long appended Note 1 ('Sur la theories des quantités positives et negatives') in which he defines arithmetical operations on 'numbers' (also called 'quantities') in rather vague terms of manipulations of rational approximations; whilst in its Note 3 ('Sur la résolution numerique des équations'), he describes a proof of the intermediate value theorem in terms of what is, in effect, a decimal algorithm, though expressed there to any base. But, vague though his account often is, Cauchy does not fudge the issue by describing arithmetic in terms of terminating decimal expansions, and then pretend that he has described arithmetic in general.

Many Babylonian clay tablets containing arithmetical tables and problems of some sophistication expressed throughout to base 60, and which date from around 2000 B.C. onwards, have been found and edited. Division appears to have been handled by reciprocation followed by multiplication; but most of the reciprocal tables only contain entries for those numbers whose reciprocals terminate (i.e., numbers whose only factors are 2 and 5). So Babylonian mathematicians also seem to have had a proper caution about the problem of arithmetic with non-terminating radix fractions.

4. THE UNIT FRACTION REPRESENTATION. Egyptian mathematics tends to be viewed with amazement by mathematicians of today because of its practice of expressing rational numbers as sums of different unit fractions:

$$\frac{p}{q} = n_0 + \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_K}, \quad \text{with} \quad 1 < n_1 < n_2 < \cdots < n_K.$$

I do not think that I need to belabour the opinion that this makes a very unpromising base on which to attempt to state, let alone prove, Dedekind's theorem. What does need belabouring is that this same practice is found through-

out Greek texts; see [**6**] Chapter 7 for details, and for a discussion of the evidence that leads me to argue that we have no good grounds for arguing that early Greek mathematics and commercial practice had anything corresponding to our common fractions p/q and their arithmetic! Unit fraction expressions are then found in Greek, Arabic, and Italian texts up to the sixteenth century; for example, astronomical texts continue to use the Egypto-Greek unit fractions side-by-side with the Babylonian sexagesimal numbers.

We can, incidentally, generate a class of unit fraction expansions using another variant of the subtraction algorithm. Write

$$\begin{aligned} x_0 &= n_0 x_1 + x_2 \quad \text{with} \quad x_2 < x_1, \\ x_1 &= n_1 x_2 + x_3 \quad \text{with} \quad x_3 < x_2, \\ x_1 &= n_2 x_3 + x_4 \quad \text{with} \quad x_4 < x_3, \text{ etc.} \end{aligned}$$

We then get

$$\frac{x_0}{x_1} = n_0 + \frac{1}{n_1} - \frac{1}{n_1 n_2} + \frac{1}{n_1 n_2 n_3} - \cdots .$$

Or, by overshooting from the second step onwards,

$$x_1 = n_1 x_2 - x_3 \quad \text{with} \quad x_3 < x_2, \text{ etc.,}$$

we can eliminate all the negative signs. While there is absolutely no evidence that anyone in antiquity used any such algorithm, similar kinds of expressions do appear in Arabic mathematics from the 12th century onwards. They were described by Fibonacci in his *Liber abaci* of 1202, and then persisted up to the 16th century, known as the *practica Italiano*. They correspond to ascending continued fractions, and sometimes have more general numerators:

$$n_0 + \frac{m_1}{n_1} + \frac{m_2}{n_1 n_2} + \frac{m_3}{n_1 n_2 n_3} + \cdots = n_0 + \cfrac{m_1 + \cfrac{m_2 + \cfrac{m_3 + \cdots}{n_3}}{n_2}}{n_1},$$

though occasionally there is an additional complication of numerators that build up in a similar way to the denominators, and so do not correspond as closely to continued fractions. In their notation,

$$\frac{m_3 m_2 m_1}{n_3 n_2 n_1} \ (\text{a variant of } \textit{practica Italiano}) = \frac{m_1}{n_1} + \frac{m_1 m_2}{n_1 n_2} + \frac{m_1 m_2 m_3}{n_1 n_2 n_3},$$

and all such expressions are usually written backwards like this, presumably a vestige of their Arabic origins.

Lagrange, in [**10**], gave a unified treatment of the three subtraction algorithms I have described so far, and others, expressed in terms of approximations. He noted the connexion of the first with continued fractions and ended with an oblique reference to ascending continued fractions, which he attributed to Lambert. Ascending continued fractions appear sporadically thereafter, but they are much more simply handled in terms of series.

5. THE EUDOXAN REPRESENTATION. I start with the celebrated *Elements* V Definition 5, attributed to Eudoxus:

> [Four] magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any whatever of the second and fourth, the former

equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

and a passage from Heath's note on this that has spawned or reinforced endless confusion:

> Max Simon remarks (*Euclid und die sechs planimetrischen Bücher*, p. 110), after Zeuthen, that Euclid's definition of equal ratios is word for word the same as Weierstrass' definition of equal numbers. So far from agreeing in the usual view that the Greeks saw in the irrational no *number*, Simon thinks it is clear from Book V that they possessed a notion of number in all its generality as clearly defined as, nay almost identical with, Weierstrass' conception of it. Certain it is that there is an exact correspondence, almost coincidence, between Euclid's definition of equal ratios and the modern theory of irrationals due to Dedekind ([**4**], vol. ii, p. 124).

The first two sentences of this second quotation represent what is, for me, an ugly disease of much scholarship: the repetition of incorrect, misleading, or meaningless stereotyped verbal formulae backed up by a liturgical parade of names. As far as I am aware, Weierstrass' description of number looks nothing like *Elements* V Def. 5; and, far from finding 'numbers in all generality' in the *Elements*, the consensus of serious investigations of the *Elements* uncovers little or no trace there of photo-real numbers; indeed, almost the only numbers found there are the positive integers, of which the unit has an ambiguous status. As remarked in the previous section, I go even further and argue, in [**6**], Chapter 7, that we find nowhere in early Greek mathematics (i.e. up to and including Archimedes) any convincing evidence for an understanding of the rational numbers, such as we derive from manipulations of common fractions p/q. So, as concerns the final sentence, while we can now easily translate V Def. 5 into Dedekind's definition of a cut, the mathematical contexts of the two definitions are even more strikingly different than the correspondence between their formulations.

Elements V is about proportionality, the equivalence relation of equality between ratios. If, as here, we are only given this equivalence relation, we can now play the formal trick of taking the equivalence classes it defines, and refer to them as ratios; but this is a late nineteenth century device, at the earliest. However, behind Book V of the *Elements*, especially when set in the context of Eudoxus' interest in cyclical calendars (for more details of this and what follows below, see the discussion in [**6**], pp. 121–30), we may be able to detect another procedure that we could try to use to describe the set of reals. This will involve leaving the subtraction algorithms of the previous examples and passing to addition.

Each point x of the positive line will generate a characteristic pattern in the way the points $x, 2x, 3x, \ldots$ interlace with the integer points $1, 2, 3, \ldots$. We can describe this, for example by specifying how many points of $\{x, 2x, 3x, \ldots\}$ lie in $[0, 1)$, how many in $[1, 2)$, etc. The labels this generates for the rational points of the line were described, in the 1870s, by Christoffel and H. J. S. Smith, and their descriptions can be extended to generate the labels for $\sqrt{n}$.

Abstracting from this, we may describe the set of reals to be all possible patterns that arise in this way. First, we need to characterise these patterns; then, define the order structure on them; finally, define arithmetic with them. The first problem is solved by an ingenious algorithm of Zeeman closely related to the idea

of rotation numbers; see [**11**] and [**12**]. The order structure is described in *Elements* V Definition 7. But I have no idea if any direct definition of their arithmetic is known or accessible. So, once again, and at yet another place, the attempt to formulate and prove Dedekind's theorem is unsuccessful.

6. A GEOMETRICAL DESCRIPTION. I finish this cycle of formulations of Dedekind's theorem with a purely geometrical version, based on a naive model of Euclidean geometry in which figures are manipulated by congruence transformations and equality is interpreted in terms of scissors-and-paste operations. This goes against Dedekind's approach, but it corresponds to what is found in much, though not all, of Euclid's *Elements*. For example, the so-called Pythagoras' theorem is a statement about decomposing and reassembling squares, and FIGURE 1 gives a proof which, though not found in the *Elements*, may have been excised from between Propositions 8 & 9 of Book II.

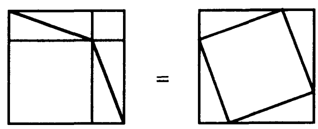

Figure 1. Pythagoras' theorem

We again make a geometrical definition of $\sqrt{n}$ as the side of a square equal to n concatinated copies of the unit square; this can be constructed, for example, by making repeated use of Pythagoras' theorem as in FIGURE 2. We now define multiplication geometrically: if n & m denote natural numbers, a & b, lines, and A & B, regions in the plane, then na or nA will denote n concatinated copies of a or A; $a.b$ will denote the rectangle with adjacent sides a and b; $a.B$ will denote the rectangular prism with base B and height a; and $A.B$ is not defined. Again this corresponds to what we find in the *Elements*. With these definitions, our original version of Dedekind's theorem is not well-posed—a rectangle cannot be compared with a line—but we can immediately adjust its formulation to $\sqrt{2}.\sqrt{3} = \sqrt{6}.1$. A proof will now consist of an argument about a figure involving these two rectangles. Many different figures are possible; FIGURE 2 illustrates one straightforward construction, using nothing more than has been described above; and the equality of the rectangles $\sqrt{2}.\sqrt{3}$ and $\sqrt{6}.1$ therein will be equivalent to the collinearity of 0, P, and Q. (This is the easy converse of *Elements* I 43, which shows that, in FIGURE 3, the shaded rectangles are equal.) Curiously, a proof now

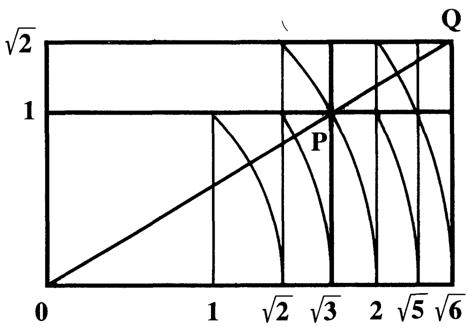

Figure 2. Geometrical formulation of Dedekind's theorem

requires arguments based on similarity—that the construction of P, starting from the line 01 is the same as the construction of Q, starting from $0\sqrt{2}$, and hence the triangles $0P\sqrt{3}$ and $0Q\sqrt{6}$ are similar—and so seems to depend on the Euclidean nature of the geometry. I remark, in passing, that Euclid's version of the parallel postulate is not actually expressed in terms of parallels, but is closer to expressing the possibility of constructing similar triangles of any given size.

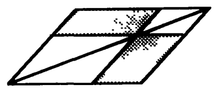

Figure 3. Elements I43

I know of no explicit reference to any formulation of Dedekind's theorem in early Greek mathematics; whether or not it is there implicitly is a delicate historical matter.

7. MATHEMATICS AND THE REAL NUMBERS. Mathematical thinking today is built on our intuitions of the real numbers, but I have tried here to illustrate how it may sometimes distort the past when we interpret their mathematics in these terms. Here are some related opinions about other related past mathematical developments:

(a) Early Greek mathematics, up to the time of Archimedes, does not seem to be arithmetised. However it is interpreted arithmetically thereafter, for example in the metrical geometry of Heron and the astronomy of Ptolemy, and many modern descriptions are now set against some assumed background of a developing idea of rational and real numbers, For example, the lurid stories of the discovery and effect of incommensurability, which is so damaging to a naive arithmetical mathematics based on the rationals, are found in later commentators but are surprisingly absent from our earlier evidence; they may, I think, be part of a later overlay. (There is a discussion of the historical evidence concerning the discovery of incommensurability in [**6**], 294–308.)

(b) Babylonian mathematics and astronomy is highly arithmetised, but it does not have the deductive structure we now associate with Greek and our mathematics. One possibly fruitful line of interpretation, which I have not seen explored anywhere, might be to develop the distinction proposed, by Knuth [**9**], between mathematical and algorithmic thinking, and see if it applies to this Babylonian material, especially the later Babylonian astronomy of the Seleucid period. In the grandest sweep of history, is it possible that the paradigm of deductive mathematics, which has dominated our view since the fourth century B.C., may have run its course, and may now be giving way to an older algorithmic style, which is now flourishing in a changed environment of automatic computation, the appeal of experimental mathematics, economic and political pressures, shifts in the school curriculum, and the increasing specialization and inaccesibility of much of mathematics today?

(c) Western mathematics since the 17th century has owed a lot of its power to the way it has been successfully and comprehensively arithmetical, though it managed to ignore the basic problems with a precise description of its underlying arithmetic until the 19th century. (This simple picture must be filled in with details of the excursions into infinitesimal and infinite numbers and non-standard analysis, which fit well with the approach to arithmetical models of the line I described in Section 1, and the mathematico-algorithmic hybrid of constructive mathematics.) I believe

that the dramatic and explosive growth of symbolism in the 17th century—for there is no symbolism before about 1600, apart from numerals and a few things that are better described as abbreviations—may be connected with a new fluency in arithmetised thinking, which itself may owe a lot to the popularisation of decimal fractions at the end of the 15th century. Stevin, for example, was a thorough-going arithmetiser: he published, in 1585, the first popularisation of decimal fractions in the West (both in Dutch, *De Thiende*, and French, *La Disme*); in 1594, he described an algorithm for finding the decimal expansion of the root of any polynomial, the same algorithm we find later in Cauchy's proof of the intermediate value theorem to which I referred above; and he argued vigorously for an arithmetical understanding of the *Elements*, including its notorious Book X. But this is another story, part of which is described in **[7]**.

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A Modified Babylonian Algorithm

Ronald J. Knill

One may infer (Mainzer, p. 44) from the nature of their approximations that the Babylonians had discovered the following algorithm for a sequence $x_1, x_2, \ldots$ of successively improving approximations to $\sqrt{x}$. The sequence is defined inductively as $x_1 = x$ and $x_k = (x_{k-1} + x/x_{k-1})/2$ for $k \geq 2$. In the case of $\sqrt{2}$, their best approximation was the fourth term of this sequence rounded to two sexagesimal places. It was accurate to within 10^{-6}. The choice of x_1 is flexible, but for the purposes of our exposition $x_1 = x$ is best. For the Babylonians it was important that x_k was rational if x was, and so could easily be approximated in decimal notation. Modern calculus texts efficiently treat this algorithm as a special case of the Newton-Raphson method, but one could object to this treatment on the grounds that it obscures the fact that the convergence of x_k to $\sqrt{x}$ is an easier consequence of algebra than of calculus, and it does not justify the fact that the convergence is quadratic (one usually states this to the students as "the number of significant digits is better than doubled by proceeding from x_k to x_{k+1}," but seldom is it proven to freshmen). Finally it does not address the issue of the non-uniformity of convergence of the sequence $x_1, x_2, \ldots$ to $\sqrt{x}$. That fact justifies the normal strategy of conditioning x by multiplying or dividing it by a suitably chosen perfect integer square to obtain a number larger than 1 and less than or equal to (say) 10, and then applying the algorithm to the result. We propose the following modification of the Babylonian algorithm for finding square roots as a good means of illucidating the above points.

Definition. *Let x be a real number larger than* 1. *Let z_k and a_k, $k \geq 1$, be defined inductively as follows*:

$$z_1 = 1, \textit{ and } a_1 = (x+1)/(x-1)$$

$$z_{k+1} = z_k(1 + 1/a_k) \tag{$*$}$$

$$a_{k+1} = 2a_k^2 - 1. \tag{$**$}$$

Proposition 1. *With z_k and a_k defined as above, then a_k is greater than* 1. *Furthermore the following hold*:

(a) $z_k^2 = x(a_k - 1)/(a_k + 1)$.

(b) $z_1, z_2, \ldots$ *is an increasing sequence.*

(c) $\lim_{k \to \infty} z_k = \sqrt{x}$.

(d) *The convergence is quadratic.*

(e) *The convergence is not uniform in $x > 1$.*

Dedicated to Anthony and John Knill

Proof: Clearly a_k is greater than 1. Item (a) and the form of formulas $(*)$ and $(**)$ imply (b) and (c). For fixed $k \geq 1$, items $(*)$ and $(**)$ imply that as $x \to \infty$, we have $a_k \to 1$ and $z_k \to 2^{k-1}$, so item (e) follows. As for (d), item (a) and $(**)$ imply that convergence of z_k^2 to x is quadratic. Since as $k \to \infty$,

$$\left(x - z_k^2\right)/\left(\sqrt{x} - z_k\right) = \sqrt{x} + z_k \to 2\sqrt{x},$$

and x is fixed, then the convergence of z_k is quadratic as well. It remains only to prove (a). Item (a) is equivalent to:

$$x = z_k^2(a_k + 1)/(a_k - 1). \tag{a$'$}$$

One proves item (a′) by induction on $k \geq 1$. But first note that item $(*)$ implies

$$z_k = z_{k+1}a_k/(1 + a_k). \tag{$*'$}$$

For $k = 1$, item (a′) is the identity $x = (a_1 + 1)/(a_1 - 1)$ for the definition of a_1. To see that the case of $k + 1$ for item (a′) follows from the k-th case, observe (using item $(*')$ to justify the second line and item $(**)$ to justify the last line):

$$\begin{aligned} x &= z_k^2(a_k + 1)/(a_k - 1) \\ &= \left(z_{k+1}a_k/(1 + a_k)\right)^2(a_k + 1)/(a_k - 1) \\ &= z_{k+1}^2 a_k^2/\left(a_k^2 - 1\right) \\ &= z_{k+1}^2(a_{k+1} + 1)/(a_{k+1} - 1). \end{aligned}$$

This finishes the proof of (a). The proof of the proposition is complete.

Let us relate the sequence z_k to the sequence x_k of the Babylonian algorithm for $\sqrt{x}$.

Proposition 2. *Formula* $(*)$ *is equivalent to*

$$z_k = x/x_k. \tag{$***$}$$

Proof: The proof is by induction on k. The case of $k = 1$ is clear since $z_1 = 1$ and $x_1 = x$. Suppose that the induction hypothesis $(***)$ is true for some $k \geq 1$. Then the following sequence of equalities proves that formula $(*)$ for z_{k+1} and $(***)$ for z_k yield formula $(***)$ for z_{k+1}.

$$\begin{aligned} z_{k+1} &= z_k \frac{a_k + 1}{a_k} && \text{(by formula } (*)) \\ &= z_k \frac{2x}{\left(x + x\dfrac{a_k - 1}{a_k + 1}\right)} \\ &= z_k \frac{2x}{x + z_k^2} && \text{(by proposition 1(a))} \\ &= \frac{x}{\dfrac{1}{2}\left(x_k + \dfrac{x}{x_k}\right)} && \text{(by the induction hypothesis)} \\ &= \frac{x}{x_{k+1}}. \end{aligned}$$

Of course, Proposition 1(a) was pivotal in this argument. To argue that $(***)$ implies $(*)$, one first redefines z_k as x/x_k and redefines a_k so that 1(a) holds, namely,

$$a_k = \frac{x + z_k^2}{x - z_k^2}.$$

Then by reversing the above sequence of equalities, one obtains formula $(*)$ from formula $(***)$.

Corollary. *With x_k as defined in the first paragraph and a_k given by $(**)$, the following hold:*

(a) $x_k^2 = x\dfrac{a_k + 1}{a_k - 1}$.

(b) $x_1, x_2, \ldots$ *is a decreasing sequence.*

(c) $\lim_{k \to \infty} x_k = \sqrt{x}$.

(d) *The convergence is quadratic.*

(e) *The convergence is not uniform in $x > 1$.*

While the focus of this paper is on the convergence properties of the Babylonian algorithm, the reader with a numerical bent might also be interested in the following observations.

I. For x normalized so that $1 < x \leq 10$, we have

$$\left|\sqrt{x} - z_k\right| < \frac{x}{z_k(a_k + 1)} \sim z_k/(a_k + 1), (k \geq 4),$$

where, for $k \geq 4$,

$$a_k > \frac{(180)^{2^{k-4}}}{2} + 1.$$

Proof: The first displayed inequality follows from 1(a) since $(\sqrt{x} - z_k) = (x - z_k^2)/(\sqrt{x} + z_k)$ and $z_k < \sqrt{x}$. The "$\sim$" means "same order of magnitude." This is a consequence of the above estimate of a_k. To see that estimate, note that a_k is least for $x = 10$. With $x = 10$, calculate a_4, then derive the estimate for $k > 4$ by induction on k, using $(**)$.

II. For $k \geq 2$, $z_k = 2^{k-2}(a_1 + 1)a_1 a_2 \ldots a_{k-2}/a_{k-1}$ [$= (a_1 + 1)/a_1$, for $k = 2$].

Proof: One uses straightforward induction on $k \geq 2$, via formulas $(*)$ and $(**)$.

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ADDITION

'Addition, is joining more numbers than one,
And putting together to make a whole sum.
Addition's the rule that learns us to count,
And the sum that's produced is called the amount.

RULE

'Write the numbers all down, as the rule comprehends,
Placing units under units and tens under tens;
Draw a line underneath, and commence at the right,
Of the unit column, the work to unite; ---
If its sum or amount should not exceed 9,
Then place it direct 'neath its own native line:
But if 9 it exceeds, then the unit you place
'Neath the column of units, (the units to grace;)
While the 10s or the figure that's to the left hand,
To the next column join, as you well understand.
Observe the same rule, 'till you come to the last,
And the whole amount write as this column you cast;

from *The Poetical Geography With the Rules of Arithmetic in Verse*, by George Van Waters, 1851.

Lines Without Order

E. A. Marchisotto

1. INTRODUCTION. The purpose of this article is to develop the concept of line in absolute or neutral geometry (Euclidean geometry without the parallel postulate), without introducing the notion of order. This development has mathematical and historical appeal, as well as pedagogical value. For a typical sophomore or junior-level college course in geometry, it provides an interesting example of the kinds of insight that can be obtained by deriving the concept of line axiomatically. It will also acquaint students with the work of a relatively unknown Italian geometer, Mario Pieri (1860–1913), within the context of the historical use of motion in geometry.

In "Of elementary geometry as a hypothetical-deductive system; monograph of point and motion," Pieri [7] constructs elementary geometry in the plane and in space on two undefined concepts and twenty postulates. He unfolds his axiomatization with a recognizable chain of definitions, axioms, and proofs, each of which flows in a natural way from what has gone before.

To develop the notion of line, Pieri introduces the primitives, point and motion, then defines collinearity in terms of motion, and line in terms of collinearity. His method of development enables students to see exactly on what undefined concepts, definitions, postulates, and theorems, the definition of line depends. His system encourages the learner to confront the subtle aspects of incidence (that eluded Euclid) in an orderly and direct way, using the idea of an admissible set of rigid motions to gain the usual incidence properties. Pieri does not reveal at first what kind of rigid motions he allows in his system. Only by examining the axioms and theorems as they occur, can students admit certain motions and eliminate others. In doing so, they engage in a discovery process—uncovering the notion of line, step by step; determining what tools they need to complete their quest; exploring the ramifications of their progress as they proceed.

Pieri's development captures much of the spirit of modern mathematics and is sufficiently simple that the proofs can be left to the students. Indeed, George Martin [4] has called Pieri's definition of line via motion: "a beautiful treatment... very modern in flavor."

2. HISTORICAL BACKGROUND. Mario Pieri was the first mathematician to give an axiomatization of elementary geometry based on the two undefined concepts of point and motion. Hilbert (1899), for example, had used six primitives (point, line, plane, between, congruent and on). Pasch (1882) had required four (point, segment, congruence, planar surface), and Peano (1894) three (point, segment, and motion). Pieri's axiomatization [7] appeared in 1899, several months before Hilbert's

was published, and accommodates all the Hilbert axioms with the exception of the Playfair postulate. In the Preface to his work, Pieri acknowledges Pasch's 1882 axiomatization of projective geometry (with Euclidean geometry constructed as a special case) as the "logical edifice that reappears in every part of what is the subject of this present script" (p. 4). He credits Peano's algebraic logic as "the most valid instrument for his present study, not only for the efficiency of the symbols in themselves but for their intellectual aspects" (p. 10), noting that Peano's 1894 system can be derived from his.

During Pieri's time, the selection of primitives for an axiom system for geometry was generally limited to joining notions that imply an idea of size or a relation among figures to the notion of point. Line (or segment) was traditionally included in the set of undefined terms. The fact that Pieri chose to define line, in lieu of taking it as primitive, was innovative.

Further, the idea of line without consideration of order or betweenness was unconventional. Once Pasch had demonstrated the necessity of making betweenness explicit in an axiom system, geometers generally postulated it or included it as an undefined concept at the beginning of their axiomatizations. Pieri chose a different strategy: Like Hilbert, Pasch, and Peano, he recognized the need to make betweenness explicit—but instead of postulating it as they did, Pieri made it the subject of a definition. To define betweenness he used the following notions of sphere and midpoint:

Sphere b_a: Given two distinct points a, b, the class of all points p such that there exists a motion that leaves a fixed and transforms p into b is called a *sphere* of center a and passing through b.

Midpoint: Given two distinct points a, b, the point m for which the sphere b_m contains a is called the *midpoint* of a and b.

Pieri then defined a point to be *interior* to a sphere if it is the midpoint of two distinct points of a sphere. Calling a sphere passing through two points a and b the *polar sphere* (a, b), he defined *betweenness* as follows: the point x is said to be *between* a and b if it is a point of the line ab and is interior to a polar sphere of a and b.

Pieri's definition of betweenness occurs late in his development of geometry, and is therefore not a factor in his treatment of lines, planes, and circles. Indeed, Pieri postulated thirteen of his twenty axioms and proved sixty-six theorems based on these axioms of position before he introduced the idea of betweenness.

Because he chose to *define* line, and did so without any consideration of order or betweenness, Pieri's exposition is interesting from a historical perspective. Further, his development is engaging from a mathematical point of view because, in using motion, he conveys a kinetic as opposed to a static understanding of geometry. Finally, Pieri's treatment is satisfying from a pedagogical standpoint because it illustrates for students the power of the axiomatic method to reveal the structure of geometry and the nature of its components.

3. PRESENTATION. The method that Pieri uses to construct geometry makes it easy to isolate the concept of line for classroom discussion. He introduces eight postulates, three definitions, and seven theorems as they are needed to develop the concept. Thus, one can easily demonstrate for students what assumptions and proofs are necessary for his definition of line.

Pieri's development is outlined below, with comments designed to illustrate at each step why what has been presented so far is inadequate to characterize a line, and how the step that follows is designed to meet that objection. His notion of line emerges in eighteen steps:

Postulate 1. Point and motion are general ideas or classes. ▽ Thus point and motion are the primitives of Pieri's system.

Postulate 2. There exists at least one point. ▽

Postulate 3. If p is a point, there exists some other point different from p. ▽

Postulates 2 and 3 establish the existence of at least two points in Pieri's system. Now he prepares to lay the groundwork for a "connection" between points and lines. Since motion is all he has to work with, he seeks a motion that will enable him to make such a "connection," i.e., a motion that establishes the incidence of points on lines. The next three postulates determine the characteristics of motion that eventually allow for the definition of line in terms of points.

Postulate 4. Every motion μ is a bijective mapping from the set of points to the set of points. ▽

Postulate 5. For every motion μ, there exists an inverse motion μ^{-1}. ▽

Postulate 6. Two motions μ and ϕ performed in succession produce the effect of one motion, their product, $\mu\phi$, i.e., μ applied to ϕ. ▽

Postulates 4, 5 and 6 provide for the existence of an identity motion, for example, $\mu\mu^{-1}$. But this is not the motion that will establish the incidence of points on lines. Since *all* points are fixed by the identity motion, there is no way to distinguish, for example, between collinear and non-collinear points strictly in terms of this motion. Thus, Pieri requires a non-identity motion in order to define a line. Hence the following:

Definition 1. Any motion different from the identity transformation is called a *proper motion*. That is, for any proper motion μ, there exists x such that $\mu(x) \neq x$. ▽

Now Pieri is ready to postulate the existence of a motion that will make the "connection" between points and lines:

Postulate 7. For every pair of distinct points, there exists at least one proper motion that holds both points fixed. ▽

This motion (which insures the existence of at least four points) is a reasonable candidate for making the "connection" between points and lines—one that has a model in Euclidean space: a rotation about a line through two fixed points a, b. Such a rotation is a proper motion (in the Pieri sense), and satisfies Postulate 7 holding a and b fixed.

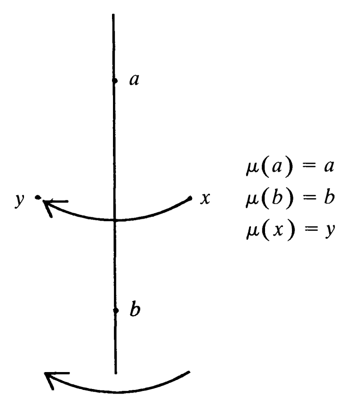


Figure 1.

The motion of Postulate 7 can be used to establish the incidence of *a pair of points* on a line. But, what about any other points on the line? Notice that the rotation in Euclidean space described above not only fixes a and b, but also fixes all points collinear with a and b. Does the motion of Postulate 7 characterize all points collinear with a and b, and only these? That is, if the motion of Postulate 7 also fixes a point c, is that sufficient to insure c is collinear with a and b? No. Again, using the model of Euclidean space, let a, b, c be non-collinear points. Consider a reflection of space through the plane abc.

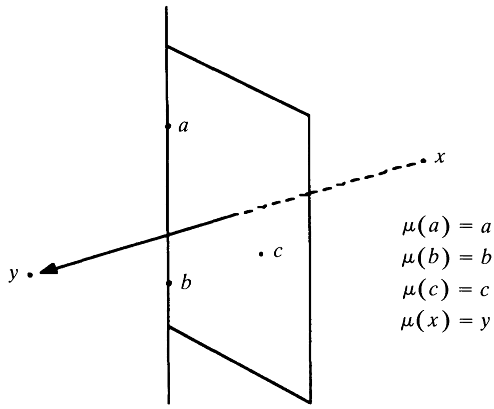


Figure 2.

This reflection is a proper motion (in the Pieri sense), satisfying all the before-stated postulates. Certainly, all points collinear with a and b are fixed by this reflection. But so is every other point of the mirror plane. In particular, this reflection fixes non-collinear points a, b, and c, and therefore is not a candidate for the motion Pieri needs to be able to define collinearity. To exclude this kind of motion, Pieri introduces the following:

Postulate 8. Let x, y, z be distinct points. If there exists a proper motion μ such that $\mu(x) = x$, $\mu(y) = y$, and $\mu(z) = z$, then every motion that leaves x and y fixed must also fix z. $\triangledown$ This postulate eliminates reflections in any three-dimensional Euclidean model of Pieri's system.

Now Pieri can define collinearity, confident that his motion of Postulate 7, with the results of Postulate 8, will provide the necessary and sufficient conditions for points to be considered collinear. He begins by defining collinear:

Definition 2. The points x, y, and z are *collinear* if there exists a proper motion that fixes each point. $\triangledown$

Pieri's first theorem prepares the way for him to be able to establish that a line will be completely determined by two points:

Theorem 1. *Three points, x, y, and z are collinear if any two of them, or all three, coincide.*

Proof: We first suppose $x \neq y$, $y = z$. By Postulate 7, there exists a proper motion μ that fixes x and y, and therefore μ fixes z. Now, suppose $x = y = z$. By Postulate 2, there exists a point $y' \neq y$. Then by Postulate 7, there exists a motion μ that fixes x and y', and therefore fixes y and z. ■

Pieri's second theorem is the converse of the definition of collinearity, and for that reason could be excluded from this development. He probably included it for pedagogical reasons. In the preface to this axiomatization, Pieri makes it clear that one of his goals in this development is "... to hasten the solution to the problem of teaching geometry." This theorem provides a working definition of non-collinearity that is easily cited and appears again and again in proofs of subsequent theorems. It also serves to confirm the kinds of motions admitted in Pieri's system.

Theorem 2. *The following are equivalent*: 1) *x, y, z, are non-collinear*; 2) *there exists no proper motion that keeps x, y, and z fixed.*

The proof of Theorem 2 follows from the definitions of collinearity and proper motion.

Definition 3. If x and y are two points, then the union xy is the set of all points collinear with x and with y. $\triangledown$

Pieri makes this definition of union so that he can establish incidence of points on lines, with no reference to ordering or positioning them there. Using the set theoretic relation of "belonging to," he proposes the following two theorems:

Theorem 3. *If x, y are distinct points, they belong to xy; and xy and yx coincide.*

The proof of Theorem 3 follows from the definition of collinearity and Postulate 7. Note that because a union is a set of points, no intuitive "picture" of line is necessary to justify the coincidence (equality) of xy and yx.

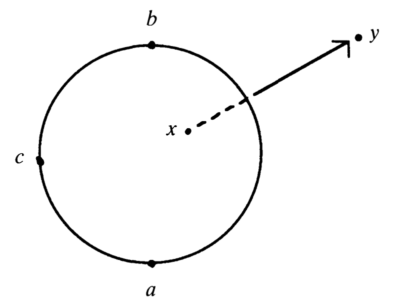
Theorem 4. *If x, y, z are points, $x \neq y$, then each of the following statements is a consequence of the other*: 1) *x, y, and z are collinear*; 2) *z belongs to xy.*

Theorem 4 allows Pieri to name unions in terms of two distinct points. Its proof follows from the definitions of collinearity and union.

At this juncture, Pieri has established that any point collinear with a and b is fixed by some proper motion that fixes a and b. But with the axioms amassed thus far, there is no way of ensuring that such a point would remain invariant under different motions that fix a and b. In other words, Pieri has not yet provided for the invariance of collinearity under motions. For example, consider the real inversive plane (the real Euclidean plane plus 1_∞, a single element at infinity). There, inversions and reflections through lines satisfy Postulates 4, 5, and 6, and are proper motions in the Pieri sense. Choose three points, a, b, c, on the circle of inversion. An inversion will fix these points, so these points are collinear in the Pieri sense (see Fig. 3(a)). Now reflect the plane through line ab. Points a and b will remain invariant, but c will not (see Fig. 3(b)). Thus under the inversion, a, b, and c are collinear (in the Pieri sense of being fixed by a motion), but under the reflection they are not. To avoid this kind of scenario, Pieri proves:

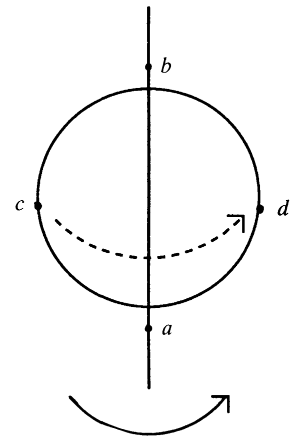
Theorem 5. *If x, y are distinct points, the union xy is the locus of all points fixed by any motion that fixes x and y.*

Proof: Let μ be a proper motion that fixes x and y. Let $z \in xy$ such that $z \neq x, y$. It suffices to show that μ fixes z. Since $z \in xy$, z is collinear with x and with y. So there exists a motion ϕ that fixes x, y and z. By Postulate 8, μ must fix z. ■



$\mu(a) = a$
$\mu(b) = b$
$\mu(c) = c$
$\mu(x) = y$

Figure 3(a)



$\mu(a) = a$
$\mu(b) = b$
$\mu(c) = d$

Figure 3(b)

Pieri defined non-collinearity in Theorem 2, but he has not yet provided for the existence of non-collinear points. He probably proves the following theorem so that by the time he makes the definition of line, he has provided for the existence of points not on a line as well as for distinct lines. In completely developing the notion of line from definitions, axioms, and theorems, such possibilities need to be made explicit.

Theorem 6. *There exist three non-collinear points, i.e., given two distinct points, there exists at least one point outside the union of them.*

Proof: Let x, y be two distinct points. We show that there exists a point p that does not belong to the union xy, i.e., that there exists no proper motion that keeps x, y, and p fixed. By Postulate 7, there exists at least one proper motion, μ, that fixes x and y. Also, there exists a point p that is not fixed by μ, since μ is a proper motion. Thus there exists no proper motion that fixes x, y, and p, because *if* such a motion should exist, then by Postulate 8 any motion that fixes x and y would have to fix p, i.e., μ would have to fix p. ■

When Pieri makes his definition of line, he wants to be able to name it using two distinct points, without any ambiguity. Thus, the final step before defining lines involves proving that a union is uniquely determined by two points:

Theorem 7. *If x and y are distinct points, and z and w are distinct points belonging to the union xy, then $xy = zw$.*

Proof: We first show $xy \subseteq zw$. Let $p \in xy$. Then there exists a proper motion μ that fixes p, x, y. Since $z, w \in xy$, there exists a proper motion that fixes z, x, y, and a proper motion that fixes w, x, y. By Postulate 8, μ fixes z and w. Therefore μ fixes p, z, w so $p \in zw$. Furthermore, since μ fixes x, y, z, w, then $x, y \in zw$ and a similar argument shows $zw \subseteq xy$. ■

Definition 4. The generic name of *line* is given to the union of any two distinct points. The term line also represents the class of all possible unions. ▽

This is one of the definitions of line proposed by Leibniz. Nothing in its statement implies an order between a, b, and c (see Theorem 3). Notice that, strictly based on the preceding postulates, definitions, and theorems, and without an appeal to intuition, Pieri has developed the notion of line and given the following insights into its nature: a line is a set of collinear points (Definitions 2, 3, Theorems 4, 5); there exist points not on a line; distinct lines exist (Theorems 2, 6); two points completely determine a line (Postulate 7, Theorems 1, 4, 7); and incidence of points on lines is invariant under motion (Postulate 8, Theorem 2). Note also, the "connection" between points and lines is a kinetic one (Postulates 1 to 8, Definitions 1, 2, Theorem 5), because the existence of a line is given by the existence of a motion that leaves points invariant.

CONCLUSION. What makes Pieri's treatment special is not only how he defines line, but the way he prepares for it. Via the axiomatic method, he builds the concept of line, establishing certain characteristics inherent to its definition and function in the axiom system. Central to his construction of line is motion. He is able to motivate the definition of line without introducing order because he takes motion as a primitive.

A footnote concerning motion can provide context to this presentation: History shows that motion was not always welcome in geometry. It was feared that motion would "bring into geometry an element foreign to it, namely, the notion of time" [**3**]. The Eleatics, with their paradoxes of motion, had shocked mathematics, and led mathematicians to try to eliminate all motion from their discipline. Aristotle, for example, had forbidden the use of motion in geometry. Euclid avoided any explicit mention of it.

Torretti [**9**] indicated how, in 1851, the philosopher Friedrich Ueberweg (1826–1871) broke new ground by proposing to base Euclidean geometry on the idea of rigid motion. A similar stand was taken by Jules G. Höuel (1823–1886), Charles Méray (1835–1911), and Giuseppe Peano (1858–1932), before Pieri. Pieri's development of geometry agrees with Klein's Erlangen Programme (1872) because it is explicitly based on the properties of a transitive group of motions. It "follows the lead" of Hermann von Helmholtz (1821–1894) and Sophus Lie (1842–1899): "But instead of relying on the familiar attributes of the 'number manifold' R^3, Pieri patiently analyzes the properties... ascribed to... motions, and to their sets of points, in order to determine fully and exactly the classical structure of geometry" [**9**].

Bertrand Russell [**8**] described Pieri's concept of motion as perhaps the simplest possible for elementary geometry. Eves [**1**] noted how Pieri's idea of motion can be "nicely adapted to the Euclidean superposition proofs" and saw in Pieri's work an anticipation of later developments in geometry:

> Pieri was considering Euclidean geometry as the study of the properties and relations of configurations of points which remain invariant under the group of direct isometries.

The use of motion as an undefined concept enables Pieri to define other geometric figures and relations quite elegantly, for example, 1) sphere (defined previously); 2) perpendicularity (if a, b, c are distinct points, (a, b) is perpendicular to (a, c) if there exists a motion that fixes ab pointwise, and fixes ac, but not pointwise).

Motion also enables Pieri to prove theorems simply and succinctly. His work in geometry should be studied for these and many other interesting presentations.

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The Sweetness of Abstraction

Abstraction sets mathematicians free
Of spatial limits and time's interludes.
Abstraction lets them add infinitudes
To reach a still vaster infinity,

To postulate points as transcendently
Unreal as pixies in solemn moods,
To fête a shadowy whole that includes
The part which equals it resplendently.

Creators of their own strange universe,
Mathematicians can transcend the earth,
Just as the spirit can transcend the flesh.

More liltingly than Irishmen speak Erse,
Their image sing of the lyric birth
Of paradoxes woven in a mesh.

—Lawrence Minet

An Identity for $\binom{2n}{n}$

Solomon W. Golomb

1. INTRODUCTION. In 1851, P. L. Chebyshev [**1**] obtained surprisingly good estimates of $\pi(x)$, the number of primes not exceeding x, by considering the prime factorization of $\binom{2n}{n}$. Chebyshev showed that $\binom{2n}{n}$ divides $\prod_{p^\alpha \le 2n} p^\alpha$, the product of the highest powers not exceeding $2n$ of the primes up to $2n$. From this, it's easy to see that

$$\binom{2n}{n} \le \prod_{p^\alpha \le 2n} p^\alpha < (2n)^{\pi(2n)}$$

where $\pi(x)$ is the number of primes up to x. It's also easy to see that $2^n < \binom{2n}{n}$. Taking logs, $n \log 2 < \log\binom{2n}{n} < \pi(2n)\log(2n)$, from which

$$k\frac{2n}{\log(2n)} < \pi(2n)$$

for some constant $k > 0$. Using the (easier) facts that $\binom{2n}{n}$ is divisible by $\prod_{n<p\le 2n} p$, and $\binom{2n}{n} < 2^{2n}$, he similarly showed that

$$\pi(n) < K\frac{n}{\log n}$$

for some other constant $K > 0$. Thus Chebyshev bounded $\pi(x)$ on both sides, showing that its order of magnitude is $x/\log x$, and even obtained the values $k > .92$ and $K < 1.105$. However, he never managed to prove the Prime Number Theorem, which is the statement $\pi(x) \sim x/\log x$ as $x \to \infty$. (The asymptotic symbol " $\sim$ " indicates

$$\frac{\pi(x)}{x/\log x} \to 1$$

as $x \to \infty$, and all logs are natural logs.)

The product $\prod_{p^\alpha \le 2n} p^\alpha$ is in fact equal to $L(2n)$, where $L(x) = L.C.M.(1, 2, \ldots, [x])$. The purpose of this note is to introduce an identity which expresses $\binom{2n}{n}$ precisely in terms of $L(n)$. This identity is

$$\binom{2n}{n} = \frac{L(2n)}{L(n)} \cdot \frac{L\left(\frac{2n}{3}\right)}{L\left(\frac{n}{2}\right)} \cdot \frac{L\left(\frac{2n}{5}\right)}{L\left(\frac{n}{3}\right)} \cdot \frac{L\left(\frac{2n}{7}\right)}{L\left(\frac{n}{4}\right)} \cdots$$

$$= \prod_{k=1}^{\left[\frac{n+1}{2}\right]} \frac{L\left(\frac{2n}{2k-1}\right)}{L\left(\frac{n}{k}\right)}, \quad \text{for all } n = 1, 2, 3, \ldots. \tag{1}$$

This may also be written as

$$\binom{2n}{n} = \prod_{k=1}^{n} L\left(\frac{2n}{k}\right)^{(-1)^{k+1}} \tag{2}$$

Taking logarithms, this also gives

$$\log\binom{2n}{n} = \sum_{k=1}^{n} (-1)^{k+1}\Psi\left(\frac{2n}{k}\right) \tag{3}$$

where

$$\Psi(x) = \log L(x) = \sum_{n \le x} \Lambda(n), \tag{4}$$

where von Mangoldt's function $\Lambda(n)$ satisfies

$$\sum_{d|n} \Lambda(d) = \log n, \tag{5}$$

and explicitly

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ is any power } (k \ge 1) \text{ of any prime } p \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

If $1 \le a < b \le 2a$, it is easily seen that

$$\frac{L(b)}{L(a)} = \prod_{a < p^{\alpha} \le b} p \tag{7}$$

where the product is of all primes which have powers (including the first power) on the interval $(a, b]$. This shows that each of the factors $L(2n/(2k-1))/L(n/k)$ in (1) is an integer, and that the first of these factors, $L(2n)/L(n)$, is a multiple of $\prod_{n<p\le 2n} p$. It is also true, from (1), that $L(2n)/L(n)$ divides $\binom{2n}{n}$, a slight strengthening of Chebyshev's observation that $\prod_{n<p\le 2n} p$ divides $\binom{2n}{n}$.

In view of (7), the formula (1) may be used to obtain the explicit prime factorization of $\binom{2n}{n}$ rather rapidly. For example,

$$\binom{10}{5} = \frac{L(10)}{L(5)} \cdot \frac{L(3)}{L(2)} \cdot \frac{L(2)}{L(1)} = (3 \cdot 2 \cdot 7)(3)(2) = 252.$$

The functions $\Lambda(n)$ and $\Psi(x)$ are basic to the study of the distribution of the prime numbers. In fact, the Prime Number Theorem is usually proved in the equivalent form $\Psi(x) \sim x$ as $x \to \infty$. In view of (4), this says that the *mean value* of $\Lambda(n)$ is 1, in the precise sense $\lim_{N\to\infty}(1/N)\sum_{n=1}^{N}\Lambda(n) = 1$.

2. PROOF OF THE IDENTITY. It is well-known and easily shown that if $F(n) = \sum_{d|n} f(d)$, then $\sum_{k=1}^{n} F(k) = \sum_{k=1}^{n} f(k)[n/k]$. From this, in view of (5), we have

$$\log n! = \sum_{k=1}^{n} \log k = \sum_{k=1}^{n} \Lambda(k)\left[\frac{n}{k}\right] = \sum_{k=1}^{n} \Psi\left(\frac{n}{k}\right), \tag{8}$$

an identity also found in [**2**].

From this, we readily observe that

$$\log\binom{2n}{n} = \log(2n)! - 2\log n! = \sum_{k=1}^{2n} (-1)^{k+1}\Psi\left(\frac{2n}{k}\right). \tag{9}$$

Since $\Psi(x) = 0$ for $x < 2$, it is only necessary to consider $1 \le k \le n$, yielding (3), from which (2) and (1) follow. For (1), we note explicitly that

$$\frac{2n}{2k-1} < 2 \quad \text{if } k > \left[\frac{n+1}{2}\right]. \qquad \blacksquare$$

The identity (2) can also be proved directly from the well-known result (see [3])

$$H_p\left(\binom{2n}{n}\right) = \sum_{k=1}^{[\log_p 2n]} \left\{\left[\frac{2n}{p^k}\right] - 2\left[\frac{n}{p^k}\right]\right\}, \tag{10}$$

where $H_p(m)$ denotes the highest power of p which divides m.

Whether $[2n/p^k] - 2[n/p^k]$ is 1 or 0 depends specifically on whether

$$\frac{2n}{2r} < p^k \le \frac{2n}{2r-1}$$

or

$$\frac{2n}{2r+1} < p^k \le \frac{2n}{2r},$$

from which (2) follows.

The facts expressed by

$$\frac{L(2n)}{L(n)} \left| \binom{2n}{n} \right| L(2n) \tag{11}$$

are the simplest special cases of the following generalization of (2):

For all integers r, $1 \le r \le 2n$, the product

$$\prod_{k=1}^{r} L\left(\frac{2n}{k}\right)^{(-1)^{k+1}} \tag{12}$$

divides $\binom{2n}{n}$ if r is *even*, and *is divisible by* $\binom{2n}{n}$ if r is *odd*. Since (12) has *the same* value for all r with $n \le r \le 2n$, which includes both even and odd values of r, this "final" value of (12) must *equal* $\binom{2n}{n}$. This equality is expressed in (2).

Similarly, the alternating sum in (3) alternately overestimates and underestimates $\log\binom{2n}{n}$.

B. Gordon [4] based an "elementary" proof of the Prime Number Theorem on the identity (8) and Stirling's Formula for $n!$. It is possible that a similar "elementary" proof could be given using (3) instead of (8).

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Newton's Identities

D. G. Mead

The usual developments of Newton's identities, the relation between the elementary symmetric functions of $x_1, x_2, \dots, x_n$ and the sums of the powers of the x_i, are unsatisfactory, for they all involve a trick of one kind or another. In this note we show that with the proper notation, the derivation of Newton's identities is both natural and simple.

The elementary symmetric functions of $x_1, \dots, x_n$ are

$$s_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k} \quad \text{for} \quad k = 1, 2, \dots, n,$$

and the Newton functions are:

$$p_k = \sum_{i=1}^{n} (x_i)^k, \qquad k = 1, 2, 3, \dots .$$

The Newton identities are:

$$p_k + \sum_{i=1}^{k-1} (-1)^i p_{k-i} s_i + (-1)^k k s_k = 0 \quad \text{if } 1 \le k \le n$$

and

$$p_k + \sum_{i=1}^{n} (-1)^i p_{k-i} s_i = 0 \quad \text{if } k > n.$$

Many authors offer different proofs for the cases $k \le n$ and $k > n$. A typical proof for the case $k > n$ proceeds as follows. With

$$f(x) = \prod_{i=1}^{n} (x - x_i) = x^n + \sum_{i=1}^{n} (-1)^i s_i x^{n-i},$$

since

$$0 = x_j^{k-n} f(x_j) = x_j^k + \sum_{i=1}^{n} (-1)^i s_i x_j^{k-i},$$

we have

$$\sum_{j=1}^{n} x_j^{k-n} f(x_j) = 0 = \sum_{j=1}^{n} x_j^k + \sum_{j=1}^{n} \sum_{i=1}^{n} (-1)^i s_i x_j^{k-i} = p_k + \sum_{i=1}^{n} (-1)^i s_i p_{k-i},$$

which are the desired relations for $k > n$. For $k \le n$ there are various algebraic proofs which involve the examination of $\sum_{i=1}^{n}(f(x)/(x - x_i))$, and others consider which the logarithmic derivative of $f(x)$. A simple compact proof of the latter type using formal power series (which yields all of the identities at once) can be found

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in [1], page 212. However, none of the proofs can be considered satisfactory, for each is devoid of motivation.

Before describing our suggested derivation, we define the notation to be employed. Let $(a_1, \ldots, a_n)$ where the a_i are nonnegative integers and $a_i \geq a_{i+1}$, represent $\Sigma x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_n}^{a_n}$ where the sum is over all permutations $(i_1, \ldots, i_n)$ of $(1, 2, \ldots, n)$ which yield distinct terms. If $a_i = 0$ for $i > t$, there will be no ambiguity if we write $(a_1, \ldots, a_t)$ instead of $(a_1, \ldots, a_n)$. With $n = 3$ we have, for example,

$$(2,1) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2, \qquad (1) = x_1 + x_2 + x_3,$$

$$(1,1) = x_1 x_2 + x_1 x_3 + x_2 x_3 \quad \text{and} \quad (1,1,1) = x_1 x_2 x_3.$$

In this notation (to be found in [2], page 82),[1] the elementary symmetric functions are $(1), (1,1), \ldots, (1,1,\ldots,1)$ and the Newton functions are $(1), (2), (3), \ldots$. Since with $n = 3$, $(1,1,1,1)$ makes no sense, we define it to be zero.

To illustrate the procedure, consider the case $k = 3$ and $n \geq 3$. We wish to find a relation among the Newton functions (1), (2), (3) and the elementary symmetric functions[1], $(1,1), \ldots, (1,1,\ldots,1)$. The key is to note that $(2)(1) = (3) + (2,1)$. To eliminate $(2,1)$ we use

$$(1)(1,1) = (2,1) + 3(1,1,1)$$

where the left side is a product of a Newton function and an elementary symmetric function. The coefficient 3 occurs since the product $x_1 x_2 x_3$ can arise from x_1 times $x_2 x_3$, x_2 times $x_1 x_3$, or x_3 times $x_1 x_2$. If we subtract the second equation from the first, we obtain the Newton identity:

$$p_3 - p_2 s_1 + p_1 s_2 - 3 s_3 = 0.$$

This can easily be generalized. To make the notation simpler, let $s_i = (1_i)$, a sequence of i ones, and if $t \geq 1$, let $(t, 1_i) = (c_1, \ldots, c_{i+1})$ where $c_1 = t$ and $c_j = 1$ for $j > 1$. To obtain the Newton identity involving $p_1, \ldots, p_k$, write t equations, where $t = \min(k-1, n)$:

$$(k-1)(1) = (k) + (k-1,1)$$
$$(k-2)(1,1) = (k-1,1) + (k-2,1,1)$$
$$(k-3)(1,1,1) = (k-2,1,1) + (k-3,1,1,1)$$

or in general

$$(k-i)(1_i) = (k-i+1, 1_{i-1}) + (k-i, 1_i) \quad \text{for } i = 1, \ldots, t.$$

If $n \geq k = t+1$, the last equation is

$$(1)(1_{k-1}) = (2, 1_{k-2}) + k(1_k)$$

while if $k > n = t$, the last equation is

$$(k-n)(1_n) = (k-n+1, 1_{n-1})$$

since the symbol $(k-n, 1_n)$, having $n+1$ entries, represents the polynomial zero. By multiplying the ith equation by $(-1)^{i-1}$ and adding the equations we obtain the Newton identities.

[1] The polynomial $(a_1, \ldots, a_t)$ is defined in [3], page 11, where it is written m_λ, with λ being the partition $a_1, \ldots, a_t$.

The Newton functions and elementary symmetric functions are easily expressed using the notation $(a_1, \ldots, a_n)$ and with this notation the derivation of the Newton identities is both easy and natural.

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How to make Pi Equal to Three
(The Sequel)

Rick Chase

In the February 1992 Monthly Rick Norwood gave a method for using relativity to make pi equal three. It is also possible to make pi equal to three on the surface of a sphere. A small circle on a sphere will have pi very close to the traditional value (measuring circumference and diameter on the surface of the sphere rather than in the interior of the sphere). On the other hand, a great circle will have a circumference equal to just twice its diameter (as measured on the surface), giving pi a value of 2. By continuity, there must be an intermediate radius that makes pi equal to three. On the surface of the earth, that radius will be approximately 2073 miles.

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On Sums of Triangular Numbers and Sums of Squares

John A. Ewell

INTRODUCTION. According to L. E. Dickson [**2**, p. 6] Fermat made the following famous comment about 355 years ago: "I was the first to discover the very beautiful and entirely general theorem that every number is either triangular or the sum of 2 or 3 triangular numbers; every number is either a square or the sum of 2, 3 or 4 squares; either pentagonal or the sum of 2, 3, 4 or 5 pentagonal numbers; and so on ad infinitum, whether it is a question of hexagonal, heptagonal or any polygonal numbers. I cannot here give the proof, which depends upon numerous and abstruse mysteries of numbers; for I intend to devote an entire book to this subject and to effect in this part of arithmetic astonishing advances over the previously known limits." [In this statement "number" means "positive integer"; "arithmetic" means "number theory"; and, the triangular, square and pentagonal numbers are respectively described by: $n(n+1)/2$, n^2 and $n(3n-1)/2$, $n = 1, 2, \ldots$].

It seems (as far as the author can tell) that certain questions about Fermat's statement regarding the polygonal numbers beyond the squares remain open to this very day. As a matter of fact, only that part of the statement regarding the squares has received a thorough and complete treatment. Thus, we begin our discussion with an historical statement of the major results on representations of numbers by sums of four or fewer squares. But, we first give a definition to facilitate statement of these results.

Definition. As usual, $\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{N} := \{0, 1, 2, \ldots\}$ and $\mathbb{P} := \mathbb{N} \setminus \{0\}$. Then, for each $k \in \mathbb{P}$ and each $n \in \mathbb{N}$,

$$r_k(n) := \left|\{(x_1, \ldots, x_k) \in \mathbb{Z}^k | n = x_1^2 + \cdots + x_k^2\}\right|,$$

$$t_k(n) := \left|\{(x_1, \ldots, x_k) \in \mathbb{N}^k | n = x_1(x_1+1)/2 + \cdots + x_k(x_k+1)/2\}\right|.$$

In 1770 Lagrange [**2**, p. 279] proved that part of Fermat's "theorem" regarding squares. [We should add that the four-square theorem was actually first stated in 1621 by Bachet [**2**, p. 275]].

Theorem 1. *Every natural number is the sum of* 4 *or fewer squares*; *that is*, *for each* $n \in \mathbb{N}$, $r_4(n) > 0$.

After this result we have two naturally-arising questions: Does there exist a "simple" description of the natural numbers which are sums of 3 squares? Two squares? Legendre [**2**, p. 261] first gave an answer to the first of these two questions. In fact, he proved the following theorem.

Theorem 2. *The set of positive integers that are not sums of three or fewer squares* $=$ $\{n \in \mathbb{P} | n = 4^k(8m + 7)$, *for some* $k, m \in \mathbb{N}\}$.

At the present time no simple proof of theorem 2 has been found. Gauss [**2**, p. 262] found a proof based on his theory of ternary quadratic forms; he also found a way to "count" the number of such representations (for numbers not of the form $4^k(8m + 7)$).

A complete answer to the second question was first given by Euler [**2**, pp. 230–231].

Theorem 3. *A positive integer* $n > 1$ *can be written as a sum of two squares if and only if when n is expressed as a product of prime-powers, every prime factor* $p \equiv 3 \pmod 4$ *occurs with even exponent.*

Can we count the number of representations of a given natural number by sums of 4 squares? Two squares? Jacobi [**2**, p. 285, p. 235] showed that in each case the count can indeed be given in terms of simple divisor functions. In order to state Jacobi's results, we need some notation: (i) For each positive integer n, $b(n)$ is the exponent of the exact power of 2 dividing n, and then $Od(n) := n2^{-b(n)}$ is the odd part of n. (ii) For each positive integer n and each $i \in \{1, 3\}$, $d_i(n) :=$ the number of positive divisors of n congruent to i mod 4. (iii) For each positive integer n, $\sigma(n) :=$ the sum of all of the positive divisors of n.

Theorem 4. *For each* $n \in \mathbb{P}$, $r_4(n) = 8(2 + (-1)^n)\sigma(Od(n))$.

Theorem 5. *For each* $n \in \mathbb{P}$, $r_2(n) = 4\{d_1(n) - d_3(n)\}$.

We now turn our attention to the problem of representing numbers by sums of three or fewer triangular numbers. Owing to the simple fact: for each $n \in \mathbb{P}$, $n^2 = (n - 1)n/2 + n(n + 1)/2$, we observe that the triangular numbers are in some sense simpler than the squares. Yet, the theory of representation of numbers by sums of three or fewer triangular numbers is *not* as well developed as the corresponding theory for representation by sums of squares. Recent work of the author [5] has helped to eliminate the gap between the two theories. And, since the methods and techniques are completely elementary, we can give a thorough treatment of this problem.

Gauss [**2**, p. 17] first proved that part of Fermat's "theorem" regarding triangular numbers. He proved the following theorem.

Theorem 6. *Every natural number is the sum of* 3 *or fewer triangular numbers; that is, for each* $n \in \mathbb{N}$, $t_3(n) > 0$.

Gauss also gave a method for "counting" the number of such representations. Again, his methods are not very accessible, as they rest on his theory of ternary quadratic forms.

After Theorem 6 we have the following naturally-arising questions: Does there exist a "simple" description of the natural numbers which are sums of 2 triangular numbers? Can we count the number of representations by sums of 2 triangular numbers? Both of these questions are answered by the following theorem.

Theorem 7. *For each* $n \in \mathbb{N}$, $t_2(n) = d_1(4n + 1) - d_3(4n + 1)$.

Proof of Theorem 7: Our proof depends on the triple-product identity

$$\prod_1^\infty (1 - x^{2n})(1 - ax^{2n-1})(1 - a^{-1}x^{2n-1}) = \sum_{-\infty}^{\infty} (-1)^n x^{n^2} a^n, \tag{1}$$

which is valid for each pair of complex numbers a, x such that $a \neq 0$ and $|x| < 1$. Michael D. Hirschhorn **[8]** showed how to deduce Jacobi's Theorem 5 from the triple product identity. The reader will doubtless note that our method is similar to that of Hirschhorn.

Separating even and odd terms on the right side of (1), and then again using (1) to replace the series in the resulting identity by infinite products, we get

$$\begin{aligned}
&\prod_1^\infty (1 - x^{2n})(1 - ax^{2n-1})(1 - a^{-1}x^{2n-1}) \\
&\quad = \sum_{-\infty}^{\infty} x^{4n^2} a^{2n} - ax \sum_{-\infty}^{\infty} x^{4n(n+1)} a^{2n} \\
&\quad = \prod_1^\infty (1 - x^{8n})(1 + a^2 x^{8n-4})(1 + a^{-2} x^{8n-4}) \\
&\qquad -(a + a^{-1}) x \prod_1^\infty (1 - x^{8n})(1 + a^2 x^{8n})(1 + a^{-2} x^{8n}).
\end{aligned}$$

With D_a denoting derivation with respect to a, we then operate on both sides of the foregoing identity with aD_a to get

$$\begin{aligned}
&-\prod_1^\infty (1 - x^{2n})(1 - ax^{2n-1})(1 - a^{-1}x^{2n-1}) \sum_1^\infty v_k(x)(a^k - a^{-k}) \\
&\quad = 2\prod_1^\infty (1 - x^{8n})(1 + a^2 x^{8n-4})(1 + a^{-2} x^{8n-4}) \\
&\qquad \times \sum_1^\infty (-1)^{k-1} v_k(x^4)(a^{2k} - a^{-2k}) \\
&\qquad -(a - a^{-1}) x \prod_1^\infty (1 - x^{8n})(1 + a^2 x^{8n})(1 + a^{-2} x^{8n}) \\
&\qquad -(a + a^{-1}) 2x \prod_1^\infty (1 - x^{8n})(1 + a^2 x^{8n})(1 + a^{-2} x^{8n}) \\
&\qquad \times \sum_1^\infty (-1)^{k-1} u_k(x^8)(a^{2k} - a^{-2k}),
\end{aligned}$$

where for convenience $u_k(x) := x^k \cdot (1 - x^k)^{-1}$, $v_k(x) := x^k \cdot (1 - x^{2k})^{-1}$, $k \in \mathbb{P}$, and x is a complex number with $|x| < 1$. Now, in (2) let $a = i$ and divide the resulting identity by $-2i$ to get

$$\prod_1^\infty (1 - x^{2n})(1 + x^{4n-2}) \sum_0^\infty (-1)^k v_{2k+1}(x) = x \prod_1^\infty (1 - x^{8n})^3,$$

or equivalently,

$$x \prod_1^\infty \frac{(1 - x^{8n})^3}{(1 - x^{2n})(1 + x^{4n-2})} = \sum_0^\infty (-1)^k \frac{x^{2k+1}}{1 - x^{4k+2}}.$$

Hence,

$$x\prod_{1}^{\infty}\frac{(1-x^{8n})^2}{(1-x^{8n-4})^2}=\sum_{0}^{\infty}(-1)^k\frac{x^{2k+1}}{1-x^{4k+2}}$$
$$=\sum_{k=0}^{\infty}(-1)^k\sum_{j=0}^{\infty}x^{(2j+1)(2k+1)}.$$

Owing to a well-known identity of Gauss [**6**, p. 284], it then follows that

$$\begin{aligned}\sum_{0}^{\infty}t_2(n)x^{4n+1}&=x\left\{\sum_{0}^{\infty}x^{2n(n+1)}\right\}^2\\&=x\prod_{1}^{\infty}\frac{(1-x^{8n})^2}{(1-x^{8n-4})^2}\\&=\sum_{k=0}^{\infty}(-1)^k\sum_{j=0}^{\infty}x^{(2j+1)(2k+1)}\\&=\sum_{m=0}^{\infty}x^{2m+1}\sum_{d\mid 2m+1}(-1)^{(d-1)/2}\\&=\sum_{n=0}^{\infty}x^{4n+1}\sum_{d\mid 4n+1}(-1)^{(d-1)/2}\\&\quad+\sum_{n=0}^{\infty}x^{4n+3}\sum_{d\mid 4n+3}(-1)^{(d-1)/2}.\end{aligned}$$

Equating coefficients of like powers of x we get: for each $n\in\mathbb{N}$,

$$\begin{aligned}t_2(n)&=\sum_{d\mid 4n+1}(-1)^{(d-1)/2}\\&=\sum_{\substack{d\mid 4n+1\\ d\equiv 1\ (\mathrm{mod}\ 4)}}1-\sum_{\substack{d\mid 4n+1\\ d\equiv 3\ (\mathrm{mod}\ 4)}}1\\&=d_1(4n+1)-d_3(4n+1),\end{aligned}$$

$$\sum_{d\mid 4n+3}(-1)^{(d-1)/2}=0.$$

This proves theorem 7.

In passing we note that the second conclusion follows easily from the following independent argument. For each $n\in\mathbb{N}$ and each divisor d (and codivisor d') of $4n+3$, exactly one of the pair (d,d') is $\equiv 1\ (\mathrm{mod}\ 4)$ and exactly one is $\equiv 3\ (\mathrm{mod}\ 4)$. Hence,

$$(-1)^{(d-1)/2}+(-1)^{(d'-1)}=0.$$

Summing over all of these pairs we obtain the desired result.

Theorems 3, 5 and 7 yield the following corollary.

Corollary 8. *A positive integer n can be written as a sum of two triangular numbers if and only if when* $4n+1$ *is expressed as a product of prime-powers, every prime factor* $p\equiv 3\ (\mathrm{mod}\ 4)$ *occurs with even exponent.*

In fact, Theorems 5 and 7 are actually *equivalent*. By Theorem 3, counting representations of positive integers by sums of two squares can be restricted to positive integers of the form $2^f(4k+1)$, $f, k \in \mathbb{N}$. The equivalence will follow from the fact that the sets

$$S = S(k) := \{(x, y) \in \mathbb{N} \times \mathbb{P} | 4k + 1 = x^2 + y^2\}$$

and

$$T = T(k) := \{(i, j) \in \mathbb{N}^2 | k = i(i+1)/2 + j(j+1)/2\},$$

$k \in \mathbb{N}$, have the same cardinality. [It's easy to verify that the function $\theta: T \to S$, defined by

$$\theta(i, j) := \begin{cases} (0, 2i+1), & \text{if } i = j, \\ (i - j, i + j + 1), & \text{if } i > j, \\ (i + j + 1, j - i), & \text{if } i < j, \end{cases}$$

is one-to-one from T onto S.]

Now, let us assume that Theorem 7 holds. Then, for each $k \in \mathbb{N}$, $|S(k)| = |T(k)| = d_1(4k+1) - d_3(4k+1)$. And, therefore, $r_2(4k+1) = |\{(x, y) \in \mathbb{Z}^2 | 4k + 1 = x^2 + y^2\}| = 4\{d_1(4k+1) - d_3(4k+1)\}$, since each solution $(x, y) \in S$ yields 4 solutions in $\mathbb{Z}^2$.

Conversely, let us assume that Theorem 5 holds. Then, for each $k \in \mathbb{N}$, $|S(k)| = r_2(4k+1)/4 = d_1(4k+1) - d_3(4k+1)$, whence $t_2(k) := |T(k)| = d_1(4k+1) - d_3(4k+1)$, as well.

Since $r_2(2^f(4k+1)) = r_2(4k+1)$, equivalence of Theorems 5 and 7 follows.

Owing to the equivalence of the two theorems, our proof of Theorem 7 is a new one for both theorems.

Concluding Remarks. We here wish to mention several recent contributions to this theory. In [**1**] G. E. Andrews derived the identity

$$\left(\sum_{n=0}^{\infty} x^{n(n+1)/2}\right)^3 = \sum_{n=0}^{\infty} \sum_{j=0}^{2n} \frac{x^{2n^2+2n-j(j+1)/2}(1 + x^{2n+1})}{(1 - x^{2n+1})},$$

which is valid for each complex number x such that $|x| < 1$. [We have here replaced the variable "q" by the variable "x," since in arithmetical discussions the letters "p" and "q" are usually reserved to denote primes.] From this identity Gauss's Theorem 6 is then easily deducible. Through this approach the theorem is thus freed from its dependence on the theory of ternary quadratic forms. Unfortunately, no such approach seems to be available for the stronger Theorem 2.

In 1982 M. D. Hirschhorn [7] and the author [**3**] independently observed that Jacobi's four-square theorem [Theorem 4] can be derived from the triple-product identity (1).

Finally, the author [**4**] showed that an easy special case of the triple-product identity implies Fermat's two-square theorem: any prime of the form $4k+1$ can be expressed as a sum of two squares. Of course, this is the major result needed to prove Euler's Theorem 3.

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A sine qua non for making mathematics exciting to a pupil is for the teacher to be excited about it himself; if he is not, no amount of pedagogical training will make up for the defect.

—*R. L. Wilder*

On the Superlinear Convergence of the Secant Method

Marco Vianello and Renato Zanovello

This note is devoted to filling a gap present in most of the numerical analysis textbooks, concerning the discussion on the superlinear convergence of the secant method.

Let us consider the secant method for the numerical solution of $f(x) = 0$ (cf., e.g., [**2**, §6.4])

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}, \qquad f(x_n) \neq f(x_{n-1}), \quad n \geq 1. \tag{1}$$

It is well known that the method converges, for sufficiently good initial approximations x_0 and x_1, if $f'(\xi) \neq 0$ and $f(x)$ has a continuous second order derivative (at least in a neighborhood of the zero ξ). It is also known (cf., e.g., [**1**, §3.5]) that the fundamental three-term recurrence relation holds

$$e_{n+1} = -\frac{f[x_{n-1}, x_n, \xi]}{f[x_{n-1}, x_n]} e_n e_{n-1}, \qquad n \geq 1, \tag{2}$$

where $e_n = \xi - x_n$, and where $f[t_0, \dots, t_m]$ denotes the mth divided difference at the points $t_0, \dots, t_m$ (cf., e.g., [**1**, §2.3]). From (2) follows

$$|e_{n+1}| = c_n |e_n|\,|e_{n-1}|$$

$$c_n = \frac{1}{2}\left|\frac{f''(\xi_n)}{f'(\eta_n)}\right|, \qquad \xi_n \in \operatorname{conh}(x_{n-1}, x_n, \xi), \qquad \eta_n \in \operatorname{conh}(x_{n-1}, x_n), \tag{3}$$

$\operatorname{conh}(t_0, \dots, t_m)$ denoting the open convex-hull of the points $t_0, \dots, t_m$.

However, the determination of the order of convergence and of the asymptotic error constant is carried out unsatisfactorily in most of the textbooks. Indeed, either the discussion is heuristic in nature, or, having assumed that

$$\lim_{n \to \infty} \frac{|e_{n+1}|}{|e_n|^p} = C \tag{4}$$

for some positive constants p and C, their respective values are deduced.

Anyway for a rigorous treatment the usual reference is the classical book by Ostrowski [**3**], where, in addition, hypotheses on the third derivative $f'''(x)$ are introduced in order to study the asymptotic behavior.

In [**4**] the superlinear convergence property is proved without resorting to the third derivative, but such a proof is long and complex because of its generality, being addressed to a wide class of iterative methods.

The purpose of this note is to provide a rigorous and quite simple proof of the superlinear convergence of the secant method, under natural assumptions on $f(x)$.

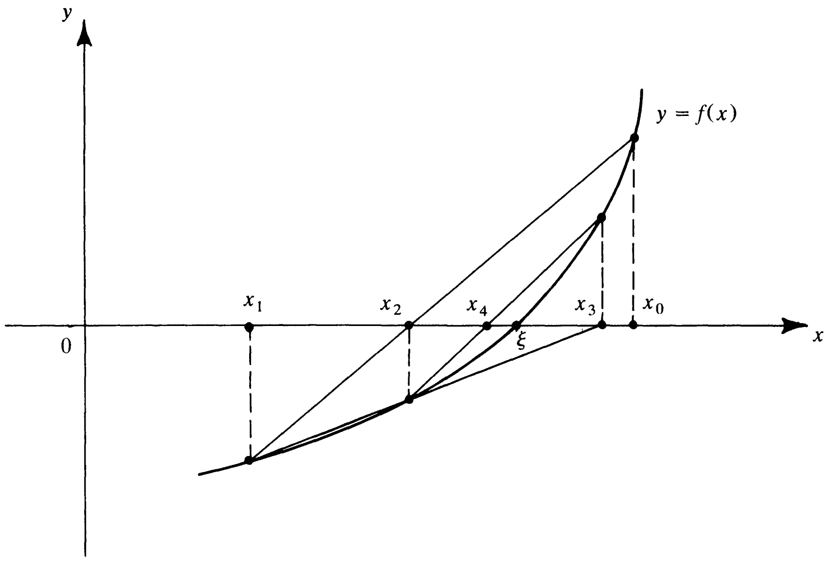

Figure 1. The Secant Method for finding roots. It is classically known that under mild restrictions, convergence is superlinear, and the errors satisfy a simple (Fibonacci-like) recurrence. There is a simple and elegant proof of this fact.

Following [**1**, §3.5, p. 103], setting $y_n = |e_n|/|e_{n-1}|^p$, $n \geq 1$ with $p > 0$, it is immediately seen from (3) that

$$y_{n+1} = c_n y_n^{-1/p}, \qquad n \geq 1 \tag{5}$$

if p is the positive solution of $t^2 - t - 1 = 0$, i.e. p is the "golden ratio" $(1 + \sqrt{5})/2$. Now, assuming $f''(\xi) \neq 0$, we'll prove that

$$\lim_{n \to \infty} y_n = \left[\frac{1}{2} \left| \frac{f''(\xi)}{f'(\xi)} \right| \right]^{1/p} = C. \tag{6}$$

Taking logarithms in (5) and defining $z_n = \ln(y_n)$, $\alpha_n = \ln(c_n)$, we get the new first order linear difference equation

$$z_{n+1} = \alpha_n - \frac{1}{p} z_n, \qquad n \geq 1 \tag{7}$$

that can be immediately solved by recurrence, obtaining

$$z_n = \left(-\frac{1}{p} \right)^{n-1} z_1 + S_n, \qquad n \geq 2, \tag{8}$$

where

$$S_n = \sum_{j=0}^{n-2} \alpha_{n-j-1} \left(-\frac{1}{p} \right)^j. \tag{9}$$

Since $p > 1$, $\{z_n\}$ (and hence $\{y_n\}$) converges if and only if $\{S_n\}$ converges.

At this point it is clear that the problem can be reduced to studying the asymptotic behavior of a sequence like

$$\sigma_n = \sum_{j=0}^{n} a_{n-j} b_j \tag{10}$$

where

$$\lim_{n\to\infty} a_n = a \quad \text{and} \quad \sum_{j=0}^{\infty} |b_j| < \infty. \tag{11}$$

The sequence defined by (10) is usually termed the *convolution* of the two sequences $\{a_n\}$ and $\{b_n\}$. It naturally appears, for instance, as general term in the Cauchy product of the series $\Sigma^\infty a_n$, $\Sigma^\infty b_n$.

Now we'll prove that

$$\lim_{n\to\infty} \sigma_n = ab, \qquad b = \sum_{j=0}^{\infty} b_j, \tag{12}$$

exploiting essentially the approach used in proving the well-known Cesaro's theorem for sequences. A second, more abstract proof of (12), which we omit for brevity, could be given by the well-known dominated convergence theorem.

Proof: Without any loss of generality we can assume $a = 0$ in (11). In fact

$$\sigma_n - ab = \sum_{j=0}^{n} (a_{n-j} - a)b_j - a \sum_{j=n+1}^{\infty} b_j$$

and by the summability of $\{b_n\}$ the second term in the right-hand side above is infinitesimal as $n \to \infty$. Let us split the sum (10) in the following way

$$\sigma_n = \sum_{j=0}^{m} a_{n-j}b_j + \sum_{j=m+1}^{n} a_{n-j}b_j, \tag{13}$$

where $m \geq 0$, $n \geq m + 1$. Fix $\varepsilon > 0$. In view of (11) we can determine two positive indexes

$$\nu_1(\varepsilon) \text{ such that } |a_k| < \frac{\varepsilon}{2\beta} \text{ for } k \geq \nu_1$$

$$\nu_2(\varepsilon) \text{ such that } \sum_{j=\nu_2}^{\infty} |b_j| < \frac{\varepsilon}{2M},$$

where M is an upper bound for $|a_n|$ and $\beta = \Sigma_{j=0}^{\infty}|b_j|$. It follows from (13) with $m + 1 = \nu_2$ that

$$|\sigma_n| < \varepsilon \quad \text{for } n \geq \nu(\varepsilon) = \nu_1 + \nu_2. \qquad \blacksquare$$

Going back to the sequence $\{S_n\}$ defined in (9) we finally obtain from (3), (5), (12)

$$\lim_{n\to\infty} S_n = \left(\lim_{n\to\infty} \alpha_n\right) \sum_{j=0}^{\infty} \left(-\frac{1}{p}\right)^j = \ln(c)\frac{p}{p+1} = \ln(c^{1/p}),$$

where $c = \dfrac{1}{2}|f''(\xi)/f'(\xi)|$ and hence by (8)

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \exp(z_n) = c^{1/p} \neq 0,$$

i.e., the secant method has order of convergence $p = (1 + \sqrt{5})/2$ and asymptotic error constant

$$C = \left[\frac{1}{2}\left|\frac{f''(\xi)}{f'(\xi)}\right|\right]^{1/p}$$

A last remark has to be made. The discussion above is based on the assumption $f''(\xi) \neq 0$. If, on the contrary, $f''(\xi) = 0$, excluding the trivial case that the method yields the root in a finite number of steps, we have $f''(\xi_n) \neq 0$ for all n. Then $\lim_{n\to\infty} \ln(c_n) = -\infty$ and hence $C = \lim_{n\to\infty} y_n = 0$. Thus the convergence order of the secant method may be greater than p. To conclude we can say, following e.g. [4], that the convergence of the secant method is superlinear.

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Any intelligent man may now, by resolutely applying himself for a few years to mathematics, learn more than the great Newton knew after half a century of study and meditation.

—*Macaulay*

How to Integrate Rational Functions

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The increasing availability of computer algebra systems has raised questions about how traditional topics in calculus are to be taught. In this note we look at integration of rational functions and propose a different approach, which has the following advantages: i) it is easily implemented on a computer or calculator algebra system, ii) it allows the students to use the computer algebra system in a meaningful way, and avoids routine calculations by hand, iii) it provides the students with some understanding of the general methods computer algebra systems actually use to integrate rational functions.

Rational function integration is important for itself and also because many integrals can be reduced to it by suitable substitutions, for example many trigonometric integrals and the so-called binomial integral [Subramaniam, Klambauer]. A rational function is traditionally integrated by expressing it in partial fractions form. This involves the following steps:

(1) Factor the denominator into linear and irreducible quadratic factors.

(2) Find the partial fraction decomposition. This involves solving a system of linear equations, with as many equations and unknowns as the degree of the polynomial in the denominator.

(3) Integrate each partial fraction. Those involving a quadratic factor require a trigonometric substitution or a reduction formula.

In the light of this recipe, consider the following integrals (of which the second and third are taken from our references):

$$\int \frac{8x^5 - 10x^4 + 5}{(2x^5 - 10x + 5)^2}\,dx$$

$$\int \frac{4x^5 - 1}{(x^5 + x + 1)^2}\,dx$$

$$\int \frac{4x^4 + 4x^3 + 16x^2 + 12x + 8}{x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1}\,dx$$

$$\int \frac{dx}{x^7 + 1}.$$

Each of these—as we shall see—has a simple antiderivative. However, in the first, the denominator is not solvable by radicals [Hungerford] and we cannot even get started, except by using numerical approximations to the roots. In the second and third the denominators factorize over the integers though this is not obvious. In the fourth the roots of the denominator are the seventh roots of unity. The partial fractions computation is quite involved. In these problems, even after factorization there is a great deal of algebra and integration left to do. Clearly this method can be quite tedious, if not impossible.

Recall that the integral of a rational function is the sum of a rational function together with a sum of logarithms and arctangents of polynomials. These are called respectively the rational and the transcendental parts of the integral. In this note we show how the rational part can be found without any integration, even when the factorization of the denominator is not known. All that is needed is the ability to calculate the g.c.d. of polynomials and to solve systems of linear equations. The algorithm is simple enough to work on any computer algebra system, even an HP-28S calculator. (In an appendix we briefly consider the HP-28S implementation.) We also consider to what extent the transcendental part can be determined. What follows is scattered between classical sources and the computer algebra literature and does not appear to be well known. We feel it is useful to write down an elementary and coherent account. Our bibliography should be consulted for a deeper study.

I. In what follows, P/Q is a rational function over the rationals; we assume that the leading coefficient of Q is one. We begin by proving the following proposition (the Hermite-Ostrogradski formula). Our proof is simpler than the ones we have been able to find (exemplified by [Davenport et al.] and [Klambauer]). We avoid the use of partial fractions.

Proposition 1. *Let P/Q be a rational function. Let $Q = \prod_{i=1}^{n} h_i^{\alpha_i}$ be the factorization of Q into linear and irreducible quadratic factors, and let $Q_1 = \prod_{i=1}^{n} h_i^{\alpha_i - 1}$ and $Q_2 = \prod_{i=1}^{n} h_i$. Then there are polynomials P_1 and P_2 such that*

$$\int \frac{P(x)}{Q(x)}\,dx = \frac{P_1(x)}{Q_1(x)} + \int \frac{P_2(x)}{Q_2(x)}\,dx. \tag{1}$$

Note that the proposition says what is intuitively clear—in a partial fractions decomposition, the repeated factors of the denominator give us the rational part and the factors without repetition give us the transcendental part.

We prove this proposition by considering two cases. The first case is when Q has only one distinct irreducible factor: either $Q(x) = (x - c)^m$ or $Q(x) = (x^2 + ax + b)^m$ where the quadratic is irreducible and $m \geq 1$.

If $Q(x) = (x - c)^m$ then our integral is

$$\int \frac{P(x)}{(x-c)^m}\,dx,$$

where $P(x)$ is a polynomial. Write $P(x) = \sum_{k=0}^{n} a_k (x - c)^k$. Then

$$\int \frac{P(x)}{(x-c)^m}\,dx = \sum_{k=-m}^{n-m} a_{k+m} \int (x-c)^k\,dx.$$

If we integrate all the terms except the one for which $k = -1$, we get the desired equation (1), since $Q_2 = x - c$ and the integrated terms have the common denominator $Q_1 = (x - c)^{m-1}$.

If $Q(x) = (x^2 + ax + b)^m$, we can essentially do the same thing, but it becomes slightly more complicated. This is the price we pay for avoiding complex arithmetic. Divide $P(x)$ by $Q_2(x) = x^2 + ax + b$: $P(x) = R(x)Q_2(x) + S(x)$, where $S(x)$ is linear. It follows that

$$\int \frac{P(x)}{Q(x)}\,dx = \int \frac{R(x)}{Q_2(x)^{m-1}}\,dx + \int \frac{S(x)}{Q_2(x)^m}\,dx.$$

There is a standard reduction formula (easily obtained by integration by parts) of the form

$$\int \frac{Ax + B}{(x^2 + ax + b)^m}\,dx = \frac{M(x)}{(x^2 + ax + b)^{m-1}} + \int \frac{N}{(x^2 + ax + b)^{m-1}}\,dx,$$

where $M(x)$ is a linear polynomial and N is constant. This formula, applied repeatedly, yields

$$\int \frac{Ax + B}{(x^2 + ax + b)^m}\,dx = \frac{M(x)}{(x^2 + ax + b)^{m-1}} + \int \frac{N}{x^2 + ax + b}\,dx,$$

where now $M(x)$ is a polynomial and N is a constant. From this formula we obtain

$$\int \frac{P(x)}{Q(x)}\,dx = \int \frac{R(x)}{Q_2(x)^{m-1}}\,dx + \frac{M(x)}{Q_2(x)^{m-1}} + \int \frac{N}{Q_2(x)}\,dx.$$

The process can be repeated on $\int R(x)/Q_2(x)^{m-1}\,dx$; this will ultimately lead to the equation (1), since $Q_2(x) = (x^2 + ax + b)$ and $Q_1(x) = (x^2 + ax + b)^{m-1}$.

In the second case, when Q has at least two distinct irreducible factors, we proceed by induction on the number k of distinct irreducible factors. Accordingly, assume that (1) holds for $k < K$ $(K > 1)$. Let $Q(x)$ have K distinct irreducible factors and let $Q(x) = \prod_{i=1}^{K} h_i^{\alpha_i}$ be the irreducible factorization of Q. Since h_1 and $\prod_{i=2}^{K} h_i(x)^{\alpha_i} = g(x)$ are relatively prime, by the Euclidean algorithm for polynomials there are polynomials $a(x)$ and $b(x)$ for which

$$P(x) = a(x)h_1(x)^{\alpha_1} + b(x)g(x).$$

Then

$$\int \frac{P(x)}{Q(x)}\,dx = \int \frac{a(x)}{g(x)}\,dx + \int \frac{b(x)}{h_1(x)^{\alpha_1}}\,dx.$$

By the inductive hypothesis and the first case, each integral on the right can be expressed in the form (1). If we write them that way and collect terms, we have the formula (1) for $\int P/Q$. The proof is complete.

We remark that if degree $P <$ degree Q then P_1 and P_2 can be found with degree $P_1 <$ degree Q_1 and degree $P_2 <$ degree Q_2. Indeed, if degree $P_2 \geq$ degree Q_2, divide P_2 by Q_2, integrate the polynomial quotient and absorb it into P_1/Q_1. Now if degree $P_1 =$ degree Q_1, then P_1/Q_1 is a constant plus a proper rational function, and the constant may be dropped from the equation. Finally, if degree $P_1 >$ degree Q_1, then P_1/Q_1 is a polynomial of degree at least one plus a proper rational function. But this is impossible, for then the limit at infinity of the derivative of the right hand side of (1) would not be zero. In fact, P_1 and P_2 are unique. We do not prove this since we don't need this fact. Finally, note that the last integral in (1) is a sum of logarithms and arctangents.

II. The real utility of the Hermite-Ostrogradski formula comes from the fact that it is possible to calculate P_1, P_2, Q_1, and Q_2 without factorizing Q (see [Horowitz] or [Klambauer].) We now show how this can be done.

It is clear that $Q_1 =$ g.c.d.(Q, Q') and $Q_2 = Q/Q_1$. Also it is easy to see that Q_1 divides $Q_1'Q_2$ whence $S = Q_1'Q_2/Q_1$ is a polynomial. If we differentiate both sides

of (1) we get

$$P/Q = \frac{Q_1P_1' - P_1Q_1'}{Q_1^2} + \frac{P_2}{Q_2} = \frac{P_1' - P_1Q_1'/Q_1}{Q_1} + \frac{P_2}{Q_2}.$$

Clearing the denominators we have $P = P_1'Q_2 - P_1S + P_2Q_1$.

In this equation, P, Q_1, Q_2, and S are known polynomials and we can solve for P_1 and P_2 by the method of undetermined coefficients. Here then is the algorithm:

Input: Polynomials P and Q, with degree $P <$ degree Q.

Output: P_1/Q_1, the rational part of $\int P/Q$, and P_2/Q_2, the integrand of the transcendental part of $\int P/Q$.

(1) $Q_1 :=$ g.c.d.(Q, Q'); $Q_2 := Q/Q_1$
(2) $S := Q_1'Q_2/Q_1$
(3) $q :=$ degree Q_1; $p :=$ degree Q_2.
(4) Write $P_1(x) := A_{q-1}x^{q-1} + A_{q-2}x^{q-2} + \cdots + A_0$ and $P_2(x) := B_{p-1}x^{p-1} + B_{p-2}x^{p-2} + \cdots + B_0$.
(5) Compute $T := P_1'Q_2 - P_1S + P_2Q_1$.
(6) Equate the coefficients of T with those of P.
(7) Solve this linear system of equations for the unknowns A_i and B_i.

If $\deg Q = d$, then in step 7 we solve a system of d equations in d unknowns, which is the same amount of work as in the method of partial fractions, except that now there is no integration left to do for the rational part. The algorithm involves only polynomial arithmetic and solving systems of linear equations. We illustrate by an example (which was done on the HP-28S). It is example #3 of the introduction.

Example:

$$\int \frac{4x^4 + 4x^3 + 16x^2 + 12x + 8}{x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1}\,dx.$$

$$\begin{aligned}
Q_1 &= g.c.d.(x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1,\\
&\qquad 6x^5 + 10x^4 + 12x^3 + 12x^2 + 6x + 2)\\
&= x^3 + x^2 + x + 1\\
Q_2 &= Q/Q_1 = x^3 + x^2 + x + 1\\
P_1 &= Ax^2 + Bx + C, \qquad P_2 = Dx^2 + Ex + F\\
T &= P_1'Q_2 - P_1S + P_2Q_1\\
&= Dx^5 + (-A + D + E)x^4 + (-2B + D + E + F)x^3\\
&\quad + (A - B - 3C + D + E + F)x^2 + (2A - 2C + E + F)x\\
&\quad + (B - C + F).
\end{aligned}$$

Equating coefficients with $P = 4x^4 + 4x^3 + 16x^2 + 12x + 8$ and solving the resulting system of equations for A, B, C, D, E, and F, we get the result

$$-\frac{x^2 - x + 4}{x^3 + x^2 + x + 1} + \int \frac{3x + 3}{x^3 + x^2 + x + 1}\,dx.$$

The last term is

$$3\int \frac{dx}{x^2+1} = 3\tan^{-1} x.$$

Examples #1 and #2 of the introduction can be worked the same way. One finds that

$$\int \frac{8x^5 - 10x^4 + 5}{(2x^5 - 10x + 5)^2}\, dx = \frac{1 - x}{2x^5 - 10x + 5}$$

and

$$\int \frac{4x^5 - 1}{(x^5 + x + 1)^2}\, dx = -\frac{x}{x^5 + x + 1}.$$

In a hand computation or when using a computer or calculator algebra system without a built-in g.c.d. function, the g.c.d. can be calculated using the algorithm of [Kung]. However, g.c.d. calculations can lead to a large increase in the size of the intermediate results, this is "intermediate expression swell" (see [Knuth] or [Collins]). See the Appendix for another way to determine the system of equations.

III. We have seen that the rational part and the integrand of the transcendental part can be found using only polynomial arithmetic and linear algebra. Since most of the computational complexity of the method of partial fractions comes from the repeated factors, this is a considerable simplification in that the denominator of the integral still to be evaluated is now square free. One could, of course, now use the method of partial fractions to evaluate this integral. We, however, show now that if the roots of the denominator are known, there is a closed formula for the transcendental part. In fact,

$$\int P(x)/Q(x)\, dx = \sum \frac{P(a)}{Q'(a)} \operatorname{Log}(x - a), \tag{2}$$

where the sum ranges over all the roots a of $Q(x)$ (including the complex ones). In this formula, we use the complex logarithm.

To establish this formula, note that we are assuming that Q has no repeated roots and we may assume degree $P <$ degree Q. Let a be a root of $Q(x)$, and write $Q(x) = (x - a)Q_1(x)$, with $Q_1(a) \neq 0$. (Note that we are using Q_1 with a different meaning now.) We wish to write

$$P/Q = \frac{A}{x - a} + \frac{P_1(x)}{Q_1(x)}$$

for a constant A and polynomial P_1 (again, we use P_1 with a different meaning). This is possible, for if we choose $A = P(a)/Q_1(a)$ then

$$\begin{aligned} P_1(x) &= Q_1(x)\left\{\frac{P(x)}{Q(x)} - \frac{P(a)}{Q_1(a)}\frac{1}{x - a}\right\} \\ &= \frac{1}{x - a}\left\{P(x) - \frac{P(a)}{Q_1(a)}Q_1(x)\right\} \end{aligned} \tag{3}$$

Since $P(x) - (P(a)/Q_1(a))Q_1(x)$ has a as a root, $P_1(x)$ is a polynomial. Also $Q'(a) = Q_1(a)$, and thus we have

$$P(x)/Q(x) = \frac{P(a)/Q'(a)}{x - a} + \frac{P_1(x)}{Q_1(x)}.$$

We now establish that

$$\frac{P_1(b)}{Q_1'(b)} = \frac{P(b)}{Q'(b)}$$

for every root b of Q_1. First note that $Q'(x) = (x - a)Q_1'(x) + Q_1(x)$, so $Q'(b) = (b - a)Q_1'(b)$. Also $P_1(b) = P(b)/(b - a)$ by (3). It follows that $P(b)/Q'(b) = P_1(b)/Q_1'(b)$. We may now repeat our process, expressing

$$P_1(x)/Q_1(x) = \frac{P_1(b)/Q_1'(b)}{x - b} + \frac{P_2(x)}{Q_2(x)}$$

where $Q_1(x) = (x - b)Q_2(x)$ and P_2 is a polynomial. We now have

$$P(x)/Q(x) = \frac{P(a)/Q'(a)}{x - a} + \frac{P(b)/Q'(b)}{x - b} + \frac{P_2(x)}{Q_2(x)}$$

with

$$\frac{P_2(c)}{Q_2'(c)} = \frac{P(c)}{Q'(c)}$$

for every root c of $Q_2(x)$. Since the degrees of the polynomials $P(x), P_1(x), P_2(x), \ldots$ strictly decrease, we eventually arrive at the formula

$$P(x)/Q(x) = \sum_{a \mid Q(a)=0} \frac{P(a)/Q'(a)}{x - a}.$$

If we integrate each term we obtain the formula (2). However, if $P(x)$ and $Q(x)$ are real polynomials, the complex roots of $Q(x)$ come in conjugate pairs and a real formula for $\int P(x)/Q(x)\,dx$ can be obtained as follows.

Let $\bar{a}$ and a be a complex conjugate pair of roots of $Q(x)$.

If $P(a)/Q'(a) = c + id$, then $P(\bar{a})/Q'(\bar{a}) = c - id$, and

$$\begin{aligned}
&\frac{P(a)}{Q'(a)}\frac{1}{x - a} + \frac{P(\bar{a})}{Q'(\bar{a})}\frac{1}{x - \bar{a}} \\
&\quad = c\left(\frac{1}{x - a} + \frac{1}{x - \bar{a}}\right) + id\left(\frac{1}{x - a} - \frac{1}{x - \bar{a}}\right) \\
&\quad = c\frac{2x - 2\operatorname{Re}(a)}{x^2 - 2\operatorname{Re}(a) + |a|^2} - 2d\frac{\operatorname{Im}(a)}{x^2 - 2\operatorname{Re}(a) + |a|^2}.
\end{aligned}$$

Write $\alpha = \operatorname{Re}(a)$ and $\beta = \operatorname{Im}(a)$. We have

$$\begin{aligned}
\int \frac{P(a)}{Q'(a)}\frac{dx}{x - a} + \int \frac{P(\bar{a})}{Q'(\bar{a})}\frac{dx}{x - \bar{a}} &= c\log\left|(x - \alpha)^2 + \beta^2\right| - 2d\int \frac{\beta\,dx}{(x - \alpha)^2 + \beta^2} \\
&= c\log\left|(x - \alpha)^2 + \beta^2\right| - 2d\arctan\left(\frac{x - \alpha}{\beta}\right).
\end{aligned}$$

Thus

$$\int \frac{P(x)}{Q(x)}\,dx = \sum \frac{P(a)}{Q'(a)} \log|x-a|$$
$$+\sum\left\{\mathrm{Re}\left(\frac{P(a)}{Q'(a)}\right)\log\left|(x-\mathrm{Re}(a))^2+(\mathrm{Im}(a))^2\right|\right.$$
$$\left.-2\,\mathrm{Im}\left(\frac{P(a)}{Q'(a)}\right)\arctan\left(\frac{x-\mathrm{Re}(a)}{\mathrm{Im}(a)}\right)\right\},$$

where the first sum is over all real roots a, while the second sum is over all pairs of complex conjugate roots $a, \bar{a}$.

This formula is often superior to the method of partial fractions. For instance, if the roots are found numerically, finding the coefficients in partial fractions will compound round-off errors, unlike this formula. Even when the roots are known in a closed form this formula is preferable.

Example. $\int dx/(x^7+1)$ (this is example #4 of the introduction). The roots of the denominator are $w_n = e^{i(2n+1)\pi/7}$ for $0 \le n \le 6$, and

$$\frac{P(w_n)}{Q'(w_n)} = \frac{1}{7}e^{-i(2n+1)6\pi/7} \quad \text{for } 0 \le n \le 6.$$

Thus

$$\int \frac{dx}{x^7+1} = -\frac{1}{7}\ln|x+1|$$
$$+\frac{1}{7}\sum_{j=0}^{2}\left\{\cos\frac{(2j+1)6\pi}{7}\ln\left(x^2-2\cos\frac{(2j+1)\pi}{7}+1\right)\right.$$
$$\left.+2\sin\frac{(2j+1)6\pi}{7}\arctan\frac{x-\cos\frac{(2j+1)\pi}{7}}{\sin\frac{(2j+1)\pi}{7}}\right\}.$$

If the roots of the denominator are known in a closed form, the transcendental part of the integral can be written in a closed form.

IV. We now show that if the roots of the denominator cannot be expressed in a closed form, then in general the integral cannot be expressed in a closed form. First, we make precise what we mean by a closed form.

Definition: A field F is said to be a radical extension of $\mathscr{Q}$ if there is a chain of fields

$$\mathscr{Q} = F_0 \subseteq F_1 \cdots \subseteq F_n = F$$

such that for i with $1 \le i \le n$, $F_i = F_{i-1}(u_i)$ with some power of u_i in F_{i-1}.

Now in (2), collecting terms with the same coefficients we have

$$\int \frac{P(x)}{Q(x)}\,dx = \sum_i b_i \operatorname{Log} R_i(x) \tag{4}$$

where each $R_i(x)$ is a polynomial.

We say that $\int P/Q$ can be expressed in *closed form* if there is a radical extension F of $\mathcal{Q}$ with b_i in F and $R_i(x)$ in $F[x]$. Note that this simply means that b_i and the coefficients of $R_i(x)$ can be expressed by repeated use of arithmetical operations and root extractions on rational numbers.

We now have this proposition:

Proposition. *Suppose $\int P/Q$ can be expressed in closed form over F. If Q is irreducible over F then $P = CQ'$ for some C in F.*

Proof: We can assume that P and Q have no common factors, and can also assume that in (4) R_i and R_i' have no common factors, for otherwise R_i would have a repeated factor S^n and this would just give us another summand $b_i n \operatorname{Log} S$. Similarly, we may assume that R_i and R_j for $i \neq j$ have no common factors. Then differentiating (4) we have

$$PR_1 \cdots R_n = Q\sum_i b_i R_1 \cdots R_i' \cdots R_n.$$

Now R_j divides all the summands on the right except (by our assumption) $R_1 \cdots R_j' \cdots R_n$. Hence R_j, and more generally $R_1 \cdots R_n$, divides Q. Since P and Q have no common factors Q divides $R_1 \cdots R_n$ and $Q = C(R_1 \cdots R_n)$ for C in F. This contradicts our assumption that Q is irreducible over F (unless $n = 1$). Hence, say, $Q = CR_1$ and $P/Q = b_1 R_1'/R_1$.

As an example, since $x^2 - 2$ is irreducible over $\mathcal{Q}$, $\int dx/(x^2 - 2)$ cannot be written without irrationals. Risch [Risch] pointed out that this integral cannot be expressed without involving $\sqrt{2}$.

For another example, we have already noted that $2x^5 - 10x + 5$ cannot be solved by radicals. Hence this polynomial is irreducible over any radical extension F of $\mathcal{Q}$.

It follows that

$$\int \frac{P(x)}{2x^5 - 10x + 5}\,dx$$

cannot be expressed in closed form unless P is a multiple of $x^4 - 1$.

The close relationship between the problem of integrating rational functions in closed form and the solvability of polynomials in radicals is hardly surprising. As the recent book [Ebbinghaus, et al.] makes clear, the hard basic questions that led to the Fundamental Theorem of Algebra arose in part from the problem of integrating rational functions. As Hardy [Hardy] put it nearly a century ago, "The solution of the problem [of integration] in the case of rational functions may be said to be complete; for the difficulty with regard to the explicit solution of algebraic equations is not one of inadequate knowledge but of proved impossibility."

In particular cases we may be able to express the transcendental part of the integral without some or even any of the roots. Consider $\int x/(x^4 + 1)\,dx$. A calculus student would substitute $u = x^2$ and get the antiderivative $(\frac{1}{2})\arctan(x^2)$. It is instructive to work this example using our method (3) above. The roots of

$x^4 + 1$ are $\pm 1/\sqrt{2} \pm i/\sqrt{2}$. We get the solution

$$\int \frac{x\,dx}{x^4+1} = \frac{1}{2}\{\arctan(\sqrt{2}x - 1) - \arctan(\sqrt{2}x + 1)\}.$$

This incidentally is how Mathematica expresses the answer, which is of course expressed over $\mathscr{Q}(\sqrt{2})$. But by using the addition formula $\arctan A + \arctan B = \arctan((A+B)/(1-AB))$ we have

$$\int \frac{x\,dx}{x^4+1} = \frac{1}{2}\arctan\frac{-1}{x^2} = \frac{-1}{2}\left(\frac{\pi}{2} - \arctan(x^2)\right) = \frac{1}{2}\arctan(x^2) + C.$$

Thus the integral of a rational function may be expressible over a smaller field than the one that contains the roots of the denominator, i.e. its splitting field. Indeed the integration of the transcendental part turns on the solvability of a polynomial different from the denominator. (See [Trager] or [Lazard].)

Definition: If $b = P(a)/Q'(a)$ for some root a of Q, we say b is a residue of P/Q.

Observe that:

1) b is a residue if and only if $P(a) - bQ'(a) = 0$ for some a such that $Q(a) = 0$, and this holds if and only if $P(x) - bQ'(x)$ and $Q(x)$ have a common root.

2) If g.c.d.$(P(x) - bQ'(x), Q(x)) = R(x)$, then the roots of $R(x)$ are precisely the roots of $Q(x)$ which have b as their residue.

We collect together terms with the same coefficients in (2) to get

$$\int \frac{P(x)}{Q(x)}\,dx = \sum_i b_i \operatorname{Log} R_i(x),$$

where the b_i are the complex numbers b such that $P - bQ'$ and Q have a common root and $R_i(x) = \text{g.c.d.}(P - b_iQ', Q)$. Thus if we can compute the residues b_i, a g.c.d. calculation (perhaps over an extension field) will give us the integral.

The problem of finding common roots of two polynomials is classical and is solved in terms of the resultant of the polynomials [Uspensky, Knuth, Griffiths, Davenport et al.]. We can avoid the resultant by realizing that if $P(x) - bQ'(x)$ and $Q(x)$ have a common factor then if we calculate their g.c.d. we will obtain as a remainder a polynomial in b which must be zero (the first remainder which is independent of x). This is a factor of the resultant. The calculation also yields the g.c.d. in terms of b.

We illustrate this by redoing our previous example $\int x/(x^4+1)\,dx$.

We need to find b such that $x - 4bx^3$ and $x^4 + 1$ have a common root. We compute their g.c.d. (the algorithm of [Kung] works nicely) and get the polynomial $1 + 16b^2 = 0$, with the g.c.d. being $1 - 4bx^2$. (The resultant is $(1 + 16b^2)^2$.) Thus $b = i/4$ or $-i/4$. We substitute these values into the g.c.d. $1 - 4bx^2$ to obtain

$$\int \frac{x}{x^4+1}\,dx = \sum_i b_i \operatorname{Log} R_i(x) = \frac{i}{4}\operatorname{Log}(1 - ix^2) - \frac{i}{4}\operatorname{Log}(1 + ix^2)$$

$$= -\frac{i}{4}\operatorname{Log}\left(\frac{x^2 - i}{x^2 + i}\right).$$

The answer is expressed over $\mathscr{Q}(i)$, the splitting field of $1 + 16b^2$. It is the further

relation between log and arctan

$$\arctan x = \left(\frac{1}{2i}\right)\log((x-i)/(x+i)) - \frac{\pi}{2},$$

which gives us the answer $(\frac{1}{2})\arctan(x^2)$ over $\mathscr{Q}$. This example illustrates that if the resultant has multiple roots then the integral may be expressible over a smaller field than the splitting field of the denominator. If not, then the problem of finding the residues is no better than the problem of finding the roots of the denominator. In that sense Hardy's observation still holds. It is worth noting that even in the case where the denominator is a cubic with three real roots, in general $\mathscr{Q}(i)$ will be required to express the integral, since the roots in general cannot be expressed in closed form using real radicals only.

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APPENDIX. We briefly discuss the HP-28S implementation. The routines for polynomial arithmetic can be found in the booklet *Mathematical Applications* published by the Hewlett-Packard Company in 1988. These can even be made to work over rational arithmetic, using routines available from the first-named author. In these routines, a polynomial is stored as a list of coefficients. The denominator in example #3 of the introduction for instance is stored as $\{1,2,3,4,3,2,1\}$. At the end of step 5 we get the polynomial T represented by the list $\{D,\ -A+D+E,\ -2B+D+E+F,\ A-B-3C+D+E+F,\ 2A-2C+E+F,\ B-C+F\}$. Now if we set all the variables except A to be zero and A to be one (and successively for the other variables) we get a matrix whose transpose is the coefficient matrix for the system of equations in step 7. This is easily implemented on the HP-28S. The first named author will be happy to provide the codes on request.

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Ha! All that time wasted! It's true but not provable.

Picture Puzzle

(from the collection of Paul Halmos)

Will he probably hit the right key?

(*See page* 796.)

> If Euclid failed to kindle your youthful enthusiasm, then you were not born to be a scientific thinker.
>
> —*Einstein*

Composite solution by all solvers. Setting $m = n = 0$ gives $f(0) = 2f(0)^2$, which implies that $f(0) = 0$ since $f(0)$ is an integer. Setting $m = 1$, $n = 0$ gives $f(1) = f(1)^2$ which implies $f(1) = 1$ since $f(1) > 0$. Now, direct application of the given identity shows that $f(2) = f(1^2 + 1^2) = f(1)^2 + f(1)^2 = 2$. Taking $m = 2$ and $n = 0$, 1 or 2 gives $f(4) = 4$, $f(5) = 5$, and $f(8) = 8$. To fill in some gaps, note that $m^2 + n^2 = k^2 + l^2$ implies that $f(m)^2 + f(n)^2 = f(k)^2 + f(l)^2$. Thus $f(3) = 3$ follows from $3^2 + 4^2 = 0^2 + 5^2$. The equations $7^2 + 1^2 = 5^2 + 5^2$, $9 = 3^2 + 0^2$, $10 = 3^2 + 1^2$, and $6^2 + 8^2 = 10^2 + 0^2$ then establish $f(n) = n$ for all $n \leq 10$. These cases form a basis for a proof by induction. Suppose that $m > 10$ and $f(l) = l$ for all $l < m$. If m is odd, write $m = 2k + 1$ and employ

$$(2k+1)^2 + (k-2)^2 = (2k-1)^2 + (k+2)^2;$$

and if m is even, write $m = 2k + 2$ and employ

$$(2k+2)^2 + (k-4)^2 = (2k-2)^2 + (k+4)^2.$$

Editorial comment. A variety of quadratic sum identities were employed by solvers in the inductive step—some falling into as many as six different modular parity cases. Those given here are special cases of

$$(ru+sv)^2 + (rv-su)^2 = (ru-sv)^2 + (rv+su)^2$$

which occurs in the classical study of representations of integers as sums of two squares.

The proposer also considered the following proposition:

Let $f\colon \mathbb{N} \to \mathbb{C}$ be unbounded and satisfy $f(a^2 + b^2) = f(a)^2 + f(b)^2$ for all sufficiently large a and b. Then $f(n)^2 = n^2$ for all sufficiently large n.

The proof is considerably more involved than that of the published version, although similar methods are employed.

Several solvers pointed out that without the assumption that $f(1) > 0$, the function $f(n) = 0$ for all n would also satisfy the conditions of the problem. Four readers solved only the analogous problem for a function defined on the non-negative *real* numbers.

Solved by the proposer and 37 others.

Collaborating editors: David F. Appleyard, Paul T. Bateman, Bruce C. Berndt, Duane M. Broline, Barry W. Brunson, Frank S. Cater, Gulbank D. Chakerian, Underwood Dudley, Gerald A. Edgar, Michael A. Filaseta, Ira M. Gessel, Richard A. Gibbs, Douglas A. Hensley, John R. Isbell, Mourad E. H. Ismail, Murray Klamkin, Daniel J. Kleitman, Frederick W. Luttmann, Frank B. Miles, Richard Pfiefer, Stephen L. Portnoy, J. O. Shallit, John Henry Steelman, Kenneth B. Stolarsky, Douglas B. Tyler, Daniel Ullman, Edward T. H. Wang, and William E. Watkins.

ANSWER TO PICTURE PUZZLE: The great probabilist Andrej Nikolajevich Kolmogorov.

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In this department the MONTHLY presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial or related results. Typescripts should be sent to Richard Guy, Department of Mathematics & Statistics, The University of Calgary, Alberta, Canada T2N 1N4.

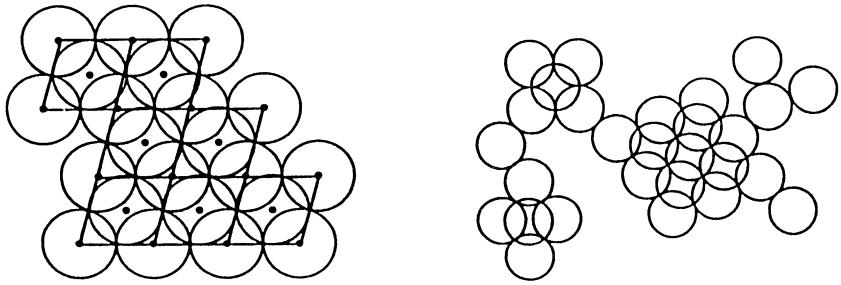
On the Intersection Points of Unit Circles

András Bezdek

Let n distinct unit circles be arranged in the Euclidean plane so that their union is a connected set. A point is called an *intersection point* if it belongs to at least two of the circles. Show that the given circles determine at least $n - 1$ intersection points. Show also that the number of intersection points is minimal only for *tree-arrangements* defined in the following paragraph:

To ease the description of tree-arrangements consider first a finite number of translates of a given 2 by 2 rhombus with angles $> 60°$ so that their union is a simple connected set, and any two rhombi are either (i) disjoint, or (ii) have a vertex in common, or (iii) have an edge in common. The family of those unit circles which are centered either at the vertices or at the centers of these rhombi (FIGURE 1(a)) is called a *cluster*. Arrange finitely many clusters (which might be generated by different rhombi) in the plane so that they form a *tree* (FIGURE 1(b)), meaning that one can label the clusters by $1, \dots, N$ so that the union of the circles of cluster i has exactly one common point with the union of the circles of the clusters $1, \dots, (i - 1)$. The circles of the clusters of a tree are said to form a tree-arrangement. A simple count shows that the number of intersection points in a cluster (and therefore in any tree) is one less than the number of circles.

To my knowledge the above problem has not been considered before, which is rather surprising, since the answer to the question concerning lines (the other most obvious configuration in the plane) is known. In fact, using a well known theorem of Sylvester it can be shown [4] that n lines in the plane determine at least $n - 1$ intersection points, unless they are all parallel to each other or they all go through the same point. Still, I must admit that a result of K. Bezdek & R. Connelly was the one which led me to raising my question. One could say that in [2] the authors considered the infinite version of the above problem. Using their terminology the *dual* of a family $\mathscr{C}$ of distinct unit discs is defined by the family $\mathscr{C}$ of all unit discs



(a) (b)

Figure 1. (a) A cluster, (b) a tree.

which are centered at points belonging to the boundary of at least two members of the family $\mathscr{C}$. Bezdek & Connelly showed that if no two of the discs are tangent and the family has no isolated disc then the density of the dual arrangement is at least that of the original arrangement (the density may be interpreted roughly as the total area of the discs divided by the area of the whole plane. For a more rigorous definition see [**2**]). We present here the elegant argument of Bezdek & Connelly in such a form that it resolves our problem in a special case:

Theorem. *Let $\mathscr{C}$ be a family of n unit circles such that their union is a connected set. If no two of the circles are tangent, then there are at least n intersection points.*

Assign to each intersection point a charge equal to 1. Distribute equally each of these charges among those circles which pass through the particular intersection point. Consider now the same distribution of charges from the circles' point of view. Suppose a particular circle c gets its least charge, say $1/k$, from the intersection point P. This means that P is contained by k circles. One of them is c. Since no two of the circles are tangent, each of the remaining $k - 1$ circles intersects c once more. By the definition of P, c gets from each of these new intersection points at least a charge $1/k$, altogether it gets a charge ≥ 1. Since the same holds for each of the n circles, we have that there are at least n intersection points. ∎

For a generalization of the Theorem see [**1**].

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PROBLEMS AND SOLUTIONS

Edited by: **Richard T. Bumby, Fred Kochman and Douglas B. West**

Proposed problems should be sent to the MONTHLY PROBLEMS address given on the inside front cover. Please include solutions, relevant references, etc. Three copies are requested.

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*An asterisk (*) after the number of a problem, or part of a problem, indicates that no solution is currently available. Partial solutions will be useful in such cases. Otherwise, the published solution is likely to be based on a solution which is complete and correct. Of course, an elegant partial solution or a method leading to a more general result is always useful and welcome. In addition, references to other appearances of MONTHLY problems or to solutions of these problems in the literature are also solicited.*

PROBLEMS

10247. *Proposed by Cristian Turcu, London, U.K.*

For a fixed real number A, define a sequence $\{X_n: n \geqslant 0\}$ by

$$X_0 = 0 \qquad \text{and} \qquad X_{n+1} = \frac{3X_n - \sqrt{5X_n^2 + 4A^2}}{2} \quad \text{for } n \geq 0.$$

(a) For which A is the sequence X_n convergent?
(b) For which A are all $X_n \in \mathbb{Z}$.

10248. *Proposed by Michael B. Handelsman, Erasmus Hall High School, Brooklyn, NY.*

Candidates Smith and Jones are the only two contestants in an election that will be deadlocked when all the votes are counted—each will receive $2n$ of the $4n$ votes cast. The ballot count is carried out with successive random selections from a single container. After exactly $2n$ votes are tallied, Smith has S votes and Jones has J votes. What is the expected value of $|S - J|$?

10249. *Proposed by O. Yumlu, Munich, Germany.*

Suppose that the inradius of an isosceles triangle and the ratio of the distances from its incenter to its vertices are given. Give a Euclidean construction of the triangle.

10250. *Proposed by Xin Li, University of Central Florida, Orlando, FL.*

Assume that $k \in \mathbb{Z}$, $k > 1$, and $\lambda \in \mathbb{R}$, $\lambda > 0$. Define

$$S(t) = \sin kt + \lambda \sin(k-1)t$$

and let $\langle t_i \rangle$ with $0 < t_1 < t_2 < \cdots < \pi$ be all zeros of $S'(t)$ in the interval $(0, \pi)$. Show that $|S(t_i)| > |S(t_{i+1})|$ for all i, i.e. that the sequence of relative maxima of $|S(t)|$ on this interval is strictly decreasing.

10251. *Proposed by J. G. Mauldon, Amherst College, Amherst, MA.*

Let $\mathscr{C}$ denote the unit cube, and let $\mathscr{P}$ be the set of all pairs $[\mathbf{a}, \mathbf{b}]$ with $\mathbf{a}$ and $\mathbf{b}$ mutually perpendicular line segments contained in $\mathscr{C}$.

(a) Evaluate $\sup\{\min\{|\mathbf{a}|, |\mathbf{b}|\} : [\mathbf{a}, \mathbf{b}] \in \mathscr{P}\}$.
(b) Deduce the area of the largest square, and the volume of the largest regular octahedron, that fit into $\mathscr{C}$.

10252. *Proposed by James S. Weber, The University of Illinois, Chicago, IL.*

An election is to be held with V voters who will rank A alternatives. It is said that alternative X is an "M-majority preference" over alternative Y if there are at least M voters who prefer X to Y. A "voter's paradox cycle" is an ordering of the alternatives $a_0, a_1, \ldots, a_{A-1}, a_A = a_0$ so that a_i is preferred over a_{i+1} for $0 \le i < A$. Prove that a voter's paradox cycle can exist for M-majority preference if and only if $AM \le V(A-1)$.

10253. *Proposed by W. Weston Meyer, General Motors Research and Environmental Staff, Warren, MI.*

Show that the quartic equation

$$z^4 - 2cz^3 + 2\bar{c}z - 1 = 0,$$

where c is a complex number with complex conjugate $\bar{c}$, has a root not on the unit circle $\{z : |z| = 1\}$ if and only if $(\Re c)^{1/3} + (\Im c)^{1/3} i$ lies outside this circle.

10254. *Proposed by E. Ehrhart, Université de Strasbourg, Strasbourg, France.*

The curve traced out by a fixed point of a closed convex curve as that curve rolls without slipping along a second curve will be called a "roulette". Let S be the area of one arch of a roulette traced out by an ellipse of area s rolling on a straight line. Prove or disprove that $S \ge 3s$, with equality only if the ellipse is a circle.

10255. *Proposed by Zalman Rubinstein, University of Haifa, Haifa, Israel*

Let $P_n(z)$ be a polynomial of degree n having no roots in the open unit disk

$$\mathscr{D} = \{z : |z| < 1\}.$$

(a) For all real η, show that the polynomial

$$P_n(z) - (1 - e^{i\eta})\frac{zP_n'(z)}{n}$$

also has no roots in $\mathscr{D}$.

(b) For $0 < p < 1$, show that

$$\int_0^{2\pi} |P'(e^{it})|^p \, dt \leq C_p n^p \int_0^{2\pi} |P(e^{it})|^p \, dt,$$

with

$$C_p = \frac{2\pi}{\int_0^{2\pi} |1 + e^{it}|^p \, dt} = \frac{2^{-p}\pi^{\frac{1}{2}}\Gamma\left(\frac{1}{2}p + 1\right)}{\Gamma\left(\frac{1}{2}p + \frac{1}{2}\right)}.$$

(c) Determine all polynomials for which the inequality in (b) becomes an equality.

NOTES

(10249) The terms "incenter" and "inradius" are abbreviations for "center and radius of the inscribed circle". **(10255)** The inequality in part (b) is an extension of Theorem 13 in N. G. deBruijn, *Nederl. Akad. Wetensch. Proc. Ser. A*, 50 (1947), 1265–1272. Recent work on inequalities of this type may be found in V. V. Arestov, *Mat. Zam.*, 48 (1990), 7–18.

SOLUTIONS

Harmonic Numbers

6616 [1989, 942]. *Proposed by Hugh M. W. Edgar, San Jose State University, CA.*

Let $d(n)$ denote the number of positive integral divisors of n and let $\sigma(n)$ denote the sum of these divisors. Let S be the set of positive integers with exactly two distinct prime factors (repeated prime factors are permitted). For $n \in S$ prove that the following three assertions are equivalent:

(1) n is an even perfect number, i.e., n is even and $\sigma(n) = 2n$;
(2) the harmonic mean of the divisors of n is integral, i.e., $nd(n)/\sigma(n)$ is an integer;
(3) $\sigma(n)$ has exactly the same prime factors as n.

Solution by David Callan, University of Wisconsin, Whitewater, WI. The equivalence of (1) and (3) was the subject of Problem 6036 [1975, 671; 1978, 830]. Thus we confine our attention to proving the equivalence of (1) and (2).

If (1) holds, Euler proved that $n = 2^{p-1}(2^p - 1)$, where both p and $2^p - 1$ are primes. Then $nd(n)/\sigma(n) = d(n)/2 = p$ and so (2) holds.

Thus it remains to show that if $n \in S$ and (2) holds, then (1) holds. Our rather lengthy proof of this will be divided into two parts after some preliminary lemmas. In Part A we show if n has exactly two prime factors and if $nd(n)/\sigma(n)$ is an integer, then n cannot be odd. In Part B we show that if n is even and has exactly one odd prime factor, and if $nd(n)/\sigma(n)$ is an integer, then n is perfect. First we give the lemmas.

Lemma 1. *Suppose p is an odd prime, q is an integer not divisible by p, and n is a positive integer. Let l denote the multiplicative order of q modulo p. Then $p|(q^n - 1)$ if and only if $l|n$, in which case $p^u \| (q^n - 1)/(q^l - 1)$ if $p^u \| n$. (The notation $p^u \| N$ means that $p^u | N$ but $p^{u+1} \nmid N$.)*

Proof: The first assertion goes back at least to Lagrange. This reduces the problem to E3445 (by taking $k = q^l$) whose solution appears later in this issue.

Lemma 2. *Suppose q is odd and n is a positive integer. Then $2|(q^n - 1)/(q - 1)$ if and only if $2|n$, in which case $2^{u-1} \| (q^n - 1)/(q^2 - 1)$ if $2^u \| n$.*

Proof: This also reduces to E3445.

Lemma 3. *Suppose k is a prime and t is any integer. Then $k|(1 + t + t^2 + \cdots + t^{k-1})$ if and only if $t \equiv 1 \pmod k$.*

Proof: If $t \equiv 1 \pmod k$, then clearly $1 + t + t^2 + \cdots + t^{k-1} \equiv 0 \pmod k$. If $t \not\equiv 1 \pmod k$, then by Fermat's theorem $(t - 1)(1 + t + t^2 + \cdots + t^{k-1}) = t^k - 1 \equiv t - 1 \pmod k$ and so $1 + t + t^2 + \cdots + t^{k-1} \equiv 1 \pmod k$.

Lemma 4. *Suppose k, l, p are primes with $p \equiv 1 \pmod l$. If $l|(1 + p + p^2 + \cdots + p^{k-1})$, then $k = l$.*

Proof: The congruence condition on p gives $1 + p + p^2 + \cdots + p^{k-1} \equiv k \pmod l$. Thus if $l|(1 + p + p^2 + \cdots + p^{k-1})$, we have $l|k$. Since k and l are primes, the conclusion follows.

PART A. Suppose $n = p^r q^s$, where p and q are distinct odd primes and r and s are positive integers. We assume that $nd(n)/\sigma(n) = m$, where m is an integer; this assertion may be written:

$$p^r q^s (r + 1)(s + 1) = m \frac{p^{r+1} - 1}{p - 1} \frac{q^{s+1} - 1}{q - 1}. \tag{4}$$

For the remainder of Part A let k be the multiplicative order of p modulo q and l be the multiplicative order of q modulo p. Thus $k|(q - 1)$ and $l|(p - 1)$.

First we show that $l|(s + 1)$. If not, then $p \nmid (q^{s+1} - 1)$ by Lemma 1. Since

$$(p^{r+1} - 1)/(p - 1) = 1 + p + p^2 + \cdots + p^r$$

is relatively prime to p, we have $p^r|m$ from (4). Since $(q^{s+1}-1)/(q-1)$ is relatively prime to q, (4) gives

$$(r+1)(s+1) = t(q^{s+1}-1)/(q-1) = t(1+q+q^2+\cdots+q^s)$$

for some integer t. Inserting this into (5) yields

$$q^s t = (m/p^r)(1+p+p^2+\cdots+p^r).$$

Hence $p^r < tq^s < (r+1)(s+1)$. Multiplying the inequalities $p^r < (r+1)(s+1)$ and $q^s < (r+1)(s+1)$, we get $p^r q^s < (r+1)^2(s+1)^2$. But this is impossible, since $3^r \geq (r+1)^2$ for $r \geq 2$, $5^s \geq (s+1)^2+1$ for $s \geq 1$, and the case $p=3$, $r=1$ leads to the absurdity $5^s \leq q^s < 2(s+1)$. Hence $l|(s+1)$ and, by symmetry, $k|(r+1)$.

Let us write $r+1 = ak$ and $s+1 = bl$. We shall show that each of the following three possibilities leads to a contradiction:

(I) $\max(a,b) \geq 3$, (II) $\max(a,b) = 2$, (III) $a=b=1$.

(I) $\max(a,b) \geq 3$. We may suppose $b \geq 3$. Since $(q^{s+1}-1)/(q^l-1)$ is relatively prime to q^s and since the multiplicities with which p divides $(q^{s+1}-1)/(q^l-1)$ and $s+1$ are the same by Lemma 1, (4) gives us that $(r+1)(s+1) = u(q^{s+1}-1)/(q^l-1)$ for some integer u. Thus from (4) we have

$$p^r q^s u = m\frac{p^{r+1}-1}{p-1}\frac{q^l-1}{q-1},$$

so that $(p^{r+1}-1)/(p-1)$ is a divisor of uq^s and

$$p^r < q^{l-1}uq^{s+1-l} < q^{l-1}u(q^{s+1}-1)/(q^l-1) = q^{l-1}(r+1)(s+1).$$

Multiplying the inequalities $p^r < (r+1)(s+1)q^{l-1}$ and $q^{s+1-l} < (r+1)(s+1)$ and using $s+1 = bl$, we obtain

$$p^r q^{(b-2)l+1} < (r+1)^2 b^2 l^2. \tag{5}$$

On the other hand it is not difficult to show that

$$p^r q^{(b-2)l+1} > (r+1)^2 b^2 l^2, \tag{6}$$

which is in contradiction to (5). In fact it suffices to prove (6) when $b=3$, since a unit increase in b increases the left-hand side of (6) by a factor q^l and increases the right-hand side of (6) by a factor $(b+1)^2/b^2 \leq 16/9$. Thus one must prove that

$$p^r q^{l+1} > 9(r+1)^2 l^2. \tag{7}$$

For $q \geq 5$ the inequality (7) follows from size considerations alone. We leave the case $q=3$ of (7) to the reader. In any event (I) cannot occur.

(II) $\max(a,b) = 2$. We may suppose $b=2$, $a \leq 2$. Here (4) takes the form

$$p^{ak-1}q^{2l-1}2akl = m\frac{p^{ak}-1}{p-1}(q^l+1)\frac{q^l-1}{q-1}.$$

Since $q^l \equiv 1 \pmod p$, it follows that q^l+1 is relatively prime to p (and to q) and so $(q^l+1)|(2akl)$. Since $a \leq 2$ and $k|(q-1)$, this yields $q^l+1 \leq 4l(q-1)$. Hence $l \leq 2$, since $(q^l+1)/(q-1) > (q^l-1)/(q-1) \geq (3^l-1)/2 > 4l$ for $l \geq 3$.

Suppose $l=1$, i.e., $q \equiv 1 \pmod p$. Then $q \geq 2p+1$ and $(q+1)|(2ak)$, while $q-1 = ck$ for some positive integer c. If $a=2$, the assertion $(q+1)|(2ak)$

becomes $(ck+2)|(4k)$. This implies that both c and the ratio $(4k)/(ck+2)$ are at most 3 and leads to possibilities that are easily eliminated; for example, if $c=1$ and $4k/(ck+2)=3$, then $k=6$, $q=7$, $p=3$, and $(p^{ak}-1)/(p-1)$ is divisible by 73. Similarly $a=1$ is impossible.

Suppose $l=2$. Then $(q^2+1)|(4ak)$ and hence $q^2+1 \leq 4a(q-1)$, which is impossible for $a=1$ and forces $q \leq 5$ for $a=2$. But when $a=2$, the assertion $(q^2+1)|(4ak)$ becomes $26|(4k)$ for $q=5$ and $10|(4k)$ for $q=3$ and gives values of k incompatible with the restriction $k|(q-1)$.

Thus (II) cannot occur.

(III) $a=b=1$. Since $r+1=k$ and $s+1=l$, we see that (4) takes the form

$$p^{k-1}q^{l-1}kl = m\frac{p^k-1}{p-1}\frac{q^l-1}{q-1}, \tag{8}$$

where both k and l exceed 1. We may assume $p<q$. We first show that k and l must both be prime. If l is a composite number greater than 4, let l_1 be a divisor of l strictly between 2 and l. Since q has multiplicative order l modulo p, we have $p \nmid (q^{l_1}-1)$. From (8) it follows that $(q^{l_1}-1)/(q-1)$ must be a divisor of kl. This yields $q^{l_1-1} < kl < qp < q^2$, a contradiction. Thus l cannot be a composite number greater than 4. If $l=4$, then similarly $(q^2-1)/(q-1)=q+1$ must be a divisor of $lk=4k$; since also $k|(q-1)$, we see that q must be one of $k+1$, $2k+1$, $3k+1$ and k must be 2, which leads to possibilities that are easily eliminated. Thus l is prime.

Write $k=l^u v$, where $u \geq 0$ and v is not divisible by the prime l. Suppose k is not prime and let d be the largest divisor of k which is less than k itself. Then $p^d \not\equiv 1 \pmod q$, since k is the multiplicative order of p modulo q. Thus $(p^d-1)/(p-1)$ is a divisor of $(p^k-1)/(p-1)$ and is relatively prime to both p and q. Thus by (8) $(p^d-1)/(p-1)$ divides $kl=l^{u+1}v$. Suppose first that l is an odd prime. By Lemma 1 we have $l^u \| (p^k-1)/(p-1)$ and so in fact $(p^d-1)/(p-1)$ divides $l^u v = k$. Hence

$$p^{d-1} < (p^d-1)/(p-1) \leq k. \tag{9}$$

If $p \geq 5$, inequality (9) is impossible, since $d \geq k^{\frac{1}{2}} > 1$ and so

$$p^{d-1} \geq 5^{d-1} \geq d^2+1 \geq k+1.$$

If $p=3$, inequality (9) gives $3^{[k^{1/2}-1]} < k$ or $k<9$; the possibilities $k=4,6,8$ are easily eliminated. Thus the assumption that k is not prime is untenable when l is odd. When $l=2$, we can only infer that $(p^d-1)/(p-1)$ divides $kl=2k$ and thus that $p^{d-1}<2k$. However an argument similar to that used for odd l again leads to a contradiction if k is not a prime. Thus, regardless of whether l is an odd prime or $l=2$, it is impossible for k to be composite.

Now that we have established that k and l are primes, we note from (8) that

$$1 > \frac{m}{kl} = \frac{p^{k-1}}{1+p+\cdots+p^{k-1}}\frac{q^{l-1}}{1+q+\cdots+q^{l-1}} > \left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right) \geq \frac{8}{15}$$

and so

$$\max(k,l) < 8kl/15 < m < kl < \min(kp,ql) < \max(kp,ql) < pq < q^2.$$

Since $m|(p^{k-1}q^{l-1}kl)$, these inequalities limit the possibilities for m to

$$q,\ p^c \text{ (with } 1<p^c<kl), \qquad p^c l \text{ (with } 1<p^c<k).$$

We call these possibilities Case α, Case β, and Case γ respectively.

Case α, namely $m = q$. Here (8) takes the form

$$p^{k-1}q^{l-2}kl = \frac{p^k - 1}{p - 1}\frac{q^l - 1}{q - 1}.$$

Since $(p^k - 1)/(p - 1)$ is relatively prime to p and $(q^l - 1)/(q - 1)$ is relatively prime to q, it follows that $(p^k - 1)/(p - 1)$ is equal to one of the following four quantities:

$$q^{l-2}, \quad kq^{l-2}, \quad lq^{l-2}, \quad klq^{l-2}.$$

We eliminate each of these possibilities in turn.

If $(p^k - 1)/(p - 1) = q^{l-2}$, then $l > 2$ and $q^{l-2} \equiv 1 \pmod{p}$. But this is impossible, since l was defined as the multiplicative order of q modulo p.

If $(p^k - 1)/(p - 1) = kq^{l-2}$, then $(q^l - 1)/(q - 1) = lp^{k-1}$, so that $lp^{k-1} \equiv 1 \pmod{q}$. Since k is the multiplicative order of p modulo q, this gives $l \equiv p \pmod{q}$. But this contradicts $l < p < q$.

If $(p^k - 1)/(p - 1) = lq^{l-2}$, Lemma 4 gives $k = l$. Since $kp^{k-1} = (q^l - 1)/(q - 1)$, we have $kp^{k-1} \equiv 1 \pmod{q}$ and thus $k \equiv p \pmod{q}$. Since both k and p are less than q, we have $k = p$. Hence $p = l$, which contradicts $l|(p - 1)$.

If $(p^k - 1)/(p - 1) = klq^{l-2}$, we have $(q^l - 1)/(q - 1) = p^{k-1}$. Thus $p^{k-1} \equiv 1 \pmod{q}$, which conflicts with the definition of k.

Thus Case α cannot occur.

Case β, namely $m = p^c$, where $1 < p^c < kl$. Here (8) takes the form

$$p^{k-c-1}q^{l-1}kl = \frac{p^k - 1}{p - 1}\frac{q^l - 1}{q - 1}$$

so that $(p^k - 1)/(p - 1)$ is equal to one of the four quantities

$$q^{l-1}kl, q^{l-1}l, q^{l-1}k, q^{l-1}.$$

Each of the first two possibilities leads to a contradiction via the use of Lemma 4. If $(p^k - 1)/(p - 1) = q^{l-1}k$, Lemma 3 shows that $p - 1$ is divisible by k. We already know that $p - 1$ is divisible by l. But

$$p^{l-1} < q^{l-1} = (1 + p + \cdots + p^{k-1})/k < p^{k-1},$$

so that $l < k$. Thus k and l are distinct prime factors of $p - 1$ and so $p > kl$, which contradicts $p^c < kl$. Finally, if $(p^k - 1)/(p - 1) = q^{l-1}$, we have $q^{l-1} \equiv 1 \pmod{p}$, which contradicts the definition of l. Thus Case β cannot occur.

Case γ, namely $m = p^c l$, where $1 < p^c < k$. Here (8) becomes

$$p^{k-c-1}q^{l-1}k = \frac{p^k - 1}{p - 1}\frac{q^l - 1}{q - 1},$$

where $c + 1 < 3^c \leq p^c < k$ and thus $k - c - 1 > 0$. Thus $(p^k - 1)/(p - 1)$ is equal either to $q^{l-1}k$ or to q^{l-1}, possibilities which are easily eliminated using Lemma 3 and the definition of l respectively. Thus Case γ cannot occur and Part A is complete.

PART B. Suppose $n = 2^r q^s$, where q is an odd prime and r and s are positive integers. We assume $nd(n)/\sigma(n) = m$, where m is an integer; this assumption may be written

$$2^r q^s (r + 1)(s + 1) = m(2^{r+1} - 1)(q^{s+1} - 1)/(q - 1). \tag{10}$$

We first show that $s = 1$ and then show that $q = 2^{r+1} - 1$, which implies that n is an even perfect number.

Suppose s is an odd integer greater than 1. Put $s + 1 = 2^u v$, where $u > 0$ and v is odd. By Lemma 2 we have $2^{u-1} \| (q^{s+1} - 1)/(q^2 - 1)$. Since also $2^{u-1} \| (s + 1)/2$, it follows from (10) that $(q^{s+1} - 1)/(q^2 - 1)$ is a divisor of $(r + 1)(s + 1)/2$. Thus we may write

$$(r + 1)(s + 1)/2 = t(q^{s+1} - 1)/(q^2 - 1) = t(1 + q^2 + q^4 + \cdots + q^{s-1}),$$

so that $q^{s-1} \le tq^{s-1} < (r + 1)(s + 1)/2$. Substitution into (10) gives

$$2^r q^s t = m(2^{r+1} - 1)(q + 1)/2.$$

Hence $(2^{r+1} - 1) | (tq^s)$ and so $2^{r+1} \le 1 + qtq^{s-1} < q(r + 1)(s + 1)/2$. Multiplying the two inequalities $q^{s-1} < (r + 1)(s + 1)/2$ and $2^{r+1} < q(r + 1)(s + 1)/2$ yields

$$2^{r+3} q^{s-2} < (r + 1)^2 (s + 1)^2.$$

But $2^{r+3} \ge 32(r + 1)^2/9$ for $r \ge 1$ and $q^{s-2} \ge 3^{s-2} \ge 3(s + 1)^2/4$ for $s \ge 5$, so that for $s \ge 5$ we have the contradiction

$$2^{r+3} q^{s-2} \ge 8(r + 1)^2 (s + 1)^2/3 \qquad (s \text{ odd} \ge 5).$$

If $s = 3$ and $r \ge 5$, we have $2^{r+3} \ge 64(r + 1)^2/9$ and $q^{s-2} \ge 3^{s-2} = 3(s + 1)^2/16$, which yields the similar contradiction

$$2^{r+3} q^{s-2} \ge 4(r + 1)^2 (s + 1)^2/3 \qquad (r \ge 5,\ s = 3).$$

The remaining four cases $s = 3$, $1 \le r \le 4$ lead to diophantine equations in q and m which are easily seen to have no integer solutions. Thus s cannot be an odd integer greater than 1.

Suppose s is an even integer greater than 1. Since $(q^{s+1} - 1)/(q - 1)$ is odd and relatively prime to q, it follows from (10) that $(q^{s+1} - 1)/(q - 1)$ is a divisor of $(r + 1)(s + 1)$, so that

$$(r + 1)(s + 1) = t(q^{s+1} - 1)/(q - 1) = t(1 + q + q^2 + \cdots + q^s)$$

for some integer t. Inserting this into (10) gives $t2^r q^s = m(2^{r+1} - 1)$, so that $(2^{r+1} - 1) | (tq^s)$. Now $q^s \le tq^s < (r + 1)(s + 1)$ and hence $2^{r+1} \le 1 + tq^s \le (r + 1)(s + 1)$. Multiplying the inequalities $q^s < (r + 1)(s + 1)$ and $2^{r+1} \le (r + 1)(s + 1)$ gives

$$2^{r+1} q^s < (r + 1)^2 (s + 1)^2.$$

But $2^{r+1} \ge (r + 1)^2$ for $r \ge 3$ and $q^s \ge 3^s \ge (s + 1)^2$ for even s, so that for $r \ge 3$ we have the contradiction

$$2^{r+1} q^s \ge (r + 1)^2 (s + 1)^2 \qquad (r \ge 3, s \text{ even}).$$

Also $2^{r+1} \ge 8(r + 1)^2/9$ for $r \ge 1$ and $q^s \ge 3^s \ge 27(s + 1)^2/16$ for $s > 2$, so that for s even, $s \ne 2$, we have the contradiction

$$2^{r+1} q^s \ge 3(r + 1)^2 (s + 1)^2/2 \qquad (s \text{ even} > 2).$$

The two remaining cases $s = 2$, $1 \le r \le 2$ lead to diophantine equations in q and m which are easily seen to have no integer solutions. Thus s cannot be an even positive integer.

As a consequence of the two preceding paragraphs, we can conclude that $s = 1$. Thus (10) becomes

$$2^{r+1}q(r+1) = m(2^{r+1}-1)(q+1). \tag{11}$$

If q were to divide m, we would have $(2^{r+1}-1)|(r+1)$, which is impossible from size considerations. Since q must divide one of the factors on the right-hand side of (11), we have $q|(2^{r+1}-1)$ and so $2^{r+1}-1 = uq$ for some odd integer u. From (11) we have $2^{r+1}(r+1) = mu(q+1)$, so that $u|(r+1)$ and $2^{r+1}(r+1) = m(2^{r+1}+u-1)$. Put $v = \lfloor \log r/\log 2 \rfloor$, so that $r = 2^{v+\theta}$ for some real θ in $[0,1)$. Since $u - 1 \le r$, we have $u - 1 \le 2^{v+\theta}$. If $u \neq 1$, the largest power of 2 dividing $2^{r+1}+u-1$ is at most 2^v. Hence $2^{r+1-v}|m$, that is $m = 2^{r+1-v}m_0$ for some positive integer m_0. This yields

$$r(r+1) = 2^{\theta}m_0(2^{r+1}+u-1),$$

which implies the absurdity $2^{r+1} \le r(r+1)$. Thus the assumption $u \neq 1$ is untenable and so $u = 1$, $q = 2^{r+1}-1$. Since q is assumed to be prime, $r+1$ must also be prime and so n has the Euclid-Euler form for an even perfect number. Thus Part B is complete and our solution is finished.

Editorial comment. The assertion of the problem is false for positive integers with three distinct prime factors. Of course no positive integer with three distinct prime factors satisfies (1). However, the integers $270 = 2 \cdot 3^3 \cdot 5$ and $672 = 2^5 \cdot 3 \cdot 7$ satisfy both (2) and (3), the integers $140 = 2^2 \cdot 5 \cdot 7$ and $6200 = 2^3 \cdot 5^2 \cdot 31$ satisfy (2) but not (3), the integers $1080 = 2^3 \cdot 3^3 \cdot 5$ and $1782 = 2 \cdot 3^4 \cdot 11$ satisfy (3) but not (2), and a squarefree number with three prime factors satisfies neither (2) nor (3).

The question of which positive integers are harmonic, i.e., satisfy (2), seems to have been first raised in [4]. Further discussion is given in [1], where it is proved that (1) and (2) are equivalent for integers of the form p^rq, where p and q are distinct primes. Also [**1**] gives a list of the 45 harmonic numbers less than 10^7. The term "harmonic number" was introduced by Carl Pomerance. See [**2**] for further references.

All known examples of harmonic numbers are even, and Ore conjectured that no odd number is harmonic. Since it is easy to see that any perfect number, whether odd or even, is harmonic, Ore's conjecture generalizes the old conjecture that there are no odd perfect numbers. In [**3**] it is proved that if an odd positive integer n is harmonic, then n has a prime-power factor greater than 10^7.

The assertion of the problem was proved also in [**5**], which the proposer brought to the attention of the editors. See also Abstract 709-A5 in *Notices A.M.S.* 20 (1973), page A-648.

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3. W. H. Mills, On a conjecture of Ore, *Proceedings of the 1972 Number Theory Conference*, University of Colorado, Boulder, 1972, 142–146.
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No other solutions were received.

An Operation on $\mathbb{Q} \cup \{e\}$

E 3427 [1991, 263]. *Proposed by R. Padmanabhan, N. S. Mendelsohn, and B. Wolk, University of Manitoba, Winnipeg, Canada.*

Let S denote the set obtained by formally adjoining an element e to the set $\mathbb{Q}$ of rational numbers. On S define a binary operation $\circ$ as follows:

$$\begin{array}{ll} p \circ q = (3 + pq)/(p + q) & \text{if } p \in \mathbb{Q}, q \in \mathbb{Q}, p \neq -q \\ p \circ q = e & \text{if } p \in \mathbb{Q}, q \in \mathbb{Q}, p = -q \\ x \circ e = x = e \circ x & \text{for all } x \in S \end{array}$$

(a) Prove that $(S, \circ)$ is an abelian group isomorphic to a subgroup of the multiplicative group of non-zero real numbers.

(b) If p is a positive rational number, put $p_1 = p$, $p_2 = p \circ p$, $p_3 = p \circ p \circ p, \cdots$. Show that $\lim_{n \to \infty} p_n$ exists and find the limit. For which values of p is the sequence $\{p_n\}_{n=1}^{\infty}$ monotonic?

Note: The specification "$= e \circ x$" was omitted in the original statement of the problem.

Solution by Marcin E. Kuczma, University of Warsaw, Warszawa, Poland. (a) The mapping ϕ: $S \to \mathbb{R} - \{0\}$ given by

$$\phi(p) = \frac{p + \sqrt{3}}{p - \sqrt{3}} \qquad \text{for } p \in \mathbb{Q} \quad \textit{and} \quad \phi(e) = 1$$

is injective and carries $\circ$ into multiplication of real numbers, as verified by straightforward calculation. Since S is closed under $\circ$ and inverses under $\circ$ exist, the image of S in $\mathbb{R}_0 = \mathbb{R} - \{0\}$ is a multiplicative subgroup of $\mathbb{R}_0$. Hence S is a group, and ϕ embeds it isomorphically in $\mathbb{R}_0$.

(b) Here we are concerned with the sequence $p_n = f_p^{n-1}(p)$ of iterates of the function

$$f_p(t) = \frac{3 + pt}{p + t} = p + \frac{3 - p^2}{p + t}$$

for a given positive rational number p. The function f_p is increasing on $(-p, \infty)$ if $p > \sqrt{3}$, decreasing on $(-p, \infty)$ if $p < \sqrt{3}$, and the number $t = \sqrt{3}$ is its unique positive fixed point. From the computation $\operatorname{sgn}(t - f_p(t)) = \operatorname{sgn}(t - \sqrt{3})$, it follows that $\sqrt{3}$ is an attracting fixed point for all of $(0, \infty)$, and $\lim_{n \to \infty} p_n = \sqrt{3}$ in all cases. Furthermore, the computation $\operatorname{sgn}((f_p(t) - \sqrt{3})(p - \sqrt{3})) = \operatorname{sgn}(t - \sqrt{3})$ shows that the convergence is monotone if $p > \sqrt{3}$, but that the sequence alternates above and below $\sqrt{3}$ if $p < \sqrt{3}$.

Solved by 32 readers and the proposers.

Density Results for Docile Sequences

E 3430 [1991, 264]. *Proposed by Paul Erdős, Hungarian Academy of Sciences, Budapest.*

(i) Let $\mathscr{A} = \{a_1, a_2, \cdots\}$ be a strictly increasing sequence of positive integers such that no sum of two or more distinct members of the sequence is equal to

another member of the sequence. Let $A(x) = \#\{i\colon a_i \le x\}$. Show that $A(x)/x \to 0$ as $x \to \infty$.

(ii) Show that if $f(x) \to \infty$ as $x \to \infty$ (no matter how slowly), then there exists a sequence $\mathscr{A}$ for which $A(x) > x/f(x)$ holds on an unbounded set of values of x.

Solution by Robert High, New York City, NY. Call a sequence satisfying the conditions of the problem *docile*. If $\mathscr{A}$ is a docile sequence, consider the derived sequence $\mathscr{B}$ with $b_j = \sum_{i=1}^{j} a_i$. Note that all sums $a_i + b_j$ with $i > j$ are distinct, since if $a_i + b_j = a_k + b_l$ with $b_l < b_j$, then subtracting b_l from both sides gives $a_k = a_i + \sum_{m=l+1}^{j} a_m$, contradicting the docility of $\mathscr{A}$.

For any positive integer n, choose $N > 2b_n$ large enough that $A(N) > 2n$. Considering all sums $a_i + b_j$ with $n < i \le A(N)$ and $j \le n$, we find at least $n(A(N) - n)$ distinct positive integers less than or equal to $N + b_n$. Thus

$$n(A(N) - n) \le N + b_n.$$

But $A(N) - n > A(N)/2$, so

$$\frac{nA(N)}{2} < N + b_n,$$

and hence

$$\frac{A(N)}{N} < \frac{2}{n} + \frac{2b_n}{nN} < \frac{3}{n}.$$

As n was arbitrary, this proves (i).

To prove (ii), let $f(x)$ be a function tending to infinity, as in the statement of the problem. We seek a docile sequence $\mathscr{A}$ such that $A(x)/x > 1/f(x)$ for an unbounded set of values of x. We will construct the desired sequence as the union of a sequence of finite docile sets S_k.

Start with $S_0 = \varnothing$, for which the sum of all elements is zero.

Having constructed S_{k-1}, construct S_k as follows. Choose an integer s larger than the sum of all the elements in S_{k-1} and a positive integer r such that $f(2rs) > 2s$. Now let $N_k = rs$ and $S_k = S_{k-1} \cup \{N_k, N_k + s, \dots, 2N_k - s\}$. Clearly S_k is docile. Since S_k contains more than $r = N_k/s$ elements,

$$\frac{A(2N_k)}{2N_k} > \frac{1}{2s} > \frac{1}{f(2N_k)}.$$

Thus $\mathscr{A} = \bigcup S_k$ has the desired properties.

Solved also by L. E. Mattics, K. Schilling, R. Stong, and the proposer.

Getting a Square Deal

6655 [1991, 372]. *Proposed by Paul Erdős, Hungarian Academy of Sciences, Budapest.*

Given a positive integer n, we wish to find distinct integers greater than n such that their product with n is a square and we wish to do this in such a way that the largest number used is as small as possible. Let $f(n)$ be this minimal value of the largest number used. For example $f(10) = 18$, since $10 \cdot 12 \cdot 15 \cdot 18 = (180)^2$ and the product of 10 with any subset of $\{11, 12, 13, 14, 15, 16, 17\}$ is not a square.

Show that if $\epsilon_1, \epsilon_2, \epsilon_3, \cdots$ is any sequence of positive numbers tending to zero, then the set

$$\{n: f(n) - n \geq n^{1-\epsilon_n}\}$$

has density zero.

Solution by the editors, based on solutions submitted by J. L. Selfridge, Northern Illinois University, DeKalb, IL. Let $P(n)$ denote the largest prime factor of the positive integer n. We shall prove the following two results, which imply the assertion of the problem:

$$f(n) - n = P(n) \text{ when } P(n) > \sqrt{2n} + 1, \tag{1}$$

$$f(n) - n \leq n^{\frac{2}{3}} + n^{\frac{1}{3}} + 1 \text{ when } P(n) \leq \sqrt{2n} + 1. \tag{2}$$

Selfridge actually proved the stronger result that $f(n) - n = O(n^{\frac{1}{2}})$ when $P(n) \leq \sqrt{2n} + 1$, but we shall content ourselves with the weaker result (2). See (5) below.

To see that (1) and (2) imply the assertion of the problem, it suffices to show that given a number ϵ in the interval $(0, \frac{1}{2})$, we have

$$\#\{n : n \leq x, P(n) > n^{1-\epsilon}\} < 4\epsilon x$$

for x sufficiently large. But, using the familiar elementary estimate on the sum of the reciprocals of the primes

$$\sum_{p \leq x} p^{-1} = \log\log x + B + O\left((\log x)^{-1}\right),$$

we have for x large

$$\begin{aligned}
\#\{n: n \leq x, P(n) > n^{1-\epsilon}\} &\leq \lfloor x^{1-\epsilon} \rfloor + \sum\{1: x^{1-\epsilon} < n \leq x;\, P(n) > x^{(1-\epsilon)^2}\} \\
&\leq x^{1-\epsilon} + \sum\{x/p: x^{(1-\epsilon)^2} < p \leq x\} \\
&\leq x^{1-\epsilon} + x \log(1-\epsilon)^{-2} + O(x/\log x) \\
&\leq x(x^{-\epsilon} + 3\epsilon + c/\log x) \\
&< 4\epsilon x,
\end{aligned}$$

where c is a certain positive constant. Thus it suffices to prove (1) and (2). We require the following lemmas.

Lemma 1. *If $n = ab$, where a and b are positive integers, then $f(n) \leq (a+1)(b+1)$.*

Proof: The product of the four integers $ab, a(b+1), (a+1)b, (a+1)(b+1)$ is obviously a square. (If $a = b$, we omit the middle two of the four integers.)

Lemma 2. *If $n = ab$, where b is odd and $b > 2a + 1$, then $f(n) \leq (a+1)b$.*

Proof: Under the hypotheses both of the integers

$$a(b+1),\ \left(a + \frac{1}{2}\right)(b-1) \tag{3}$$

are greater than ab and less than each of the two integers

$$\left(a + \frac{1}{2}\right)(b+1),\ (a+1)(b-1), \tag{4}$$

while each of the integers in (4) is less than $(a+1)b$. Further, the product of the integers ab, $(a+1)b$, and the four integers listed in (3) and (4) is equal to $\{a(a+\frac{1}{2})(a+1)(b-1)b(b+1)\}^2$. (If the two integers in (3) or the two integers in (4) are equal, which happens when $b = 4a+1$ or $b = 4a+3$ respectively, both of them should be deleted.)

To prove (1) we note that if p is a prime dividing n, the inequality $p > \sqrt{2n}+1$ is equivalent to the inequality $p > 2n/p + 1$. For if $p > \sqrt{2n}+1$, then $(p-1)^2 > 2n$ or $p^2 > 2n + 2p - 1$ or $p > 2n/p + 2 - 1/p > 2n/p + 1$. On the other hand, if $p > 2n/p + 1$, then $p > 2n/p + 2$ or $p^2 - 2p > 2n$ or $(p-1)^2 > 2n + 1 > 2n$, so that $p > \sqrt{2n}+1$. Thus if $p > \sqrt{2n}+1$, we may apply Lemma 2 with $a = n/p$, $b = p$ to give $f(n) \le (n/p+1)p = n + p$. But the inequality $p > \sqrt{2n}+1$ implies $p^2 \nmid n$ and so $f(n) \ge n + p$. Thus (1) is proved.

To prove (2) when $P(n) \le \sqrt{2n}+1$, suppose $n = p_1 p_2 \cdots p_s$, where $p_1, p_2, \ldots, p_s$ are primes and $p_1 \ge p_2 \ge \cdots \ge p_s$. Put $d_0 = 1$, $d_1 = p_1$, $d_2 = p_1p_2, \ldots, d_s = p_1p_2\cdots p_s$. Let r be the positive integer such that $d_{r-1} < n^{\frac{1}{3}}$, $d_r \ge n^{\frac{1}{3}}$. If $r = 1$, then $d_1 = p_1 \ge n^{\frac{1}{3}}$. If $r > 1$, then $p_1 < n^{\frac{1}{3}}$ and so all prime factors of n are less than $n^{\frac{1}{3}}$. Thus if $r > 1$, we have $n^{\frac{1}{3}} \le d_r < n^{\frac{2}{3}}$. In either case d_r is a divisor of n lying in the interval $[n^{\frac{1}{3}}, n^{\frac{2}{3}})$. By Lemma 1 we have $f(n) \le (d_r+1)(n/d_r+1)$. Since d_r lies in $[n^{\frac{1}{3}}, n^{\frac{2}{3}})$, we have by convexity

$$f(n) - n \le d_r + n/d_r + 1 \le n^{\frac{2}{3}} + n^{\frac{1}{3}} + 1.$$

Thus (2) is proved.

By a more involved argument Selfridge obtained the very precise inequality

$$f(n) - n \le 3(\sqrt{8n+1}+1)/4 \text{ if } P(n) \le \sqrt{2n}+1 \text{ and } n \ne 2,3,8,10,32, \quad (5)$$

a much stronger result than (2). The obvious fact that $f(p(p-1)) - p(p-1) = 2p > 2\sqrt{p(p-1)}$ for any prime p shows that the order of magnitude of the estimate in (5) cannot be improved. More significantly, if $n = p(2p-1)$, where both p and $2p+1$ are primes, then it is not hard to see that

$$f(n) - n = 3p = 3(\sqrt{8n+1}+1)/4.$$

Thus Selfridge's impressive inequality (5) is as sharp as can be (assuming that there are infinitely many primes p such that $2p+1$ is also prime).

Of course $f(n) - n$ can sometimes be of lower order of magnitude than $n^{\frac{1}{2}}$. For example, if m is any positive integer greater than 1 and if $n = m^6 - 4m^2$, then the identity

$$(m^6 - 4m^2)(m^6 - 3m^2 - 2)(m^6 - 3m^2 + 2)$$
$$= (m^2-2)^2(m^2-1)^2m^2(m^2+1)^2(m^2+2)^2$$

shows that

$$f(n) - n \le m^2 + 2 = n^{\frac{1}{3}} + 2 + O(n^{-\frac{1}{3}}).$$

Editorial comment. When n itself is a square, the definition of $f(n)$ is somewhat ambiguous. On the one hand, it would be reasonable to adopt the convention that $f(n) = n$ when n is a square. On the other hand, the wording of the problem seems to suggest that we are expected to use a non-empty set of integers greater than n. If this second interpretation is adopted, it is immediate that when n is a square m^2, then $f(n) = f(m^2) \le (m+1)^2 \le m^2 + 3(\sqrt{8m^2+1}+1)/4$. Thus Selfridge's inequality (5) holds trivially when n is a square. However, Selfridge

observed that if $m \neq 1, 2, 3, 4, 6$, then in fact

$$f(m^2) = \min_{n > m^2} f(n) < (m+1)^2. \tag{6}$$

For example, $f(25) = f(27) = 35$, $f(49) = f(50) = 63$, $f(81) = f(88) = 99$. For large m a proof of (6) can be given which depends on the residue class of m modulo 12. For example, if $m \equiv 8 \pmod{12}$, then from Lemma 2

$$\begin{aligned} f(m^2) &\le f\left(\frac{2m-1}{3}\frac{3m+2}{2}\right) \le \frac{2m+2}{3}\frac{3m+2}{2} \\ &= m^2 + 5m/3 + 2/3 < (m+1)^2, \end{aligned}$$

while if $m \equiv 10 \pmod{12}$, then

$$\begin{aligned} f(m^2) &\le f\left(\frac{2m-2}{3}\frac{3m+4}{2}\right) \le \frac{2m+1}{3}\frac{3m+4}{2} \\ &= m^2 + 11m/6 + 2/3 < (m+1)^2. \end{aligned}$$

If we were to adopt the convention that $f(m^2) = m^2$, Ronald Graham remarked that then the function f never takes the same value twice. Whatever interpretation of $f(m^2)$ is used, it is easy to see that if $n_1 \neq n_2$ and if neither n_1 nor n_2 is a square, then $f(n_1) \neq f(n_2)$.

Solved also by L. E. Mattics and the proposer.

Inscribed Triangles Are Circumscribed

E 3443 [1991, 438]. *Proposed by Călin Popescu, St. Michiels Brugge, Belgium.*

Let A_i, $i = 0, 1, \ldots, 5$, denote the vertices of a hexagon inscribed in a circle and let B_i denote the intersection of the straight lines A_iA_{i+2} and $A_{i+1}A_{i+3}$, for $i = 0, 1, \ldots, 5$, the indices being computed modulo 6. Prove that, if the triangles $A_0A_2A_4$ and $A_1A_3A_5$ have the same orthocenter, then the straight lines B_iB_{i+3}, $i = 0, 1, 2$, are concurrent. (The orthocenter of a triangle is the intersection of its three altitudes.)

Solution by Robin J. Chapman, University of Exeter, Exeter, U.K. The hypothesis concerning the orthocenters is superfluous. The result is a straightforward corollary of some classical theorems of projective geometry. As the triangles $A_0A_2A_4$ and $A_1A_3A_5$ are inscribed in the same conic, they also circumscribe a conic C [**1**, p. 169], and so C is circumscribed by the hexagon $B_0B_1B_2B_3B_4B_5$. Now by Brianchon's Theorem [**1**, p. 175], the lines B_0B_3, B_1B_4, and B_2B_5 are concurrent, as required.

Editorial comment. The fact the assumption about the orthocenters is superfluous was also noted by Jordi Dou and Jiro Fukuta. As observed by other solvers, if O is the center of a circle Γ of radius R, and H is another point interior to Γ, then all triangles inscribed in Γ and having H as orthocenter are circumscribed about the ellipse with foci O and H and with major axis of length R. The auxiliary circle of this ellipse (i.e., the circle of radius $R/2$ centered at the midpoint of OH) is the common nine-point circle of all these triangles. The properties of this configuration are discussed thoroughly in [**2**].

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2. H. F. Baker, *An Introduction to Plane Geometry*, Cambridge, 1943, Chapter XIII.

Solved also by J. Anglesio (France), J. Dou (Spain), J. Fukuta (Japan), N. Komanda, O. P. Lossers (The Netherlands), G. Velissarios (Greece), and the proposer.

Another Proof That 2 Is an "Odd" Prime

E3445 [1991, 552]. *Proposed by Ronald A. Jansen and Pieter Moree, University of Leiden, The Netherlands.*

(a) Suppose p is an odd prime and $k \equiv 1 \pmod p$. Prove that for any positive integer n the highest power of p dividing n is equal to the highest power of p dividing $1 + k + k^2 + \cdots + k^{n-1}$.

(b) Suppose $k \equiv 1 \pmod 4$. Prove that for any positive integer n the highest power of 2 dividing n is equal to the highest power of 2 dividing $1 + k + k^2 + \cdots + k^{n-1}$.

Solution by Robin J. Chapman, University of Exeter, Exeter, U.K. Let p and k be fixed. Given an integer m, let $v(m)$ denote the largest integer r such that p^r divides m. Also let $f(n) = \sum_{i=1}^{n-1} k^i = (k^n - 1)/(k-1)$.

(a) Let $l = \lfloor k/p \rfloor$, so $k = 1 + pl$. By the binomial theorem, $f(n) = n + \sum_{j=2}^{n} \binom{n}{j}(pl)^{j-1}$, and therefore it suffices to show that $v\left(\binom{n}{j}\right) + j - 1 > v(n)$ for $2 \le j \le n$. Since $\binom{n}{j} = n(n-1)\cdots(n-j+1)/j!$, we have $v\left(\binom{n}{j}\right) \ge v(n) - v(j!)$. However, $v(j!) = \sum_{i=1}^{\infty} \lfloor j/p^i \rfloor < \sum_{i=1}^{\infty} j/p^i = j/(p-1)$. Since $p \ge 3$ and $j \ge 2$, we thus have $v(j!) < j - 1$, so $v\left(\binom{n}{j}\right) > v(n) - j + 1$, as desired.

(b) Let $l = \lfloor k/4 \rfloor$, so $k = 1 + 4l$. Using $f(n) = n + \sum_{j=2}^{n} \binom{n}{j}(4l)^{j-1}$, it suffices to show that $v\left(\binom{n}{j}\right) + 2j - 2 > v(n)$ for $2 \le j \le n$. With $p = 2$, the computation above for $v(j!)$ shows $v(j!) < j$. Hence $v\left(\binom{n}{j}\right) > v(n) - j \ge v(n) - (2j - 2)$, as desired.

Editorial comment. This problem appears as Proposition 1 in F. R. Beyl, Cyclic subgroups of the prime residue group, this MONTHLY 84(1977), 46–48, by a different proof. Beyl comments there, "The first appearance for this proposition may be Chevalley" in a 1955 paper "where heavy machinery is used for the proof." Some solver rediscovered Beyl's argument; others used induction or appealed to "heavy machinery" in various ways.

Solved by 39 readers and the proposers.

A Characterization of the Identity Function

E 3458 [1991, 754]. *Proposed by Umberto Zannier, Venezia, Italy.*

Let $\mathbb{N}_0$ denote the set of non-negative integers. Suppose that $f\colon \mathbb{N}_0 \to \mathbb{N}_0$ is a function such that $f(1) > 0$ and $f(m^2 + n^2) = f(m)^2 + f(n)^2$ for all $m, n \in \mathbb{N}_0$. Show that f is the identity function.

Composite solution by all solvers. Setting $m = n = 0$ gives $f(0) = 2f(0)^2$, which implies that $f(0) = 0$ since $f(0)$ is an integer. Setting $m = 1$, $n = 0$ gives $f(1) = f(1)^2$ which implies $f(1) = 1$ since $f(1) > 0$. Now, direct application of the given identity shows that $f(2) = f(1^2 + 1^2) = f(1)^2 + f(1)^2 = 2$. Taking $m = 2$ and $n = 0$, 1 or 2 gives $f(4) = 4$, $f(5) = 5$, and $f(8) = 8$. To fill in some gaps, note that $m^2 + n^2 = k^2 + l^2$ implies that $f(m)^2 + f(n)^2 = f(k)^2 + f(l)^2$. Thus $f(3) = 3$ follows from $3^2 + 4^2 = 0^2 + 5^2$. The equations $7^2 + 1^2 = 5^2 + 5^2, 9 = 3^2 + 0^2, 10 = 3^2 + 1^2$, and $6^2 + 8^2 = 10^2 + 0^2$ then establish $f(n) = n$ for all $n \leq 10$. These cases form a basis for a proof by induction. Suppose that $m > 10$ and $f(l) = l$ for all $l < m$. If m is odd, write $m = 2k + 1$ and employ

$$(2k+1)^2 + (k-2)^2 = (2k-1)^2 + (k+2)^2;$$

and if m is even, write $m = 2k + 2$ and employ

$$(2k+2)^2 + (k-4)^2 = (2k-2)^2 + (k+4)^2.$$

Editorial comment. A variety of quadratic sum identities were employed by solvers in the inductive step—some falling into as many as six different modular parity cases. Those given here are special cases of

$$(ru+sv)^2 + (rv-su)^2 = (ru-sv)^2 + (rv+su)^2$$

which occurs in the classical study of representations of integers as sums of two squares.

The proposer also considered the following proposition:

Let $f\colon \mathbb{N} \to \mathbb{C}$ be unbounded and satisfy $f(a^2 + b^2) = f(a)^2 + f(b)^2$ for all sufficiently large a and b. Then $f(n)^2 = n^2$ for all sufficiently large n.

The proof is considerably more involved than that of the published version, although similar methods are employed.

Several solvers pointed out that without the assumption that $f(1) > 0$, the function $f(n) = 0$ for all n would also satisfy the conditions of the problem. Four readers solved only the analogous problem for a function defined on the non-negative *real* numbers.

Solved by the proposer and 37 others.

ANSWER TO PICTURE PUZZLE: The great probabilist Andrej Nikolajevich Kolmogorov.

REVIEWS

Edited by **Darrell Haile**
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Gödel's Theorem in focus. By S. G. Shanker. Routledge, London and New York, 1990, vi + 261 pp.

Reviewed by **C. Smoryński**

One of the most widely, if not deeply, known results of 20th century mathematics must surely be Gödel's Theorem (actually: pair of theorems) on incompleteness. The popularity of Gödel's result has never sat very well with mainstream mathematicians who, generally speaking, know little logic and can comprehend neither its meaning nor its appeal. Their attitude was summed up beautifully by one unknown who wrote that students who hear of Gödel's Theorem either recover from it or else go on to become experts on mathematical logic. To the yuppy logicians of the late 60s and early 70s, to whom acceptance by the mathematical community had to be achieved by providing combinatorially wicked constructions in recursion theory or set theory, Gödel's Theorem proved something of an embarrassment: its proof just isn't that difficult. At least, by then it wasn't, but only specialists were in a position to know that logic had progressed far beyond Gödel (if "progressed" is, indeed, the right word for a field then dominated by a veritable teratology of counterexamples). Even logic-friendly mathematicians who had occasionally taught logic in the 50s blithely assumed they could teach a modern course in the subject after a couple of weeks' preparation. In the late 70s the Paris-Harrington Theorem (on the arithmetic independence of a variant of the finite Ramsey Theorem) came along and the new yuppies tried to use Gödel's Theorem as a lever to lift Paris-Harrington into the view of the general mathematician and win respect and recognition for their field: with Paris-Harrington one had not only an interesting result, but one with a difficult proof: or, reflecting less on the logical attitude of a few years earlier, a result that not only had a difficult proof, but was also genuinely interesting—indeed, from the beginning the contrast was even more invidious: Gödel's Theorem was deemed uninteresting because it was *artificial*. Many logicians still believe this. It would seem that, even after 60 years, we are still having difficulty putting Gödel's Theorem into focus.

Gödel's Theorems may be stated as follows.

First Incompleteness Theorem. *Any consistent formal theory T which contains enough arithmetic is incomplete. In fact, T does not prove the sentence φ expressing its own unprovability; and if T is a bit more than consistent, T does not prove $\neg\varphi$.*

Second Incompleteness Theorem. *No consistent formal theory T which contains enough arithmetic can prove the sentence expressing T's consistency.*

The words, beyond those with intuitively obvious meaning ("consistent," "theory," etc.), which are problematic and which have drawn the attention of philosophers are "formal", "contains", "enough", "the", and "expressing". Obviously, a first step toward bringing Gödel's Theorems into focus is to explain these terms. Equally obviously, a first step towards this is to present the proof. An analysis thereof will reveal what effectivity is implicit in the gloss "formal," how much arithmetic is "enough" and the sense in which this must be "contained" in T, etc. Mathematically speaking, each of these words has acceptable, precise meanings—meanings which are not universally accepted. The problem is that one wants to draw philosophical conclusions from Gödel's Theorems, and the strict mathematical definitions seem to raise more problems than they settle. Under the most commonly accepted notion of "expressibility" for the Second Theorem, for example, there are many *inequivalent* sentences expressing the consistency of T within the language of T. Which one *really* expresses this consistency? A philosopher just will not accept the attempt to brush the problem aside by saying that it doesn't matter because *all* of these sentences are unprovable. He may point out that, under a slight weakening of the conditions governing "expressibility," there can be provable assertions of consistency—and that the dropped requirement is a seemingly *ad hoc* technical one. To make matters worse, there are now proofs of the Second Theorem for theories so weak that it is questionable if the sentence "expressing" consistency can properly be said to express it in anything even approaching an intuitive sense. It is a small wonder that some logicians decry Gödel's unprovable sentences as artificial—produced by mere coding tricks—and celebrate instead the combinatorial independence results of Paris, Harrington, and their followers. These independent sentences are also arithmetically "expressed" via coding tricks; but, if one is not wedded to any reductionist programme requiring one to stick to a narrow arithmetic language, they are naturally enough stated in a standard mathematical language making no mention of syntax. The Paris-Harrington Theorem does for Peano Arithmetic everything that the First Incompleteness Theorem does. Moreover, it does so clearly and unambiguously. There are no terms that have to be explained to the logically-illiterate mathematician or to the trouble-making philosopher. Everything about the Paris-Harrington Theorem lies right on the surface for all to appreciate.

Alas, Paris-Harrington is not Gödel. No one could conceive of the Paris-Harrington Theorem as proof that mind is not a machine, as a refutation of the logicist philosophy of mathematics, or of having any epistemological significance at all. Whether Gödel's Theorems have genuine epistemological significance is not clear, but they do *seem* to have some relevance to the subject, and if they indeed are relevant we cannot avoid having to answer the philosophers' queries about "expressibility." In any event, Gödel's Theorems seem to have an extra-mathematical significance that no other results in mathematics have—not even the controversial non-quantitative applications of catastrophe theory, which, incidentally, have never achieved the wide audience that Gödel's Theorems have.

If Gödel's Theorems have clear mathematical significance—the clarity readily available in the textbooks—and a questionable philosophical significance, they also have a genuine historical significance. Their historical significance has, however, been muddied, at least in the US, by a loss in translation and the repetition of error. Putting Gödel's Theorems into focus must also include explaining their historical background. What problem did Gödel address himself to and what specifically did each of his incompleteness theorems accomplish? I have attempted to set the record straight elsewhere (*CWI Quarterly* 1 #4 (1988), pp. 3–59) and

other than to state the obvious—that Hilbert's programme called for a consistency proof for mathematics and that Gödel showed this couldn't be done—I will not do so here. I should like to remark, however, that the history of mathematics is sadly neglected in our textbooks. A knowledge of history gives us both a broader perspective on our field and, occasionally, a better appreciation of individual results. This latter is certainly true of Gödel's Theorems: at the very least, knowledge of their rôles in killing Hilbert's programme and in providing tools for Kleene's subsequent development of recursion theory would forestall any thoughts that Gödel's Theorems have been supplanted by the Paris-Harrington Theorem, as some ahistorical logicians seem to believe.

Attempts to put Gödel's Theorems into sharper focus, or at least to explain them to the nonspecialist, abound. The "grand old man" of Gödelian exegesis on the American scene is the little monograph by E. Nagel and J. R. Newman entitled *Gödel's Proof*. Its popularity with everyone but me is unquestioned. Personally, I feel Nagel and Newman cloud things up a bit, not only by their attempt to make the proof appear more difficult than it is, but also in their discussion of epistemological issues. By far the most popular attempt to broadcast the truth as revealed by Gödel's Theorems is Douglas Hofstadter's *Gödel, Escher, Bach; An Eternal Golden Braid* which book, perhaps, goes too far in its appreciation of Gödel's results, which book some logicians dislike because of minor technical errors (as if any work of broad scope did not possess errors), but which book I must admit I find quite enjoyable. My personal favourite, however, is Rudy Rucker's *Infinity and the Mind*, which I recommend without reservation to anyone with the requisite technical skills, i.e. any professional mathematician or advanced undergraduate. Where these books are aimed at a more-or-less mathematically literate public, Shanker's compendium *Gödels Theorem in focus* is aimed at the professional philosopher and is a whole new story.

Gödel's Theorem in focus is a mixed bag—a *very* mixed bag. It is, with one exception, a collection of previously published papers on three subjects—logic, history, and philosophy. The logical and historical papers are quite good and I recommend them whole-heartedly, but I am only enthusiastic about one of the philosophical papers. That said, I must also warn the reader, however, that neither the good philosophical paper nor the historical papers are completely relevant.

The sole logic article included in the book is, naturally enough, Gödel's famous paper itself, given in the excellent English translation that first appeared in Jean van Heijenoort's *From Frege to Gödel; A Source Book in Mathematical Logic*. The typography of the present printing is not up to the standard of the earlier Harvard University Press version, but this is, perhaps, a trifling point as the paper is eminently readable. Indeed, except in matters of generality, Gödel's paper remains to this day the most readable exposition of his proof of the First Incompleteness Theorem in print. Shanker made a wise choice in including the original instead of a later exposition. If he is to be faulted at all, it is for not having also included Solomon Feferman's "Arithmetization [sic] of metamathematics in a general setting", which is the next most important paper after Gödel's on the subject and which first fully clarified the problem of the gloss-words "formal", "enough", etc.

The historical articles include a brief biography of Gödel (by John Dawson), an overview of Gödel's contributions to logic (by Stephen Kleene), a study of the initial reactions to Gödel's Theorems (by Dawson), and a sort of scientific-personality profile (by Feferman). These articles are informative and well-written, but they offer, as in the title of Dawson's biographical sketch, "Kurt Gödel in sharper focus" more than "Gödel's Theorems in focus". For the latter, Hilbert's pro-

gramme—the background to and problem addressed by Gödel's Theorems—ought to have been given more attention than a mere passing mention, especially as this is vitally important for philosophical discussions of Gödel's results.

When compiling the above mentioned *Source Book*, van Heijenoort inadvertently did logical scholarship a disservice by declaring Hilbert's paper "On the infinite" to offer the clearest description of his programme Hilbert ever made. This remark, neither true nor false, is a kind of half-truth that has licensed American philosophers to ignore those papers of Hilbert's that van Heijenoort did not have translated and base their discussions of Hilbert's programme on this one confused paper. To a small extent, Michael Resnik's contribution to the volume, "On the philosophical significance of consistency proofs", suffers from this defect. The main thrust of his paper, however, is unaffected by this historical inaccuracy and I mention his paper in this respect merely to illustrate the point above with a readily cited and relevant example of an established philosopher who has been misled by van Heijenoort.

As for Resnik's paper, Shanker chose wisely in reprinting it. For it is a prime example of the philosophical concern over the meaning of the word "expressing" in the statements of Gödel's Theorems. Resnik takes as his point of departure the provable assertions of consistency alluded to earlier (which, incidentally, Feferman constructed in the unincluded paper cited above), and asks what bearing they have on Hilbert's programme. Do they hold out hope for a consistency proof for mathematics? The answer, of course, is "no." Feferman's statements, albeit technically useful, are genuinely artificial and for philosophical purposes irrelevant. I am torn between simply dismissing Resnik's paper as wrong-headed, and grudgingly acknowledging that one must explain the sense in which Feferman's consistency statements are irrelevant and that this is indeed Resnik's purpose. In any event, I find Resnik's paper preferable to non-technical discussion of the mind as a machine or not as a machine, as can be found in, e.g., Nagel and Newman's little monograph cited earlier. Nonetheless, an anthology bearing the title that the present work does ought to have included an example of such a discussion.

Shanker's own contribution, the only one written specifically for the book, is a delight. It is well-written and exhibits insights into the philosophy of mathematics rare in a philosopher. The man is a scholar who has read the relevant works and whose writing reveals that he has understood them. Unfortunately, it requires an act of faith to get far enough into the paper to realise this. Let me explain. The paper starts off like gangbusters, making the interesting point that there are two kinds of impossibility proofs—those like Abel's which open up new vistas and those like Wantzel's which simply "close a chapter in mathematics". Wantzel's result, the impossibility of the trisection of the angle, is a beautiful example of a result of purely historical significance. The problem had been around for ages when Wantzel came along, translated it into algebraic terms, and then appealed to Abel's results. And that was that. It is a bit disconcerting at this point in Shanker's discussion to see the question raised as to whether Gödel's Theorem is a breakthrough *á la* Abel or simply a roadblock *á la* Wantzel. The problem with Shanker's paper, at least to anyone who is not a Wittgenstein fan, is that the whole paper is a huge counterfactual explaining what Wittgenstein would have said about Gödel's results had he chosen to write seriously about them and not merely dismiss them in passing with a remark that convinced everyone that he didn't understand them. Thus, the above question is not a mark of ignorance, as one is tempted to read it, but a reflexion of what Wittgenstein would have asked back then when the answer was not yet clear. Still, it is a bit jarring to read it and I must confess that

only on being forced to re-read the article carefully for the present review did my initial negative reaction to the paper change to an appreciation of it. After Gödel's paper, I find it the most fascinating one in the volume.

It must be said that Shanker's article on Wittgenstein on Gödel does not so much put Gödel's Theorems into focus as it does put Wittgenstein into focus. Here in microcosm is a reflexion of the main flaw of the book, or, at least, of a book which would call itself *Gödel's Theorem in focus*: Gödel's Theorems are not very well focussed on. Some of the contributions are of secondary relevance, and some topics of primary relevance (Feferman's work, Hilbert's programme) are only tangentially touched upon. The bottom line is that I would recommend the volume as a handy, but very incomplete, anthology for philosophers of mathematics and logicians. I would not recommend it to the general mathematician, however curious he may be about Gödel's Theorems, simply because he will not find much illumination in it. Rucker's above cited book is a much better choice.

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A Course in Modern Geometries. By Judith N. Cederberg, Springer-Verlag, New York, 1989, xii + 232 pp.

Reviewed by **Gudlaugur Thorbergsson**

Until the last century geometry meant the study of the space we live in. Judith Cederberg's book develops not only this geometry but, as the plural in the title indicates, several others as well. What geometry means nowadays is not easy to make precise. Here it means an axiomatic system based on points, lines, planes, and the incidence relations among them. Euclidean, affine, hyperbolic, elliptic, and projective geometry are such systems and all are considered in the book. As is appropriate in an elementary textbook, only the two-dimensional versions of these geometries are studied.

Euclid's geometry is the paradigm of the axiomatic approach in mathematics. The fifth and most famous of the postulates was considered unsatisfactory by Euclid himself. In its best known formulation this so-called parallel axiom (or Playfair's axiom) states: through a given point not on a given line, exactly one line can be drawn that does not intersect the given line. Equivalent statements are (a) the sum of the angles in a triangle is equal to π and (b) given any three noncollinear points, there exists a circle passing through them.

Much effort was spent trying to prove the fifth postulate from the other four. It was only in the last century that Lobachevsky (1829) and Bolyai (1832) realized that this cannot be done. They developed a geometry, now called hyperbolic, which satisfies all of Euclid's postulates except that one assumes, instead of the fifth, that at least two lines can be drawn parallel to a given line through any point not on the given line. This was a very bold step that went against the ideas of the time. It is clear from Gauss's correspondence that he knew about this geometry, but did not want to publish it because he feared that a theory contradicting some of the most influential contemporary philosophers would not be understood.

There are many surprising theorems in hyperbolic geometry. The sum of the angles in a triangle is less than π and depends on the area of the triangle.

Furthermore the sum decreases as the area increases. For this reason Gauss tried to verify whether the geometry of the space we live in is Euclidean or hyperbolic by measuring the sum of angles in large triangles with mountain tops as vertices. The experiment was not conclusive because the difference from π was within the error limits of the measurements.

Elliptic geometry was also discovered during the last century. Here the fifth postulate is replaced by the assumption that all lines intersect and the second postulate is reinterpreted to mean that a line segment can be prolonged to a boundaryless although not necessarily infinite line. The reader may wonder what this means, but it must be remembered that Euclid's language, or modifications of it, are not very precise by modern standards! One must also add either that a line separates the plane or that two distinct points lie on a unique line. The first possibility gives us an axiomatic system that is satisfied by the two-dimensional round sphere with the great circles as lines. Notice that there are infinitely many great circles joining antipodal points so two distinct points do not lie on a unique line. The other possibility gives us an axiomatic system satisfied by the sphere with antipodal points identified. This geometry is called one-sided elliptic geometry because it shares with the well-known Möbius band the property that concepts such as "left" and "right" do not make sense. This brings us close to projective geometry. Roughly the difference between one-sided elliptic geometry and projective geometry is that the latter does not have concepts of measurement (such as length and area). What the geometries have in common is the same set of points and lines.

So far we have been describing the axiomatic method (also called the synthetic method) that is the content of the first two chapters of the book. As in the case of elliptic geometry one can also work with "models" that satisfy the axioms. These models belong to the realm of analytic geometry, which has Cartesian coordinates of two or three (or higher) dimensional space as basic construction. The models are then either subsets of the coordinate spaces or obtained from them by identifying points. It can of course happen that the models have more properties than can be proved from the axioms—one of the most beautiful topics in projective geometry (and the subject not covered in the book I miss most) is the proof that, in an appropriate axiomatic system, coordinates can be introduced which establish an equivalence between the geometry and the projective plane one obtains from the sphere by identifying antipodal points.

Analytic geometry dominates the second half of the book. In chapter three it is shown that the isometries of the Euclidean plane can be represented as linear transformations of three-space that leave the plane $x_3 = 1$ invariant. (Cederberg's definition of an isometry assumes it can be represented by a 3×3 matrix; I would prefer to define the isometries of the Euclidean plane as those of its transformations that preserve distance and then prove they can be so represented.) Matrices are used to classify the isometries into translations, rotations, reflections and glide reflections and the seven possible frieze groups are listed. These are the groups of isometries of the plane that leave a line invariant and whose translations form an infinite cyclic subgroup.

The fourth and last chapter is on projective geometry. Roughly speaking the projective plane is the affine plane with "a line added at infinity." One of the reasons for introducing this geometry is that it simplifies statements and proofs from Euclidean and affine geometry. For example all regular conics are equivalent in the projective plane; the parabola is a conic touching and the hyperbola a conic intersecting the line at infinity. The subject is first considered from the axiomatic

point of view; for example there is a purely synthetic treatment of conics. An analytic model is then developed and the cross ratio, a substitute for the concept of distance in Euclidean geometry, is introduced and its connection with harmonic sets explained. The chapter (and book proper) ends with a proof that all of the geometries discussed in the previous chapters may be viewed as subgeometries of projective geometries. This serves to unify the treatment and to emphasize the central importance of projective geometry.

There are several appendices. In the first Euclid's definitions, postulates, and first thirty propositions of Book I are reprinted. Later appendices include the axiom systems of Hilbert and Birkhoff for the Euclidean plane. There is a good bibliography and every chapter ends with suggestions for further reading.

This is a very well written book that presents a good mixture of axiomatic and analytic geometry and its history. It should serve as an excellent textbook in a geometry course for junior or senior mathematics majors. By this I do not want to say that it will be easy reading—many students will certainly find this book difficult—but those willing to invest the necessary work should profit from the choice of material and the method of presentation. Geometry courses are not common at colleges today. Other courses better prepare students for graduate school and the subject is somewhat isolated within mathematics. (However, as Cederberg points out, interest is increasing in part because of the development of computer graphics.) One of the best reasons for using the book in such a course is its emphasis on the history of the subject. Such a course is very much in the spirit of what we expect from a college education and should be taken by all mathematics majors.

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TELEGRAPHIC REVIEWS

Edited by
Arnold Ostebee and Paul Zorn
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Books and software submitted for review should be sent to *Book Reviews Editor, American Mathematical Monthly, St. Olaf College, Northfield, Minnesota 55057.*

Reference, S*, P, L*. *UNIX in a Nutshell: System V Edition, Revised and Expanded for SVR4 and Solaris 2.0.* Daniel Gilly. O'Reilly & Assoc, 1992, xv + 402 pp, $9.95 (P). [ISBN: 1-56592-001-5] A concatenation of quick reference guides for major UNIX tools: shells (Bourne, Korn, C); editors (emacs, vi, ex, sed, awk); text formatting with nroff/troff (mm, ms, me macros); preprocessors (tbl, eqn, pic); and development tools (sccs, rcs, make, sdb, dbx). Based on System V and Solaris (Sun) 2.0. A handy summary reference. LAS

Finite Mathematics, T(13-14). *Applied Finite Mathematics, Fifth Edition.* Howard Anton, Bernard Kolman, Bonnie Averbach. Saunders College, 1992, xv + 600 pp, $40 net. [ISBN: 0-15-502942-8] Begins with elementary linear algebra. Treats linear programming, and basic combinatorics and probability. Statistics chapter includes normal approximation to binomial distribution and chi-square test. Applications such as finance, insurance, game theory, genetics. (*First Edition*, TR, January 1975; *Second Edition*, TR, June-July 1978; *Third Edition*, TR, November 1983.) AD

Finite Mathematics, T(13). *Mathematics with Applications for the Management, Life, and Social Sciences, Fourth Edition.* Bernard Kolman, Howard Anton, Bonnie Averbach. Saunders College, 1992, xv + 992 pp, $44 net. [ISBN: 0-15-555228-7] Something for everybody, from matrices to the mathematics of finance, from linear programming to probability. With such a smorgasbord of topics from finite mathematics, it is a bit surprising to find 330 of the 880 pages of textual material devoted to calculus. (*Second Edition*, TR, April 1984; *Third Edition*, TR, August-September 1988.) AWR

Education, S(15-16). *First-Grade Book.* Grace Burton, *et al.* Addenda Ser., Grades K–6. NCTM, 1991, viii + 24 pp, $9.50 (P). [ISBN: 0-87353-311-9] Activities and sample lessons in the areas of patterns, number sense, data sense, and geometry. Connections with literature, science, and Logo algorithms. Children estimate numbers of ladybugs, make graphs, and use geoboards to solve triangle problems. Idea starter for pre-service teachers. MW

Education, P, L. *Symbolic Computation in Undergraduate Mathematics Education.* Ed: Zaven A. Karian. MAA Notes No. 24. MAA, 1992, x + 181 pp, $20 (P). [ISBN: 0-88385-082-6] 22 papers reflecting on issues arising as symbolic computation begins to spread in undergraduate courses: philosophy, pedagogy, and caveats; course examples (calculus, linear algebra, differential equations, combinatorics, graph theory, probability and statistics); and aids for getting started. A valuable "state-of-the-art" survey. LAS

Education, P, L. *Data Analysis and Statistics Across the Curriculum.* Gail

Burrill, *et al.* Addenda Ser., Grades 9–12. NCTM, 1992, viii + 88 pp, $15 (P). [ISBN: 0-87353-329-1] Another in the continuing NCTM series of guides for implementing the *Standards.* Introduces statistics via data (linear and nonlinear); concludes with chi-square and student projects. LAS

History, P, L. *A History of Inverse Probability: From Thomas Bayes to Karl Pearson.* Andrew I. Dale. Stud. in History of Math. & Physical Sci., V. 16. Springer-Verlag, 1991, xx + 495 pp, $59. [ISBN: 0-387-97620-5] Begins with a brief biographical sketch of Bayes and an examination of "Essay towards solving a problem in the doctrine of chances," and ends with a brief discussion of K. Pearson's thoughts on (and feelings about) Bayes' Theorem. Includes little biographical detail and little in way of sociological or historical perspective. Does include discussion of influential papers on inverse probability from Bayes, Condorcet, Laplace, Poisson, Venn, Laurent, Pearson, and many more. MK

Foundations, P. *The Mathematical Philosophy of Bertrand Russell: Origins and Development.* Francisco A. Rodriguez-Consuegra. Birkhäuser, 1991, xiv + 236 pp, $68.50. [ISBN: 0-8176-2656-5] Study of development of Russell's mathematical philosophy from mid-1890's through publication of *The Principles of Mathematics* in 1903. Examines influence of other mathematicians and philosophers on Russell and analyzes his early works, including unpublished manuscripts. KES

Combinatorics, P. *Probabilistic Combinatorics and Its Applications.* Eds: Béla Bollobás, *et al.* Proc. of Symp. in Appl. Math., V. 44. AMS, 1991, xv + 196 pp, $56. [ISBN: 0-8218-5500-X] New developments in the probabilistic method in combinatorics and graph theory. Topics include construction of random-like graphs, Martingle techniques and discrete isoperimetric inequalities, rapidly mixing Markov chains and finite Fourier methods, applied to problems on the discrete cube, chromatic number, and convex body volume approximation. JPH

Number Theory, T(13-16: 1), S*, L.** *A First Course in Number Theory.* Hugh M. Edgar. Wadsworth, 1988, xiii + 138 pp. [ISBN: 0-534-08514-8] Engaging, readable, humorous, yet rigorous text suitable for either a standard course or for independent study. Short yet complete; includes an introduction to the p-adic absolute value (along with applications such as the p-adic Newton's method for successive approximations). Excellent range of exercises; notes at the end of the chapters give excellent references for further study and current research. Appendix includes solutions/hints to selected exercises, basic proof techniques, and some abstract algebra topics. SB

Group Theory, T(18: 1), S, P. *Geometry of Defining Relations in Groups.* A. Yu. Ol'shanskii. Math. & Its Applic., V. 70. Kluwer Academic, 1991, xvi + 505 pp, $185. [ISBN: 0-7923-1394-1] By use of diagrams to represent groups defined by relations, the book is "the systematic implementation of the non-standard formula 'algebra-geometry-algebra' resulting in the use of elementary topological and geometric techniques" to solve a number of difficult problems, many of long standing, in group theory. Assuming a modest formal background, the first two chapters present the relevant group theory and a third does the same for topology. Remaining ten chapters are largely a summary of recent work of the author and his students, including a proof of the Novikov–Adian theorem and solutions to problems of Schmidt, Markov, von Neumann, and P. Hall. Bibliography, index. Note price! JS

Algebra, P. *Semigroups.* H. Jürgensen, F. Migliorini, J. Szép. Akademiai Kiado, 1991, v + 121 pp, $23 (P). [ISBN: 963-05-6046-1] Combines some work of the authors and other researchers emphasizing finite semigroups. Focuses on the decomposition of a semigroup as a union of a family of subsets, as opposed to a decomposition based on products (Krohn–Rhodes theory). Assumes familiarity with basic semigroup theory. MC

Algebra, T(15-17: 1), S, L. *Introduction to the Galois Correspondence.* Maureen H. Fenrick. Birkhäuser, 1992, xi + 195 pp, $49.50. [ISBN: 0-8176-3522-X] Essentially self-contained but best suited to follow an introductory course in algebraic structures. After a preliminary chapter on groups and rings followed by one on field extensions, the Galois correspondence is presented in full. Applications are made to insolvability, geometric constructions, Wedderburn's theorem on finite division rings, and Dirichlet's theorem on primes in an arithmetic progres-

sion. Exercises, bibliography, index. JS

Real Analysis, P, L**.** *Divergent Series, Second Edition.* G.H. Hardy. Chelsea, 1991, xvi + 396 pp, $28.50. [ISBN: 0-8284-0344-1] Ignoring questions of convergence yields some whiz-bang proofs in analysis. Euler, of course, was adept at this, though far from the only offender. Hardy avers that every divergent series has a "reasonable" sum about which conclusions may be drawn. He gives several different definitions of summability (e.g., under Cesàro-summability, based on the average of partial sums, $1 - 1 + 1 - \ldots = 1/2$), and demonstrates when it is appropriate to draw analytical conclusions from these other notions of summability. Erudite, witty, and laced with history and historical motivation, the final work of a master, this beautiful book belongs in every mathematics library. SK

Complex Analysis, P. *Finiteness Theorems for Limit Cycles.* Yu. S. Il'yashenko. Transl. of Math. Mono., V. 94. AMS, 1991, ix + 288 pp, $196. [ISBN: 0-8218-4553-5] Proves the finiteness theorem: a polynomial vector field on the real plane has a finite number of limit cycles. Note price. MLR

Complex Analysis, S(18), P. *Introduction to Complex Analytic Geometry.* Stanisław Lojasiewicz. Transl: Maciej Klimek. Birkhäuser, 1991, xiv + 523 pp, $118. [ISBN: 0-8176-1935-6] A comprehensive "toolkit," at the advanced graduate level, on complex analytic geometry—i.e., local analytic and geometric properties of zero sets of analytic functions, always in the complex domain. Three initial chapters review necessary basics in algebra, topology, and complex analysis, all at graduate level. Largely self-contained: author aims to provide necessary background without resort to "well-known" results. No exercises. *First Edition*, in Polish, was published in 1988. PZ

Partial Differential Equations, P. *Inverse Scattering and Applications.* Eds: D.H. Sattinger, C.A. Tracy, S. Venakides. Contemp. Math., V. 122. AMS, 1991, xiii + 133 pp, $41 (P). [ISBN: 0-8218-5129-2] Proceedings of a conference on inverse scattering held at the University of Massachusetts in 1990. Thirteen papers on a variety of topics including inverse scattering, inverse conductivity, numerical methods, monodromy, quantum scattering, and the Bethe ansatz. Preface and summary. JS

Partial Differential Equations, T(16), S. *Partial Differential Equations of Evolution.* Jaroslav Barták, *et al.* Math. and Its Applic. Ellis Horwood, 1991, 261 pp, $52. [ISBN: 0-13-651449-9] By evolution one means that the solution has time dependence. This text presents a systematic treatment of four basic linear partial differential equations: the wave (telegraph) equation, the heat equation, the fourth order beam and plate equations, and all first order equations. Methods used include characteristics, Laplace transforms, and separation of variables. All in all, a fairly thorough treatment of the subject. No exercises. MPR

Numerical Analysis, T(16-17: 1, 2), L. *Numerical Methods for Differential Equations: Fundamental Concepts for Scientific and Engineering Applications.* Michael A. Celia, William G. Gray. Prentice Hall, 1992, xii + 436 pp. [ISBN: 0-13-626961-3] Methods for initial value problems, boundary value problems, and partial differential equations. Develops methods from fundamental concepts such as characteristics and finite difference approximations. Includes finite element methods, accuracy considerations, and dynamic grids. RWN

Numerical Analysis, P. *Rational Approximations and Orthogonality.* E.M. Nikishin, V.N. Sorokin. Transl. of Math. Mono., V. 92. AMS, 1991, viii + 221 pp, $90. [ISBN: 0-8218-4545-4] Rational approximation of analytic functions. Chapters discuss rational approximation of numbers, Padé approximants and orthogonal polynomials, asymptotic properties of orthogonal polynomials, simultaneous Padé approximants, and potential theory. LC

Operator Theory, P. *Estimates and Asymptotics for Discrete Spectra of Integral and Differential Equations.* Ed: M. Sh. Birman. Adv. in Soviet Math., V. 7. AMS, 1991, x + 204 pp, $118. [ISBN: 0-8218-4106-8] Seven papers on spectral theory given to the Leningrad Seminar on Mathematical Physics (1989–1990). For the most part, devoted to investigations of the spectrum of the Schrödinger operator perturbed by some relatively compact operator. KS

Analysis, P. *Dirichlet Forms and Analysis on Wiener Space.* Nicolas Bouleau, Francis Hirsch. Stud. in Math., V. 14. Walter de Gruyter, 1991, x + 325 pp, $69. [ISBN: 3-11-012919-1] "Introduction to the ideas, phenomena, and methods of analysis in infinite-dimensional spaces, in particular

Wiener spaces, and stochastic differential equations. Emphasis is on the interaction between two important tools: the Malliavin calculus and the theory of Dirichlet forms and spaces." Exercises at end of each section. Extensive bibliography. Rather dry exposition. BH

Algebraic Geometry, P. *The Curves Seminar at Queen's, Volume VIII.* Anthony V. Geramita. Papers in Pure & Appl. Math., No. 88. Queen's Univ, 1991, 233 pp, (P). Contains three expository articles on curves on cubic surfaces in P^3, algebraic curves and differential equations, and Strano's theorem. Also contains four research papers on flat families, White surfaces, ruled surfaces, and an algorithm for computing conductors. SP

Differential Geometry, S(16-17), L. *Elements of the Geometry and Topology of Minimal Surfaces in Three-Dimensional Space.* A.T. Fomenko, A.A. Tuzhilin. Transl. of Math. Mono., V. 93. AMS, 1991, vii + 142 pp, $100. [ISBN: 0-8218-4552-7] Exposition of some of the basics of minimal surface theory. Begins with soap films and Steiner's problem, presents some classical examples such as catenoids and the helicoid, and then discusses and proves some general properties. Designed to introduce the field and encourage further study. Some exercises; bibliography. OJ

Differential Geometry, P. *The Geometry of Supermanifolds.* Claudio Bartocci, Ugo Bruzzo, Daniel Hernández-Ruipérez. Math. & Its Applic., V. 71. Kluwer Academic, 1991, xix + 242 pp, $77. [ISBN: 0-7923-1440-9] The authors "wish to unfold a consistent and systematic, if not exhaustive, investigation of the structure of geometric objects—called supermanifolds—which generalize differentiable manifolds by incorporating, in a sense, anti-commuting variables." Intended to be a mathematics text rather than a physics text. JO

Differential Geometry, P. *The Differential Invariants of Generalized Spaces, Second Edition.* Tracy Yerkes Thomas. Chelsea, 1991, x + 241 pp, $27.50. [ISBN: 0-8284-0336-8] Republication of a work from 1934 which offered a then current account of recent developments in differential geometry. Invariants of various tensors are explored using local coordinates. OJ

Geometry, S.** *Not Knot.* Charlie Gunn, Delle Maxwell. 15 minute video. *Supplement.* David Epstein, Charlie Gunn. Jones & Bartlett, 1991, 48 pp, (P). [ISBN: 0-86720-240-8] A visual tour of hyperbolic space tiled with cells formed from the complement of the linked knot known as the Borromean rings—the part of space that is "not knot." Visually stimulating and mathematically challenging: merits repeated watching accompanied by careful reading of the *Supplement* which provides a complete script illustrated with key frames, interspersed with extensive "Q & A" to explain the rapid pace of unfamiliar, technical ideas. LAS

Topology, P. *Subfactors and Knots.* Vaughan F.R. Jones. CBMS Reg. Conf. Ser. in Math., No. 80. AMS, 1991, ix + 113 pp, $43 (P). [ISBN: 0-8218-0729-3] The record of the author's expository lectures delivered at the Naval Academy in 1988. The lectures cover a variety of topics—von Neumann algebras, braid groups, links, and statistical mechanics—and their relationship to knots. Extensive bibliography. SG

Mathematical Modelling, T?(16-17: 1), P. *Object-Oriented Systems Analysis: A Model-Driven Approach.* David W. Embley, Barry D. Kurtz, Scott N. Woodfield. Prentice Hall, 1992, xvi + 302 pp. [ISBN: 0-13-629973-3] Introduction to Object-Oriented Systems Analysis (OSA) which is based on a model-driven approach using object-oriented techniques for creating and maintaining large complex analysis models. Discusses methods for capturing and organizing information about objects and their relationships, object behavior models, object interaction models, model integration. RM

Systems Theory, P. *Lecture Notes in Control and Information Sciences-166: Local Disturbance Decoupling with Stability for Nonlinear Systems.* L.L.M. van der Wegen. Springer-Verlag, 1991, 135 pp, $24 (P). [ISBN: 0-387-54543-3] Develops a local theory for problems of the following sort: for a feedback system S, and desired equilibrium conditions where there are controlled inputs and uncontrolled inputs ("disturbances"), find a compensator which takes feedback and controlled inputs so that the disturbances do not influence the outputs and the equilibrium is exponentially stable with respect to the modified drift dynamics. RM

Probability, S(18), P. *Limit Theorems for Large Deviations.* L. Saulis, V.A. Statulevičius. Math. & Its Applic., V. 73.

Kluwer Academic, 1991, viii + 232 pp, $88. [ISBN: 0-7923-1475-1] Investigates probabilities of large deviations for sums of independent and dependent random variables, polynomial forms, multiple stochastic integrals or stochastic processes and fields, and some statistics. Theorems on large deviations proved by cumulant method. Shows that mixed cumulants of a random process can be estimated by various mixing functions. KB

Stochastic Processes, S(18), P. *Markov Processes: An Introduction for Physical Scientists.* Daniel T. Gillespie. Academic Pr, 1992, xxi + 565 pp, $44.50. [ISBN: 0-12-283955-2] Most stochastic processes books assume a prior knowledge of probability, look at some simple processes (Markov chains, Poisson processes and their immediate generalizations), and then pass to the more general settings of the Markov process. The author takes a far different approach as he first develops that part of probability that he needs, then carefully develops the general theory of Markov processes and finishes with the above examples. Too much material for even a one-year course and no exercises make this a questionable choice for a text, but the careful exposition and development make it an invaluable reference. TAV

Stochastic Processes, P. *Functional Equations in Probability Theory.* Balasubrahmanyan Ramachandran, Ka-Sing Lau. Prob. & Math. Stat. Academic Pr, 1991, xvii + 249 pp, $64.95. [ISBN: 0-12-437730-0] In the Preface, the authors admit that they are interested in just a limited collection of functional equations, mostly variations and extensions of the integrated Cauchy function equation, with some discussion of stability and semistability of processes. Highly technical. Extensive bibliography. TAV

Elementary Statistics, T(13: 1, 2). *An Introduction to Statistics with Data Analysis.* Shelley Rasmussen. Ser. in Stat. Brooks Cole, 1992, xix + 707 pp, $47.25. [ISBN: 0-534-13578-1] Covers data analysis using graphical and tabular techniques, basic probability models, confidence intervals, hypothesis testing (including ANOVA and chi-squared tests), correlation, and regression with an emphasis on data collection techniques and experimental design. Appropriate chapters include nonparametric procedures with explanations of when these are preferable to classical analysis. Examples and exercises use real data sets. Many chapters have a Minitab appendix. Requires only high school algebra. KB

Statistical Methods, T(17: 1). *Statistical Methods in Reliability Theory and Practice.* Brian D. Bunday. Math. & Its Applic. Ellis Horwood, 1991, 252 pp, $66. [ISBN: 0-13-853797-6] Well-written series of handouts for a lecture course in statistics given to reliability engineers. Includes chapters on stochastic processes and Bayesian methods. Outline solutions are provided for all exercises. Presumes some background in elementary statistics. RSK

Statistical Methods, S, C. *FAST*PRO: Software for Meta-Analysis by the Confidence Profile Method.* David M. Eddy, Vic Hasselblad. Academic Pr, 1992, xxiii + 225 pp, $295 (P), user manual and PC software. [ISBN: 0-12-230621-X] Software program and manual for the confidence profile method. Written for use on IBM-compatible personal computers. Manual describes use of software with numerous examples, overviews the confidence profile method providing background for software, gives tutorial introduction as well as more detailed, technical appendices. Technical support available from authors. MK

Computational Statistics, P. *The Frontiers of Statistical Computation, Simulation, & Modeling, Volume I of Proceedings of the ICOSCO-I.* Eds: Peter R. Nelson, *et al.* Ser. in Math. & Management Sci., V. 25. American Sciences Pr, 1991, vi + 338 pp, $98.75 (P). [ISBN: 0-935950-27-3] First of three volumes from 1987's First International Conference on Statistical Computing in Izmir, Turkey. Paper topics include random variate generation for binomial, Poisson, and Gumbel distributions as well as a more general combining method for random number generation, robust estimation of multivariate parameters, simulation in hypothesis testing, goodness-of-fit problems, and survey sampling. MK

Computational Statistics, P, L. *Statistical and Scientific Databases.* Ed: Zbigniew Michalewicz. Ser. in Comput. & Their Applic. Ellis Horwood, 1991, xii + 532 pp, $59. [ISBN: 0-13-850652-3] First text on statistical and scientific database (SSDB) management. Grew out of conferences held between 1981 and 1990. Chapters written by a variety of speakers at these conferences. Topics include proper-

ties of SSDB's; data analysis requirements; visualization; data models such as GRASS and MEFISTO; data integration; query languages; relational models; dynamic maintenance; query optimization; security; statistical expert systems. MK

Statistics, P. *Statistical Inference for Spatial Processes.* B.D. Ripley. Cambridge Univ Pr, 1991, viii + 148 pp, $19.95 (P); $37.50. [ISBN: 0-521-42420-8; 0-521-35234-7] Paperback release of 1988 hardcover copy (TR, June-July 1989). RWJ

Algorithms, S(16-17), P. *Topics in Distributed Algorithms.* Gerard Tel. Inter. Ser. on Parallel Computat., V. 1. Cambridge Univ Pr, 1991, x + 240 pp, $44.50. [ISBN: 0-521-40376-6] This monograph, an extension of the author's Ph.D. dissertation, takes an in-depth look at a selected set of advanced topics in the field of distributed computer systems. The three primary areas of investigation are synchronization of ABD (Asynchronous Bounded Delay Networks), verification of the correctness of distributed system software, and the modular design of distributed algorithms using a technique called the "building block" approach developed by the author. All areas are studied in great detail, with numerous examples. GMS

Computer Systems. *Essential System Administration.* Æleen Frisch. O'Reilly & Assoc, 1991, xxiii + 440 pp, $29.95 (P). [ISBN: 0-937175-74-9] A guidebook for UNIX system administrators explaining UNIX customs, routine administration (startup, shutdown, adding accounts, backup, accounting), and management issues (file systems, printers, terminals, modems, networks, security). Illustrates shell procedures useful for automating routine tasks. Well-written; very useful. LAS

Computer Systems, P*, L. *TEX By Topic: A TEXnician's Reference.* Victor Eijkhout. Addison-Wesley, 1992, viii + 307 pp, $29.25 (P). [ISBN: 0-201-56882-9] A comprehensive reference for TEX, organized by chapters into related groups of commands, each explained in context with numerous effective examples. Thoroughly indexed and cross-referenced; filled with insight and practical ideas. LAS

Applications (Biological Science), S (15-18), P, L. *Modelling Biological Populations in Space and Time.* Eric Renshaw. Stud. in Math. Biology, V. 11. Cambridge Univ Pr, 1991, xvii + 403 pp, $110. [ISBN: 0-521-30388-5] Analyzes problems using both deterministic and stochastic models. Theoretical mathematics in separate sections. Covers birth-death processes, time-lag models, competition, predator-prey, spatial predator-prey, fluctuating environments, spatial population dynamics, epidemic processes, and linear and branching architectures. DH

Applications (Physics), P. *Modern Theory of Anisotropic Elasticity and Applications.* Eds: Julian J. Wu, T.C.T. Ting, David M. Barnett. SIAM, 1991, 377 pp, $68.50 (P). [ISBN: 0-89871-289-0] Twenty-six papers on the mathematical properties of materials or media that stretch more easily in some directions than others. BC

Applications (Physics), T(18), S, P. *Twistor Geometry and Field Theory.* R.S. Ward, Raymond O. Wells, Jr. Mono. on Math. Physics. Cambridge Univ Pr, 1991, x + 520 pp, $34.50 (P); $89.50. [ISBN: 0-521-42268-X; 0-521-26890-7] A collection of papers beginning with an introduction to differential geometry and progressing through such topics as anomalies in quantum field theory, the role of stratification in anomalies, and knots and their links to biology and physics. MU

Applications (Physics), T(18: 1, 2), S, L. *Strings, Conformal Fields, and Topology: An Introduction.* Michio Kaku. Grad. Texts in Contemp. Physics. Springer-Verlag, 1991, xiv + 535 pp, $49.95. [ISBN: 0-387-97496-2] Although the present volume attempts to be self-contained, the non-expert would be well-advised to first consult the author's previous book *Introduction to Super Strings* (TR, January 1989). MU

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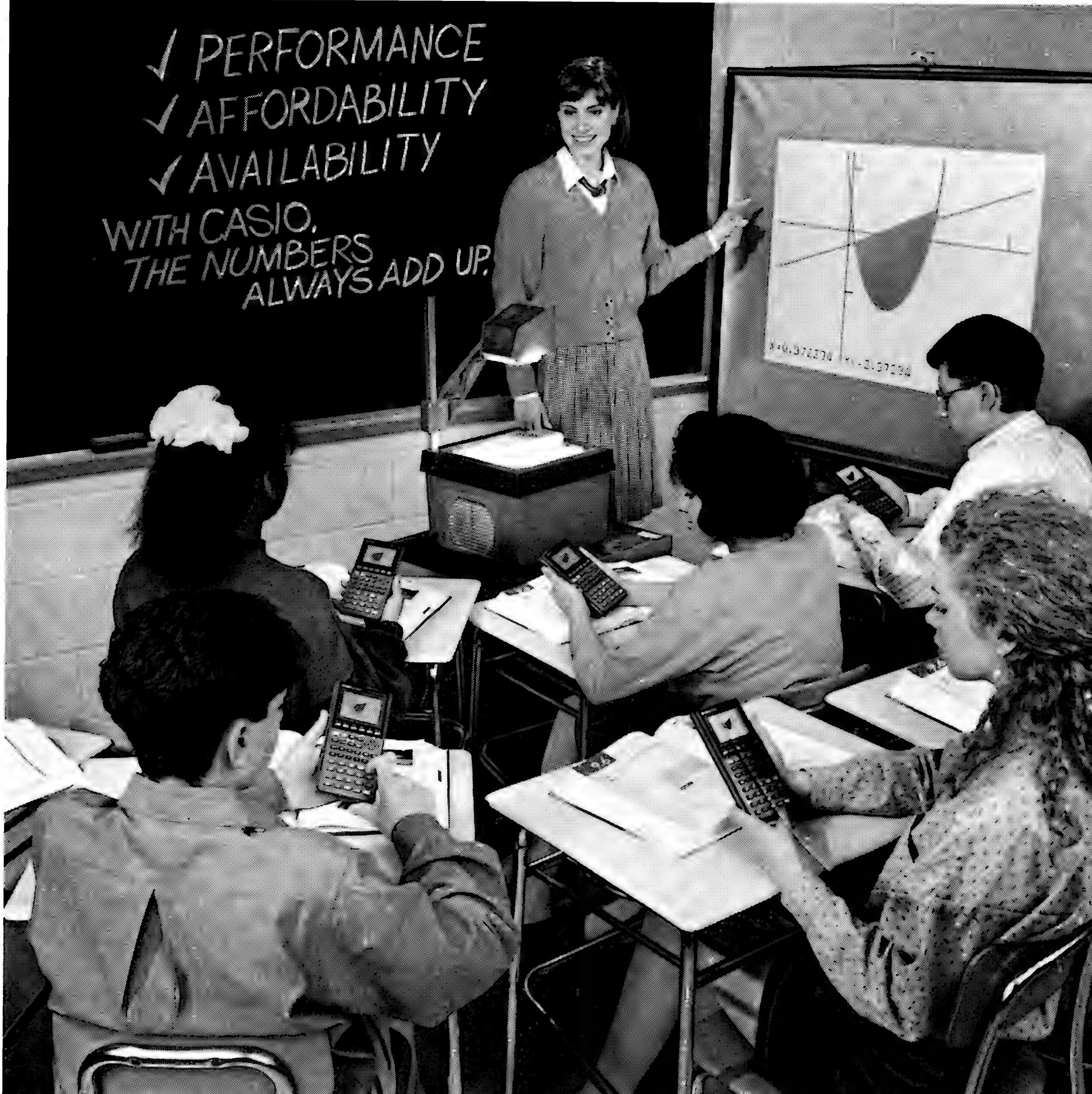
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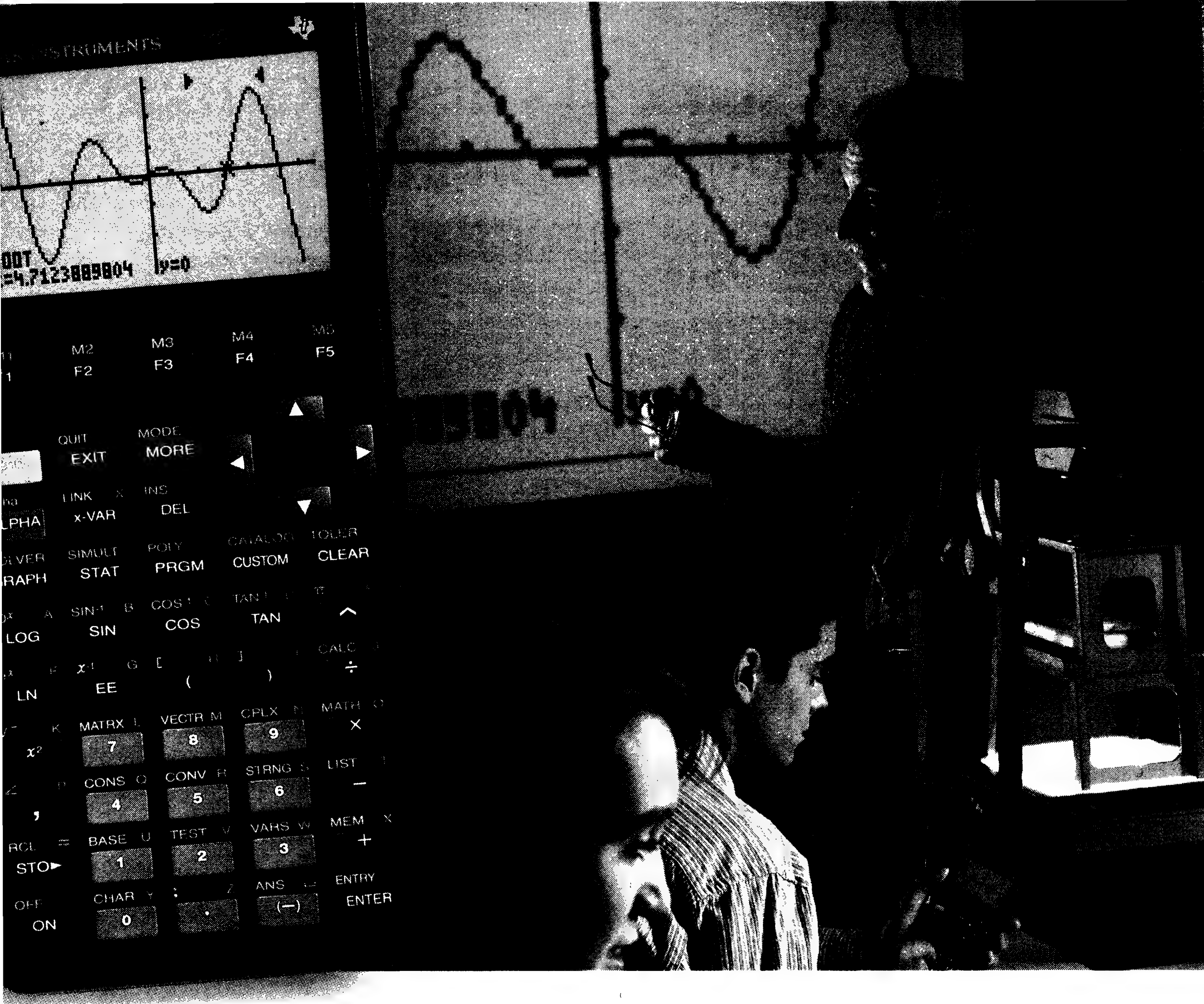

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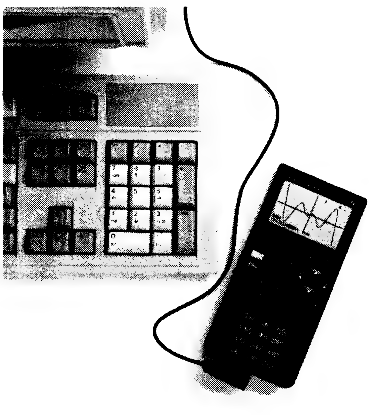

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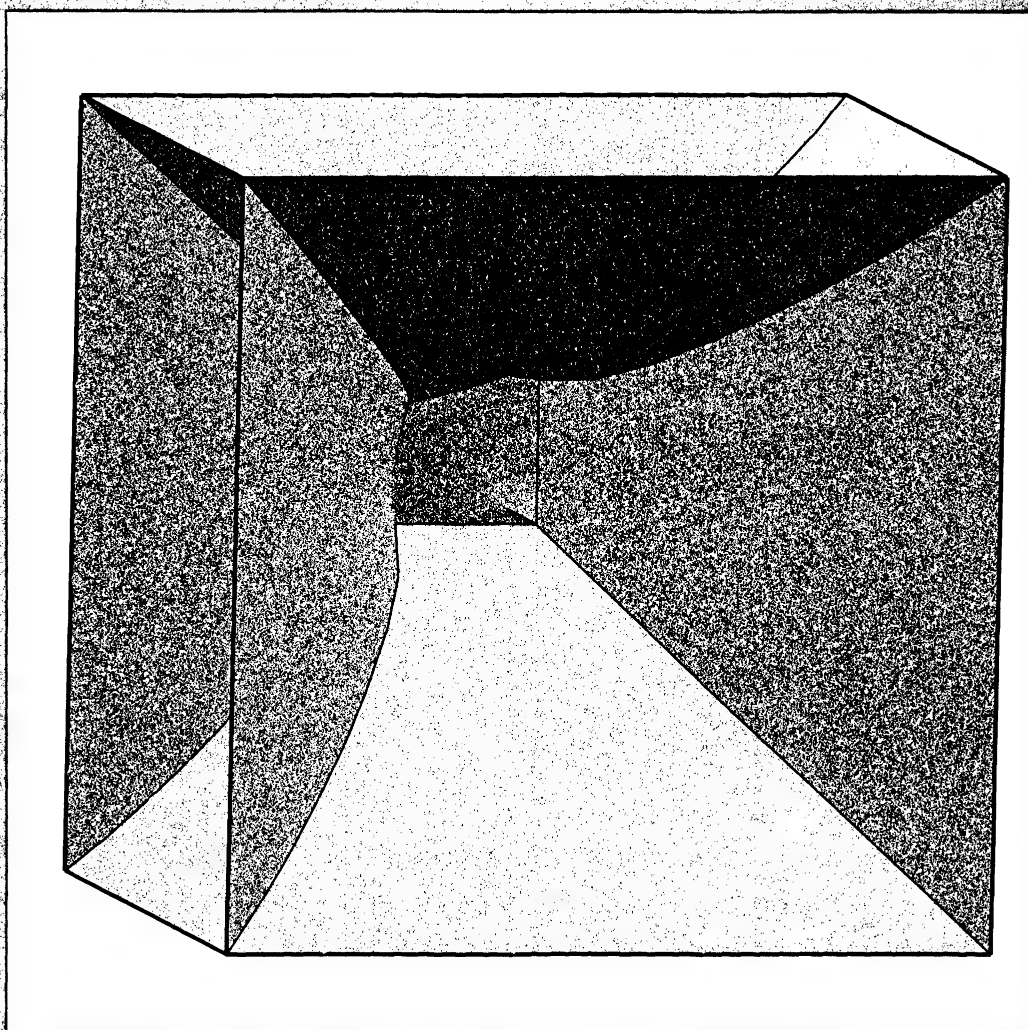

The Opaque Cube (p. 866)

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Articles may be expositions of old results or presentations of new ones They may concern all of mathematics or one small area, a broad development or a single application, historical reminiscences or one important event While some articles may contain the author's new research, the novelty of material and generality of the results is far less important than the clarity of exposition and general interest Discussing one illuminating case of a well known result is far better than providing all the details of an obscure but new proposition Articles in the *Monthly* are supposed to inform and to entertain, they are meant to be read rather than archived

Notes are short and possibly informal articles A note may concern a clever new proof of an old theorem, a novel way to present tired material, or a lively discussion of a philosophical (but still mathematical) issue. Also any topic is suitable, so long as it is related to mathematics Because a note is short, the first few sentences are the most important part They should explain the purpose and invite the reader in Photographs or diagrams often will attract the reader's attention.

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ARTICLES

FEATURES

COMMENTS

It's that time of year again, half way through the recommendation season. We pull out our pens (or more likely our computers) to practice what used to be an art—writing a subtle, honest, and helpful evaluation. For many, however, the satisfaction of crafting a thoughtful letter has turned to the dread of piecing together a few stale paragraphs of hackneyed phrases. The only craftsmanship here is inventing ways to make a substantive statement about someone you have met only once at a conference.

What's behind this glut of words? Several authors have recently chided young mathematicians who, eager and desperate to find jobs, apply to hundreds of institutions. Their mentors and colleagues, eager and desperate to help, write letters that tell of breathtaking mathematical feats in exaggerated terms. (It is remarkable that some universities award Ph.D.'s to "the best student we have had in the past 20 years"—every year.) But with jobs so scarce, can we blame them? And after all, we have always written letters for mathematicians entering the job market (although they used to be much shorter—try looking at a dossier from 20 years ago). Sending 200 copies of the same letter is bad for trees, but it has little ill effect on the writer.

The real cause of the glut comes at a later stage. We lack trust and the self-confidence to exercise judgment. Departments and deans demand an ever increasing number of letters of recommendation at every career decision—for renewal, for tenure, for promotion. At many universities, the numbers are daunting: 4 letters for the first appointment, 2 or 3 for renewal, 12 for tenure, and 12 for promotion to Full. Thirty letters for each career. Do we need 12 letters (4 for the department and 8 for the dean) to make a tenure decision? If we are unconvinced by the first 6 (from those who know the candidate best), should we be convinced by the second 6 (from those who know the candidate in passing)? Why is the dean more convinced by *outside* letters than by *inside* opinions? And why does the *department* rely so heavily on "experts", who know the candidate slightly, rather than on close associates and colleagues, who know the candidate well? (Indeed, some place more faith on a few sentences from an anonymous referee of a grant proposal than on the testimony of a colleague; the referee is the "expert" after all.)

Renewals and tenure decisions and promotions are matters of judgment, not endorsement. Forcing a dozen people to pretend to know a young mathematician's work and abilities subverts a system that once worked well. (Do we need to send people copies of all publications if they know the work well?) The pretense forces writers to substitute fussy details about the mathematics for pithy comments about the person. Letters explain the technical terms in theorems, but fail to comment on the talents that proved them. Relying only on the judgments of "experts in the field" gives unfair advantage to those fields that have the greatest sense of self-importance, and the *least* sense of perspective.

Letters of recommendation play an important role in making decisions about a mathematician's career. Departments should treat those letters, however, as partial evidence in a complicated judgment; they should have confidence to make that judgment themselves rather than to let the letters do it for them. Deans who cannot trust departments have a serious problem, but they will not solve it by asking for more letters; they only produce more pieces of paper, with less content. We should remind those deans that sometimes, more is less and less is more. In this case, fewer letters surely will produce more information . . . and better decisions.

—John Ewing

Giants[1]

Cathleen S. Morawetz

I could have chosen to speak about the progress and changes in applied mathematics that have taken place in the years since the MAA was founded. I have chosen instead to speak about these particular giants of applied mathematics not only because they represent a certain period and a certain influence but because they attained their distinction in very different ways. These people divide into two groups. Those who did nothing or nearly nothing but applied mathematics and those who divided their time between pure and applied mathematics. The first kind is exemplified par excellence by Sir Geoffrey Taylor and Theodore von Karman (he might be annoyed to have the label mathematician) and the examples of the second kind are John von Neumann, Norbert Wiener and Kurt Friedrichs. I will also say if time permits a few words about my father, John L. Synge since there is no question that I learned a lot from him about attitude and action in applied mathematics. I might add that he was chairman of mathematics on this campus from 1943 to 1947.

Before describing the first two of these men—I would like to say a word or two about the subject applied mathematics. This is a term that has different meanings attached to it by both its friends and enemies. Some people like to call it mathematics of the real world (an unattractive expression but at least fairly general), others think of it as being useful and still others use the term as equivalent to lack of rigor.

I wish I could avoid the expression "applied" altogether, but it's there and the meaning that I attach to it is:

(1) It is mathematics.

(2) It is connected to some other science including engineering science.

I then proceed to strip it down and I exclude statistics. If I did not want to talk about v. Neumann I would also exclude computer science. And I think computer scientists would most definitely agree.

The other sciences range all over: medicine, cryptography, economics, and I think we can be happy to embrace as much as we can.

What distinguishes pure mathematics is that it is exploring mathematics for itself. But I have yet to see in the flesh a pure mathematician who is not ecstatic with delight if someone can apply his result to some other science.

So I have picked my giants mainly on the basis of this distinction between pure and applied. But I have also picked them on the basis of my own knowledge and I confess the possibility of reminiscing.

I must have been interested at an early age in the struggles of the little department of applied mathematics in Toronto that my father chaired. As an

[1]This article is based on an address given by the author at the 75th anniversary of the founding of the MAA, Columbus, Ohio, in August 1990.

undergraduate I remember asking him how many applied mathematicians there were in North America and he replied with the question "You mean excluding those who just do Laplace Transforms." I have forgotten how many constituted the remainder but it was an insignificant minority of the mathematics community.

Let me start with the oldest of my group and I'll bet an unsung hero in most mathematical halls. Geoffrey Ingram Taylor, born in 1886, was the grandson of George Boole, so mathematics cannot have been strange to him. He also had a mathematical aunt, Boole's youngest daughter who published her first paper in geometry in her old age. (I imagine that her father was her teacher). Geoffrey Taylor took the natural science tripos not the mathematical tripos at Cambridge. I have wondered if he was not discouraged from pure mathematics by the situation of his distinguished grandfather who got his first position at the advanced age of 36 in that outpost of the British empire, the University of Cork in Ireland. Taylor received a fellowship at Cambridge in 1908 and there he stayed the rest of his life. In 1911 he was scientific crew for a trip to the Arctic on the H.M.S. Scotia. There he not only enjoyed the making of measurements in the middle of nature but he got started on his study of turbulence. He was fascinated by the outpouring of smoke from the ship's funnel.

I recently had the opportunity to get equally fascinated. Twice a day a local steamer plies its way past my summer cottage. And this is what I see on a windless day, Figure 1, and I suppose that is what Taylor saw.

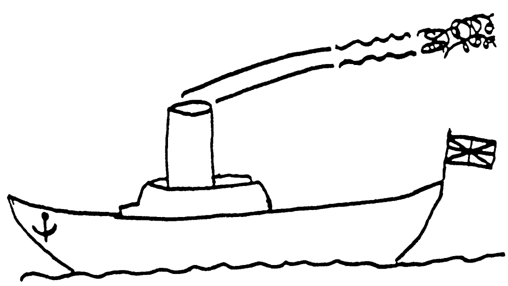

Figure 1.

In the beginning you have a plume of smoke, a layer of air and under it a layer of hot air and smoke which has lower density and under that again air.

Let us just look at one surface separating gas at different densities as in Figure 2a. Just think of these two layers between two walls separated by a surface S as in Figure 2b. If we displace the surface by changing its level then the increased pressure on the surface is the difference in the weight of the water above the surface and thus proportional to $\delta y(\rho_2 - \rho_1)$ where δy is the difference in level. The acceleration of the surface will be proportional to this force.

$$\frac{d^2}{dt^2}\,\delta y = k(\rho_2 - \rho_1)\,\delta y.$$

If $\rho_2 > \rho_1$ there is exponential growth in time proportional to $\sqrt{k}\,(\rho_2 - \rho_1)$. If $\rho_2 < \rho_1$ things are stable.

So we are not surprised that the upper layer of the smoke plume is unstable. But it is really worse than that. Disturb the surface by tilting it as in Figure 2c. Then the pressure will form a torque if $\rho_2 > \rho_1$ that makes the surface tilt still more. So it is *more unstable*. I won't bore you with the large number of equations

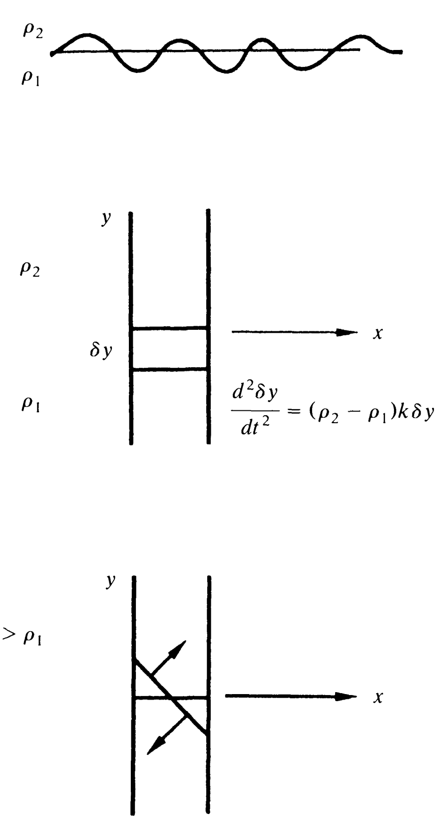

Figure 2a, 2b, 2c (Top to Bottom).

you have to write down to do this problem fully. You assume there is a ripple of some prescribed wave length in the surface, again as in Figure 2a and you find the corresponding exponents in time growth. There is *no* growth if $\rho_2 < \rho_1$ but if we have the Taylor situation, every wave length (sufficiently small) produces exponential growth at a rate proportional to the *inverse* of the wave length.

So this pair of layers is not just *unstable*. The initial value problem is *ill-posed* (but that is a long story whose answer is coming slowly now). And so the smoke becomes turbulent very soon.

Having settled the instability in this case (Rayleigh-Taylor), Taylor studied many other problems of stability. Then he went on to formulate a theory for what happens after the flow becomes turbulent.

He introduced the fundamental concept of mixing length and various correlations. Work that influenced Wiener's work and Taylor himself in turn was influenced by Wiener. Primarily his interest in applied mathematics was in understanding nature by mathematical means and for him much of nature was fluid.

I met him in his lab in 1953 in Cambridge. He delighted in showing me a big trough with a huge kind of paddle for making waves. (I should add he had an incredibly able technician to help him make things.) He also had an uncanny ability to find the right mathematical models always reducing his answers to something currently computable.

One of the amusing tales of my visit is that he could not invite me as a woman, to lunch in the Commons, a completely masculine stronghold, so instead he

arranged a lunch for a small group in a student's room—something that in America would have at that time been completely forbidden.

I saw him again, I think in 1972, in Poland at a conference in his honor. To please him he was taken sailing, his lifelong hobby, and being myself an aficionado of that sport I was particularly impressed that a man of 84 could jibe so elegantly.

By the way, many people who know all about Taylor instability do not know that he designed an anchor, still in common use; it folded better and could be stored less awkwardly. And it also held. He left a legacy not only in his own work but in the work of his scientific children and grandchildren especially G. K. Batchelor. There are not many like him and although in principle one can do fancier experiments followed by modelling followed by very fancy computations—it is hard to imagine that any one being will be able to span it all as he did.

I turn next to **Theodore von Karman** born in 1881 in Hungary where he was trained as an engineer and when I claim him for applied mathematics it is because of the role he played in bringing mathematics into aeronautical engineering. In his autobiography he describes the poor engineering education he received and how he decided to study mechanics with Prandtl in Gottingen. But there he also studied mathematics and physics. He did some fundamental work with Born on crystal lattices but in the end went to Aachen as a professor of aeronautical engineering finally going to Caltech in 1930. Anybody who has studied flight or vortices or many other aspects of fluid dynamics knows von Karman's work.

For example, I learned about the von Karman vortex street as an undergraduate in Toronto. It had been discovered by Hiemenz in Gottingen that no matter how smoothly a cylinder was honed to be circular, flow past it was always oscillatory and the cylinder oscillated too. One never found the classical 2D flow of the complex variable application, Figure 3a. Von Karman postulated that vortices were

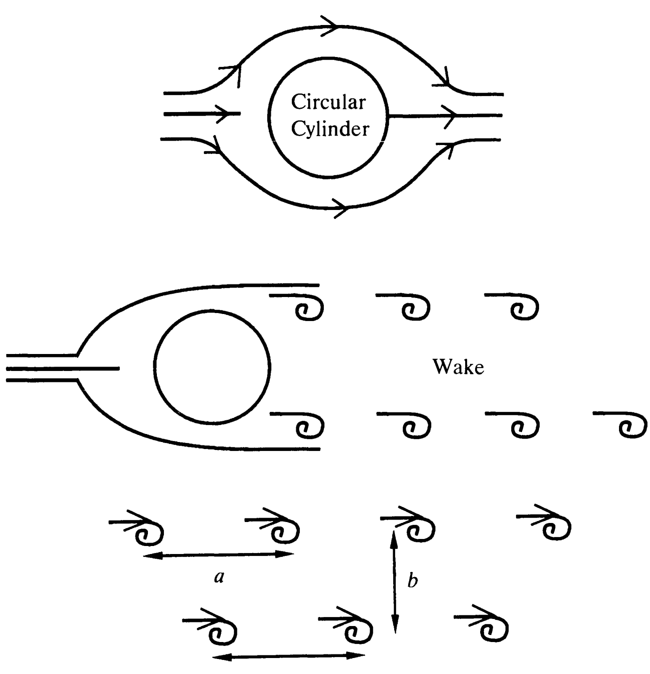

Figure 3a, 3b, 3c.

Cathleen S. Morawetz and her father, John L. Synge, on his 90th birthday.

shed alternately on each side and a wave could not be avoided as in Figure 3b. He then proposed a simple model of a vortex street with regular spacings and equal strength vortices, Figure 3c.

He found the very elegant result that if the spacing and the width of the street do not satisfy a very simple ratio condition then the array is unstable.

Thus this pattern is the one that is seen. All others being unstable will not be seen.

Enamored by the elegance of this theory I wanted to go to Caltech myself but since Caltech did not admit women in 1945 I landed up in Wiener's class at MIT instead. But I met von Karman later. I had written a paper on the so-called limiting line which had been a proposed way of explaining why most transonic flow has shocks. I attributed the idea to von Karman in part without checking a reference and then proceeded to show that it could not be the explanation. I was actually following up some work of Friedrichs. The next time von Karman came to N.Y. he invited Courant and Friedrichs, Lax as a representative Hungarian and me to lunch in his hotel. Every few minutes he turned to me and asked *where* I had seen the limiting line proposition. I felt terrible and learned my lesson but from then on was treated very well by von Karman.

His biggest role was as the father of our space program and the developer of rocketry. But wherever he could, he brought mathematics to bear. I think Friedrichs, or perhaps Courant told me that once when an admirer asked him about his successful mastery of a problem he patted a large pile of calculations meanwhile muttering "Physical intuition, physical intuition."

Let me turn now to the "other kind" of a giant and let me begin with Norbert Wiener. Probably most of you have read his autobiography. There is no question

that his formative years were his childhood years and that his tastes and ambitions were a product of his struggle to be independent of his father.

After graduating at the age of 14 from college, he tried graduate school in zoology but his clumsiness and bad eyesight made him realize that the experimental science of that time was not for him. He tried philosophy then logic and finally ended in mathematics. His enormous talent took time to develop and it was not until he had made many starts, that he settled down by his own account to really study mathematics at the age of 24. Finally, he was appointed to the faculty of MIT. In the late twenties he went for a second time as a young post-doc to Gottingen where his, I was going to say arrogance but perhaps it's better to say his particular mixture of self-esteem and lack of self-esteem led Hilbert to try to "cure" him with scorn. The Hilbert entourage followed suit and Wiener understandably retained a lifelong dislike of many of them especially Richard Courant. When I told Wiener at a Sunday lunch at his house that I was leaving MIT to join Courant's group as my new husband was in the New York area, I was not aware of this bad situation and couldn't understand why Wiener refused to carry on the conversation. Thus, to say the least, I had very little contact with Wiener. But as a fresh graduate student I had briefly tried out his course in Noise and Random Processes. It was clearly for those who knew more mathematics than I had learned in Toronto in applied mathematics and I dropped out. In spite of the Gottingen story Wiener admired Hilbert and continued to see him as, to quote his autobiography, "the sort of mathematician I would like to become, combining tremendous abstract power with a down-to-earth sense of physical reality."

To quote Mark Kac writing in 1964 on Wiener's work:

> "The simplest and most celebrated example of a stochastic process is the Brownian motion of a particle. Wiener conceived in (1921) the idea of basing the theory of Brownian motion on a theory of measure in a set of all continuous paths. This idea proved enormously fruitful for probability theory. It breathed new life into old problems."

This work drew strongly on Taylor's work that came from pictures of "plumes of smoke" and which had led Taylor to introduce his special correlations. As Norman Levinson put it there were two reasons for Wiener's interest in Taylor's work—one was that it inspired him to try turbulence as a model for his problem of integration in function spaces. And the other was that it suggested his own auto and cross correlation functions for his generalized harmonic analysis.

I reread the introduction to Wiener's "Cybernetics." Looked at from a human point of view one perceives the incredibly high scientific aspirations of Wiener. Resting, I would like to say comfortably (but even I knew as a student his uneasiness), on a long history of successful accomplishment Wiener wanted to conquer the brain with mathematics. In the course of it he worked very hard with able physicians and experts in neurology to become not only as knowledgable as he could but able to interact. His great ambition was to put together the extant knowledge of computers, feedback systems and physiology coupled with the emerging subject of signal processing—all subjects he had contributed to fundamentally. The course I went to in 1945 was a spinoff of these interests.

In his introduction, read now 43 years later, he takes a rather high position for the role of his new subject. Still cybernetics has stood the test of time with engineers. His view of the relation of science and society is interesting if pessimistic.

"The best we can do is to see that a large public understands the trend and the bearing of the present work, and to confine our personal efforts to those fields, such as physiology and psychology, most remote from war and exploitation. As we have seen, there are those who hope that the good of a better understanding of man and society which is offered by this new field of work may anticipate and outweigh the incidental contribution we are making to the concentration of power (which is always concentrated, by its very conditions of existence, in the hands of the most unscrupulous). I write in 1947, and I am compelled to say that it is a very slight hope."

There is no good source of biographical information for my next giant, John von Neumann. This situation is being repaired and we can look forward to a full biography in the next couple of years. v. Neumann was, like Wiener, a mathematical prodigy as a child. His father, however, was an enlightened banker and somehow or other when v. Neumann was ready to go to university a compromise between banking (big business) and mathematics (purest of sciences) was worked out. v. Neumann went to Zurich to study chemistry (possibly applicable in the eyes of the family).

I never met von Neumann. Occasionally I saw him in Richard Courant's company. And that affected my life a lot because as I understand it, it was at v. Neumann's recommendation that the first big university computer was placed with Courant's group at N.Y.U. That was probably not disconnected from the fact that Courant and those around him shared v. Neumann's view that mathematics would become an arid subject if it lost contact with science and engineering.

Those who knew v. Neumann always remark on the speed of his brain. He grasped things immediately. His interests ranged over everything. I don't know whether one should call his early work in quantum mechanics applied. One might say he set it up as a part of pure mathematics. His early elegant work in game theory received little attention until after the Second World War. But even before the Second World War broke out he became involved in ballistics in anticipation.

von Neumann, as everyone knows played a big role in the development of the atomic bomb. And it was in that connection that he made his mark in fluid dynamics.

Despite some striking contributions to the field (lots of us are hard at work these days on the paradoxes he uncovered in shock reflection) it led him quickly into big computation (big for its day) and hence into the whole area of large scale computing: The universal machine, the coding, the programming. Some ideas were around but he cleaned them up and set the whole thing on a logical and expandable footing. As Peter Lax has suggested he would have developed parallel computing if he had lived long enough. I have really no time to bring up his many contributions, to economics, to Monte Carlo methods etc. It might be said of v. Neumann that his sweep was so broad that it included most of applied mathematics.

I would only *like* to reiterate his philosophy that mathematics would become an esoteric arid branch of science if it lost its connections. I think he would be happy to see today how modern mathematics is knitting bonds within its many branches and even more with other sciences.

I turn now to my teacher **Kurt Friedrichs**, or as he was known to those around him, Frieder.

Born in Kiel in 1901 he entered university in Dusseldorf. Following the German practice he studied a variety of topics in a variety of places (including the

philosophy of Husserl and Heidegger). Finally, he came to "the Mecca of mathematics," Gottingen in 1922.

His relation to mathematics, pure and applied, is best described by his own expression that "he was like a dancing bear on a stove, first hopping on his pure foot till it got too hot and then on his applied foot." In fact, if one looks over his work one finds a pretty random distribution. His first official applied mathematical work was as von Karman's assistant in Aachen. He took the position according to his own explanation to Constance Reed because Courant thought that in the late twenties Friedrichs being so shy and withdrawn would have a hard time competing against pure mathematicians for an academic position in Germany. He became shortly the youngest professor at the University of Braunschweig. He left Germany to join Courant in America partly from disgust with the Nazis but also to be able to marry Nellie Bruell, forbidden under the Nazi racist rules.

From then on he worked in elasticity, fluid dynamics, quantum field theory, plasma physics alternately with partial differential equations, asymptotics, spectral theory and other subjects too pure to be applied but too applied to be quite pure. Friedrichs liked to say that applied mathematics was whatever the physicist had discarded as no longer exciting.

As a graduate student at New York University I first worked at editing the book on "Supersonic Flow and Shock Waves" by Courant and Friedrichs. That was my good luck. I learned the main ideas from Courant and the exceptions and the necessity to be accurate from Friedrichs.

After passing my orals after my first child I went to Friedrichs for a thesis topic. I thought it would be in fluid dynamics but he showed me a whole bunch of topics mainly I think on spectral theory. He asked me if I could get excited about one; that was an essential part of my taking it on. I could not but we agreed I would work on one. But when my second child was on the way, the gods (Courant, Stoker and Friedrichs) decided my contract work could be developed quickly into a thesis. It was on stability of implosions (used as neither I nor Friedrichs knew for detonating the atomic bomb. It was connected to the collapse of supernovae under self-gravitation.) Beautiful special solutions can be found using the group invariance of the equations. Friedrichs was hopping on his pure foot and at times it was very hard to get him to think about fluids. Incidentally the idea of an implosion had been considered by v. Neumann, G. I. Taylor and by the German aeronautical scientist Guderley and for all I know the Russians. My stability result was very modest but it did give me a thesis and the asymptotic theory involved gave me a good start for other problems. I kept on learning from Friedrichs but I never did get involved in either quantum mechanics or spectral theory.

Once Friedrichs got a physical problem properly and clearly mathematized (as say with either fluid dynamics or magneto-hydrodynamics), he then went after it with every tool he knew and wrestled it to the ground. When I was helping him with his Selecta just a few years before he died he kept saying "oh let's not pick that one. So and so did it much more cleanly later." He somehow had trouble realizing how important his innovations had been.

One of the things that surprised me about Friedrichs was his indifference to the role of the big computer and even his own contributions to difference schemes as a useful tool for finding answers as opposed to existence theorems. I tried once to draw him out on that subject but got nowhere—which is the way it was when Friedrichs did not want to follow a particular line of thought. I wish now that I had asked him a lot more.

My father is another example of the applied mathematician of this century. Born in 1897 and trained in Trinity College Dublin he came to Canada as a young man and started working in mechanics. He was diverted into differential geometry and the exciting new subject of relativity by the influence of Veblen. His lifelong interest was in the intersection of geometry and physics. He was fascinated like Taylor at the way nature worked and he fought hard for the turf of applied mathematics through the thirties. The war threw a lot of mathematicians into applied problems (in Canada that was 1939) and he fitted naturally. None of us should forget how frightened our world was at the thought that Hitler would win and his terrible ideas would prevail. Today some may look back and ask how we could have helped with weapons of destruction. But by and large there was very little pacifism and applied mathematicians for the most part were heavily engaged. It was in that period I first studied mathematics and its utility was of paramount interest. Only later did I capture the sense from my father of the beauty of nature transformed into mathematics and from Friedrichs the beauty of proof of the resulting mathematics.

My father's work has ranged from ideal steering mechanisms to general relativity with the latter being his main stomping ground. A particular physics-geometry approach led him to invent the first finite element method accompanied by estimates. But the item I would like to tell you about is his excursion into dentistry and his feelings about it, since they have a universal application. In the thirties he was approached by a dentist, H. K. Box, about the problem of traumatic occlusion caused by biting. What's that? In Figure 4, we have a rigid tooth lying in its rigid

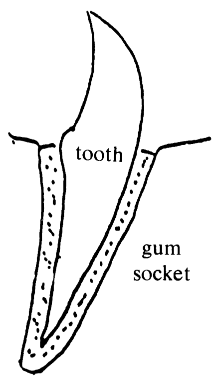

Figure 4.

socket and separated from each other by the periodontal membrane which is transmitting the force of the bite and also the pain of traumatic occlusion if the membrane defects. Clearly a problem in elasticity and, mindful of George Bernard Shaw's saying that even the Archbishop of Canterbury is 90% water, my father decided to tackle the problem with a model of a thin incompressible elastic membrane to represent the periodontal membrane. He worked hard and got some results but in 1972, 40 years later when he received the Boyle medal in Dublin he reflected:

> "I have a social conscience of sorts. When Dr. Box told me about traumatic occlusion, I lacked the strength of mind to tell him that I had other things to

do. So I engaged on this work as a social duty. But as the mathematical argument took shape, my professionalism took over and I was fascinated by this problem in which the geometry of the tooth and the physics of the membrane were combined. The final result calls for a sardonic laugh. On the one hand, you have a paper of over forty pages, published nearly forty years ago, full of intricate formulae developed (if I may say so) with considerable skill. On the other hand, you have humanity suffering still, I presume, from traumatic occlusion."

So one must be wary of trying to do good in mathematics.

I'd like to close with a parting shot in the dark. Of all the sciences mathematics has had the least impact on biology. v. Neumann died before the spectacular developments of molecular biology had started and that is really even true of Wiener. Both were challenged by problems of biology but as it turned out somewhat peripheral problems. Can we look forward in the next decade to new giants: to a new G. I. Taylor or a new von Karman thoroughly immersed in biology as they were in mechanics who will bring to the v. Neumann or Wiener of the day the deep and as yet not formulated mathematical problems of biology? I think that's an extremely interesting future to look forward to.

Thank you.

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One of the facts which the historian of the future will not fail to note regarding our present epoch is the way in which mathematicians have turned from applied mathematics. Mathematicians may be divided into three classes in respect to their attitude towards applied mathematics: (a) those who have nothing to do with applied mathematics and do not want to, regarding it as an inferior type of intellectual exercise; (b) those who would like to be better acquainted with applied mathematics, but cannot find time for prolonged study of what is not their major interest; (c) those primarily interested in applied mathematics, studying the pure almost solely for its repercussions on the applied...

The eighteenth century was the age of class (c); the twentieth century is the age of class (a). The nineteenth was the age of transition.

—*John L. Synge*
Monthly, 1939, p. 155

Euclidean Quadratic Fields

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1. INTRODUCTION. An **algebraic number** α of degree n is any root of an irreducible polynomial of degree n with coefficients in the rationals Q. The **algebraic field** $Q(\alpha)$ is the smallest subfield of the complex numbers which contains α. The **algebraic integers** $I(\alpha)$ are those elements of $Q(\alpha)$ which are roots of monic polynomials with (ordinary) integer coefficients.

Computation in $I(\alpha)$ is in general unlike computation in the integers Z, since usually there is no analogue of the uniqueness of prime factorization. Historically the (false) assumption that prime factorization is unique in every algebraic field proved to be a stumbling block for various distinguished mathematicians, among them Gabriel Lamé. (A nice discussion is given by Edwards [**6**, Chap. 4].)

The problem of determining the algebraic fields which do have unique factorization is still not completely solved. However, in certain fields, known as **Euclidean fields**, it is possible to define an analogue of Euclid's algorithm, and in such cases this guarantees unique factorization. The algebraic fields of degree 2 which have this property are called **Euclidean quadratic fields**. Work of Davenport and others, culminating in 1952, showed that there are just 21 of them.

The well-known book by Hardy and Wright [**8**] is a standard reference on Euclidean quadratic fields. In 14 of the 21 cases they present proofs that $Q(\sqrt{d}\,)$ is Euclidean. The reader naturally wonders whether the proofs in the 7 remaining Euclidean cases are difficult. Hardy and Wright also prove that there are no other Euclidean cases with $d < 0$ and only finitely many for $d \not\equiv 1 \bmod 4$.

In this paper, we use a uniform geometric method both to prove that $Q(\sqrt{d}\,)$ is Euclidean in all 21 cases, and to show that it is not Euclidean in any other case with $d < 0$ or $d \not\equiv 1 \bmod 4$. The constructions are straightforward and give geometric insight into the arithmetic of the relevant quadratic fields. We hope that the reader shares in the pleasure which they have given us. We wish to thank Peter Waterman for help with the figures drawn using *Mathematica* graphics.

In the next section we prove some basic properties of $I(\sqrt{d}\,)$. Readers familiar with algebraic integers and norms should skip to the following section, which describes a representation of $Q(\sqrt{d}\,)$ in the Euclidean plane.

2. INTEGERS AND NORMS IN QUADRATIC FIELDS. If $Q(\alpha)$ is a quadratic field, then $\alpha := (r + s\sqrt{d}\,)/t$ for some integers r, s, t and d, with $d \neq 0, 1$ and squarefree, $s \neq 0$ and $t > 0$. Indeed, $Q(\alpha) = Q(\sqrt{d}\,)$ in this case. We call this the quadratic field with **discriminant** d. If the discriminant is negative, the field is **complex**, and otherwise **real**. It is convenient to refer to it as $Q(\sqrt{d}\,)$, and to refer to its set of algebraic integers as $I(\sqrt{d}\,)$.

Every $\beta \in Q(\sqrt{d}\,)$ has the form $\beta := (a + b\sqrt{d}\,)/c$, where a, b and c are integers, $c > 0$ and $gcd(a,b,c) = 1$. When $\beta \in I(\sqrt{d}\,)$ it satisfies a monic quadratic equation with integer coefficients, say $\beta^2 - m\beta + n = 0$. There are two cases.
(1) If $b = 0$ then $a^2 - mac + nc^2 = 0$, so $c | a^2$. But $gcd(a,c) = 1$ and $c > 0$, so $c = 1$ and $\beta = a$. Conversely, any $a \in Z$ is in $I(\sqrt{d})$. (2) If $b \neq 0$, the monic quadratic equation with rational coefficients which is satisfied by β is easily seen to be unique. Also, we note that

$$\left[c\beta - \left(a + b\sqrt{d}\right)\right]\left[c\beta - \left(a - b\sqrt{d}\right)\right] = \left(c\beta - a\right)^2 - b^2 d = 0,$$

so $\beta^2 - (2a/c)\beta + (a^2 - b^2d)/c^2 = 0$, whence $m = 2a/c$ and $n = (a^2 - b^2d)/c^2$. Thus $c|2a$ and $c^2|(a^2 - b^2d)$. Let $g := gcd(a,c)$. Then $g|a$, $g|c$ and $g^2|(a^2 - b^2d)$, so $g^2|b^2d$. But d is squarefree, so $g|b$. Therefore $g|gcd(a,b,c)$, so $g = 1$. Now $g = 1$ and $c|2a$ implies $c|2$ so $c = 1$ or 2. (i) If $c = 1$ then $m = 2a$, $n = a^2 - b^2d$ and $\beta = a + b\sqrt{d}$. Conversely, taking $m = 2a$ and $n = a^2 - b^2d$ shows that any such β is in $I(\sqrt{d}\,)$. (ii) If $c = 2$ then a is odd and $4|(a^2 - b^2d)$, so $b^2d \equiv a^2 \equiv 1 \bmod 4$. Then b is odd, so $d \equiv 1 \bmod 4$. Conversely, with a and b odd and $d \equiv 1 \bmod 4$, taking $m = a$ and $n = (a^2 - b^2d)/4$ shows that $\beta = (a + b\sqrt{d}\,)/2$ is in $I(\sqrt{d}\,)$. In view of these results, whenever $d \equiv 1 \bmod 4$ it is convenient to write all the irrational algebraic integers in the form $(a + b\sqrt{d}\,)/2$ by taking a and b to be any integers with the same parity. Thus we can summarize the situation as follows:

The set of algebraic integers of $Q(\sqrt{d}\,)$ is

$$I(\sqrt{d}\,) = \begin{cases} \{a + b\sqrt{d} : a, b \in Z\}, & \text{if } d \not\equiv 1 \bmod 4, \\ \{(a + b\sqrt{d}\,)/2 \colon a, b \in Z,\ a \equiv b \bmod 2\}, & \textit{if } d \equiv 1 \bmod 4. \end{cases}$$

Note that if $d \not\equiv 1 \bmod 4$ then $d \equiv 2$ or $3 \bmod 4$ because d is squarefree. We note here that every element of $Q(\sqrt{d}\,)$ is an algebraic integer divided by a positive integer.

For any $\beta := (a + b\sqrt{d}\,)/c \in Q(\sqrt{d}\,)$ where a, b and c are integers with $c > 0$, we define the **norm** to be

$$N(\beta) := |a^2 - b^2d|/c^2.$$

(Some authors omit the absolute value operation from this definition.) The identity

$$\left(a^2 - b^2d\right)\left(e^2 - f^2d\right) = \left(ae + bdf\right)^2 - \left(be + af\right)^2 d$$

ensures that norm is multiplicative; that is, if β and γ are any two elements of $Q(\sqrt{d}\,)$ then

$$N(\beta\gamma) = N(\beta)N(\gamma).$$

(When $d = -1$, this corresponds to Euler's identity showing that the product of two integers, each the sum of two squares, is itself the sum of two squares.)

In particular, if $\beta \in I(\sqrt{d}\,)$ satisfies the unique quadratic equation $\beta^2 - m\beta + n = 0$, where m and n are integers, then $N(\beta) = |n|$. Thus, the norm of any algebraic integer in $I(\sqrt{d}\,)$ is a nonnegative integer. However, an element of $Q(\sqrt{d}\,)$ with norm equal to a nonnegative integer is not necessarily an algebraic integer: $(3 + 4i)/5$ is an example in $Q(i)$.

A **unit** of $I(\sqrt{d}\,)$ is any element with norm 1. Any two elements $\beta, \gamma \in I(\sqrt{d}\,)$ are **associates** if there exists a unit ϵ such that $\beta = \gamma\epsilon$: in that case $N(\beta) = N(\gamma)$. Conversely, if $N(\beta) = N(\gamma)$ then $N(\beta/\gamma) = 1$, so β/γ is a unit if it is in $I(\sqrt{d}\,)$,

and then β and γ are associates. If $\beta, \gamma \in I(\sqrt{d})$ are such that $\gamma/\beta \in I(\sqrt{d})$, then β is a **factor** of γ : we write this as $\beta|\gamma$. In particular, if β and γ are associates then $\beta|\gamma$ and $\gamma|\beta$.

A **prime** of $I(\sqrt{d})$ is any $\pi \in I(\sqrt{d})$, with $N(\pi) > 1$, such that if $\pi = \beta\gamma$ with $\beta, \gamma \in I(\sqrt{d})$ then necessarily one of β and γ is a unit, and so the other is an associate of π. Let us call the prime π **strong** if it has the property that $\pi|\beta\gamma$ implies $\pi|\beta$ or $\pi|\gamma$ when $\beta, \gamma \in I(\sqrt{d})$. (Edwards [**6**] uses the term **irreducible** where we follow Hardy and Wright [**8**] in using the term **prime**; Edwards reserves the term **prime** where we propose to use the term **strong prime**.) For fields where unique factorization holds, it turns out that every prime is a strong prime. We will later give some examples of primes which are not strong.

3. REPRESENTING $Q(\sqrt{d})$ IN THE EUCLIDEAN PLANE. A simple geometric representation of $Q(\sqrt{d})$ in the Euclidean plane is given by the mapping

$$(a + b\sqrt{d})/c \to (a/c, b/c),$$

under which $Q(\sqrt{d})$ is represented by Q^2, the **rational points** in the plane. We call this the **plane embedding** of $Q(\sqrt{d})$. In order to preserve more algebraic properties of $Q(\sqrt{d})$, Lenstra [**11**] uses a more complicated embedding for algebraic fields, but the simple embedding we have just defined suffices for this paper.

Under the plane embedding of $Q(\sqrt{d})$, the algebraic integers in $I(\sqrt{d})$ correspond to the **lattice points** Z^2 when $d \not\equiv 1 \bmod 4$. When $d \equiv 1 \bmod 4$ they correspond to the lattice points Z^2 and the **midlattice points** $Z^2 + (1/2, 1/2) := \{(a + 1/2, b + 1/2) : a, b \in Z\}$.

For any $\lambda \in I(\sqrt{d})$, we define the **unit neighborhood** of λ in $Q(\sqrt{d})$ to be the set

$$U(\lambda) := \{\beta \in Q(\sqrt{d}) \colon N(\beta - \lambda) < 1\}.$$

Suppose $\lambda \in I(\sqrt{d})$ maps into (x, y) under the plane embedding of $Q(\sqrt{d})$: we use $U(x, y)$ to denote the image of $U(\lambda)$ under the plane embedding, and refer to $U(x, y)$ as a **unit neighborhood** in the plane. Then

$$U(x, y) = \{(r, s) \in Q^2 \colon |(r - x)^2 - (s - y)^2 d| < 1\}$$

so $U(x, y)$ consists of the rational points in the interior of a region of the Euclidean plane bounded by an ellipse when $d < 0$ (in fact a circle when $d = -1$) or bounded by a pair of conjugate hyperbolas when $d > 0$. The boundary has eccentricity $\sqrt{1 + 1/d}$, and center (x, y) which is a lattice point or a midlattice point, the latter possibility arising only when $d \equiv 1 \bmod 4$.

4. EUCLIDEAN QUADRATIC FIELDS. The quadratic field $Q(\sqrt{d})$ is **Euclidean** if, corresponding to any given $\gamma, \delta \in I(\sqrt{d})$, with $\delta \neq 0$, there are $\lambda, \rho \in I(\sqrt{d})$ such that

$$\gamma = \lambda\delta + \rho \quad \text{and} \quad N(\rho) < N(\delta).$$

By iteration, this property yields a Euclidean algorithm for γ and δ. Since $N(\rho/\delta) < 1$, if $\beta := \gamma/\delta$ is any element of $Q(\sqrt{d})$, there is a $\lambda \in I(\sqrt{d})$ such that $N(\beta - \lambda) < 1$. In this sense, when the field is Euclidean, each element of $Q(\sqrt{d})$ is close to one of its algebraic integers.

Hence $Q(\sqrt{d})$ is Euclidean exactly when each $\beta \in Q(\sqrt{d})$ is in the unit neighborhood $U(\lambda)$ of some $\lambda \in I(\sqrt{d})$. Equivalently, $Q(\sqrt{d})$ is Euclidean precisely if each point $(r, s) \in Q^2$ lies in some unit neighborhood $U(x, y)$ in the plane, with center (x, y) which is a lattice point or, if $d \equiv 1 \bmod 4$, a midlattice

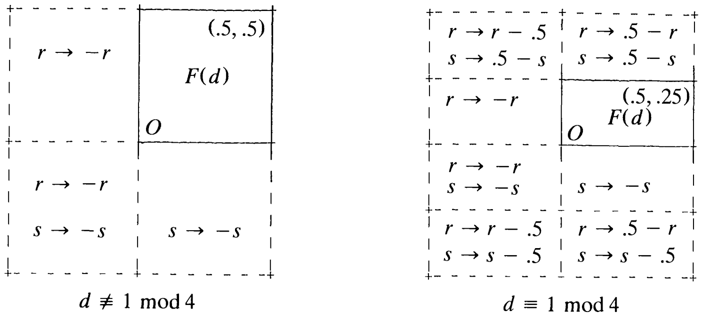

Figure 1. Symmetries of the fundamental rectangular region.

point. From the symmetry of Q^2, Z^2 and $Z^2 + (1/2, 1/2)$, we see from Figure 1 that we can restrict attention to the rational points comprising the **fundamental rectangular region** $F(d)$ for $Q(\sqrt{d}\,)$, defined by

$$F(d) := \begin{cases} \{(r, s) \in Q^2 \colon 0 \leqslant r \leqslant 1/2, 0 \leqslant s \leqslant 1/2\} & \text{if } d \not\equiv 1 \bmod 4, \\ \{(r, s) \in Q^2 \colon 0 \leqslant r \leqslant 1/2, 0 \leqslant s \leqslant 1/4\} & \text{if } d \equiv 1 \bmod 4. \end{cases}$$

Hence we have the criterion:

The field $Q(\sqrt{d}\,)$ is Euclidean precisely when the fundamental rectangular region $F(d)$ is covered by unit neighborhoods in the plane.

5. COMPLEX EUCLIDEAN QUADRATIC FIELDS.

If $Q(\sqrt{d}\,)$ is complex, its discriminant is negative so its unit neighborhoods in the plane are bounded by ellipses.

Theorem 5.1. *There are five complex quadratic fields which are Euclidean. Their discriminants are* -1, -2, -3, -7 *and* -11.

Proof: (This proof parallels that of Hardy and Wright [**8**] and illustrates our method.)

Case 1: $d < 0$, $d \not\equiv 1 \bmod 4$.

When $|d| < 3$, the fundamental rectangular region $F(d)$ lies entirely within the unit neighborhood $U(0,0)$, since every $(r, s) \in F(d)$ satisfies $r^2 + s^2|d| \leqslant 1/4 + |d|/4 < 1$. Consider congruent ellipses, centered at $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$, each with horizontal semimajor axis 1 and eccentricity e, $e^2 = 1 + 1/d$. The point $P = (1/2, 1/2)$ lies outside all of these ellipses when e is large enough. Certainly P lies outside every unit neighborhood $U(x, y)$ when $|d| > 3$ since $r^2 + s^2|d| > 1$. Hence the only complex Euclidean quadratic fields with discriminant $d \not\equiv 1 \bmod 4$ are those with $d = -1$ and -2.

Case 2: $d < 0$, $d \equiv 1 \bmod 4$.

When $|d| < 12$, the unit neighborhood $U(0,0)$ contains all of $F(d)$, since every $(r, s) \in F(d)$ satisfies $r^2 + s^2|d| \leqslant 1/4 + |d|/16 < 1$.

Now consider a pair of congruent ellipses, centered at $(0,0)$ and $(1/2,1/2)$, each with horizontal semimajor axis 1 and eccentricity e. A straightforward calculation shows that $F(d)$ lies in the union of their interiors just if $e^2 \leqslant 4\sqrt{3} - 6$, while for all greater values of e the point $P = (1/2, \sqrt{3} - 3/2)$ lies outside both ellipses. The point $(1/2, 7/30) \in F(d)$ is sufficiently close to P that it lies outside every unit neighborhood $U(x,y)$ when $|d| \geqslant 15$, for then $(1/2 - x)^2 + (7/30 - y)^2|d| \geqslant 16/15 > 1$ for each ellipse. Therefore, the only complex Euclidean fields with discriminant $d \equiv 1 \bmod 4$ are those with $d = -3$, -7 and -11. ■

When $d = -15$, Figure 2 shows these ellipses in boldface and parts of other unit neighborhoods in lightface. The enlargement shows the region where $0.25 \leqslant x \leqslant 0.55$ and $0.15 \leqslant y \leqslant 0.25$.

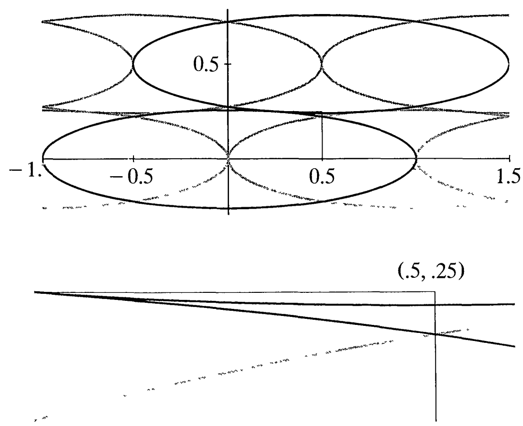

Figure 2. $d = -15$.

It follows from Theorem 1 that if $Q(\sqrt{d})$ is a complex field which is not Euclidean then $d \leqslant -5$ if $d \not\equiv 1 \bmod 4$, and $d \leqslant -15$ if $d \equiv 1 \bmod 4$. In fact, it need not even have the unique factorization property. We can show this as follows. When $d \leqslant -5$, the only units in $Q(\sqrt{d})$ are ± 1. Suppose $\beta, \gamma, \delta \in I(\sqrt{-5})$ satisfy $\beta = \gamma\delta$ and $N(\beta) = 9$. Then $N(\gamma)N(\delta) = 9$, but no element of $I(\sqrt{-5})$ has norm 3, so one of γ and δ must be a unit, and therefore β must be a prime of $I(\sqrt{-5})$. Thus $(2+\sqrt{-5})(2-\sqrt{-5})$ and 3^2 are two distinct prime factorizations of 9 in $Q(\sqrt{-5})$, and no two of the three primes involved are associates. Indeed, it follows that none of them is a strong prime in this field. (However, $Q(\sqrt{-5})$ does contain strong primes: it turns out that $3 + 2\sqrt{-5}$ is an example.) Similarly, any element of $I(\sqrt{-15})$ with norm 4 is prime and

$$\left[(1+\sqrt{-15})/2\right]\left[(1-\sqrt{-15})/2\right]$$

and 2^2 are two distinct prime factorizations of 4 in $Q(\sqrt{-15})$. As before, it follows that $(1+\sqrt{-15})/2$, $(1-\sqrt{-15})/2$ and 2 are primes which are not strong in this field.

6. REAL EUCLIDEAN QUADRATIC FIELDS. When d is positive, the unit neighborhoods of $Q(\sqrt{d})$ in the plane are infinite X-shaped regions bounded by

conjugate hyperbolas. The left and right boundaries of $U(x, y)$ satisfy the equation

$$(r - x)^2 - d(s - y)^2 = 1,$$

while the top and bottom boundaries satisfy the equation

$$(r - x)^2 - d(s - y)^2 = -1.$$

Theorem 6.1. *The sixteen real quadratic fields with discriminants* $d = 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57$ *and* 73 *are Euclidean.*

Proof: In each case we need the unit neighborhood $U(0,0)$ only to cover the point $(0,0)$, so we restrict our attention to the rest of $F(d)$.
Case 1: $2 \leqslant d < 5$, $d \not\equiv 1 \bmod 4$, or $5 \leqslant d < 20$, $d \equiv 1 \bmod 4$.
The top boundary of $U(1,0)$ passes through $(0, \sqrt{2/d})$ and $(0.5, \sqrt{1.25/d})$, thus covering the rest of $F(d)$ when $d = 2, 3, 5, 13$ and 17. So these five fields are Euclidean and henceforth we restrict our attention to that part of $F(d)$ lying above $U(1,0)$.
Case 2: $6 \leqslant d < 8$, $d \not\equiv 1 \bmod 4$ or $21 \leqslant d < 32$, $d \equiv 1 \bmod 4$.
The right boundary of $U(-1,0)$ passes through $(0,0)$ and $(0.5, \sqrt{1.25/d})$ and its top boundary passes through $(0, \sqrt{2/d})$ and $(0.5, \sqrt{3.25/d})$. Note that $U(1,0)$ and $U(-1,0)$ cross at $(0.5, \sqrt{1.25/d})$ which is not rational since $d > 5$ is squarefree. Thus $U(-1,0)$ covers the rest of $F(d)$ when $d = 6, 7, 21$ and 29. So these four fields are Euclidean, and henceforth we restrict our attention to the portion of $F(d)$ lying above $U(-1,0)$.
We now come to the cases not proved in Hardy and Wright [8].
Case 3: $33 \leqslant d \leqslant 41$, $d \equiv 1 \bmod 4$.

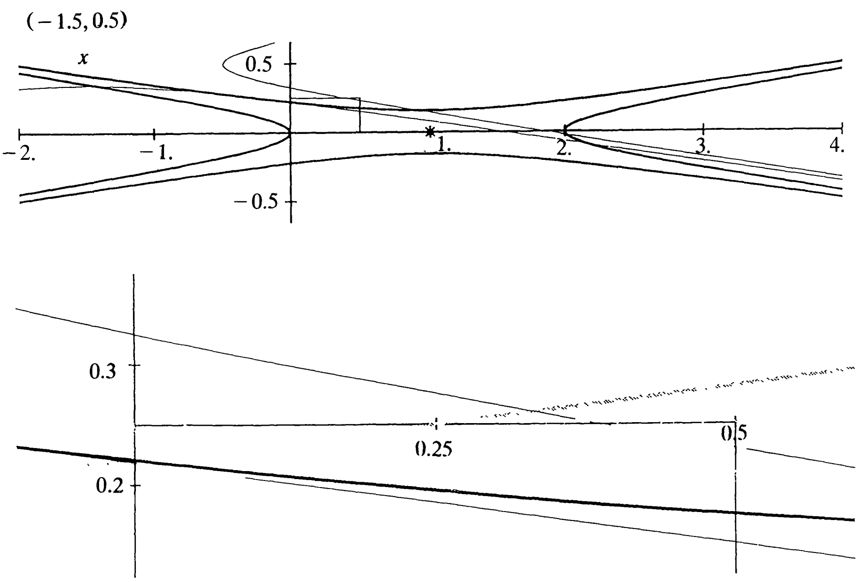

Figure 3. $d = 41$.

The southeast (SE) arm of $U(-1.5, 0.5)$ covers the rest of $F(d)$, when $d = 33$, 37 and 41. For each d, you can easily solve the quadratic equations on your pocket calculator, as we did. Thus these three fields are Euclidean.

Figure 3 depicts the situation when $d = 41$. We show all four branches of $U(1,0)$ and the relevant branches of $U(-1.5, 0.5)$. The enlargement also includes the top branch of $U(-1,0)$ in lightface.

Case 4: $d = 11$.

The rest of $F(11)$ is covered by the SW arm of $U(2,1)$ and the NE arm of $U(-5,-1)$, so $Q(\sqrt{11})$ is Euclidean.

The overlap of these two arms increases with increasing r, and is about 0.0006 when $r = 0$.

Case 5: $d = 19$.

The rest of $F(19)$ is covered by the SW arm of $U(3,1)$; the NW arms of $U(2,0)$, $U(6,-1)$, $U(7,-1)$ and $U(19,-4)$; and the SE arms of $U(-2,1)$, $U(-7,2)$, $U(-90,21)$ and $U(-430,99)$. So $Q(\sqrt{19})$ is Euclidean.

At $r = 0$, the arm width of $U(-430,99)$ is about 0.0005. It covers the gap of about 0.0002 between the arms of $U(3,1)$ and $U(-90,21)$.

Case 6: $d = 57$.

The rest of $F(57)$ is covered by the SW arm of $U(2.5, 0.5)$, the SE arm $U(-6,1)$, and the NW arms of $U(2,0)$ and $U(5.5,-0.5)$. So $Q(\sqrt{57})$ is Euclidean.

The top of the $U(-6,1)$ arm and the bottom of the $U(5.5,-0.5)$ arm touch, but do not cross, at $(0.25, 0.25)$: under central reflection about this point, each boundary curve (indeed, each neighborhood) is the image of the other. The width of overlap of these two arms on $F(57)$ increases with increasing r, and is greater than 0.000045 when $r = 0$.

Case 7: $d = 73$.

The rest of $F(73)$ is covered by the NW arm of $U(2,0)$; the SE arm of $U(-10.5, 1.5)$; the NE arms of $U(-10,-1)$ and $U(-27,-3)$; and the SW arms of $U(2.5, 0.5)$, $U(7,1)$ and $U(28.5, 3.5)$. So $Q(73)$ is Euclidean.

The top of the $U(-27,-3)$ arm and the bottom of the $U(28.5, 3.5)$ arm touch, but do not cross, at $(0.75, 0.25)$: under central reflection about this point, each boundary curve is the image of the other. The width of overlap of these arms on $F(73)$ increases with decreasing r, and is greater than 0.00000034 when $r = 0.5$. The bottom of the $U(-27,-3)$ arm crosses the top boundary of $U(-1,0)$ at $(0.2552, 0.1878)$, then crosses the bottom of the $U(-10.5, 1.5)$ arm at $(0.4604, 0.2119)$, and then crosses the top of the $U(2,0)$ arm at $(0.4743, 0.2135)$. The bottom of the $U(2,0)$ arm crosses the top boundary of $U(-1,0)$ at $(0.1667, 0.1798)$. From these internal crossings, checking that $U(2,0)$ and $U(-10.5, 1.5)$ cover the portion of $F(73)$ between $U(-1,0)$ and $U(-27,-3)$ is straightforward. ■

7. NON-EUCLIDEAN QUADRATIC FIELDS.

Theorem 7.1. *The quadratic field $Q(\sqrt{d})$ with positive discriminant $d \not\equiv 1 \bmod 4$ is not Euclidean unless $d = 2$, 3, 6, 7, 11 or 19.*

Proof: We show that for any positive discriminant $d \not\equiv 1 \bmod 4$, other than the six specified, either $(1/2, 1/2)$ or $(0, t/d)$, for some positive integer t, is not in any unit neighborhood $U(x,y)$. Hence $Q(\sqrt{d})$ cannot be Euclidean.

Suppose that $(0, t/d) \in U(x,y)$. Then $|x^2 - d(y - t/d)^2| < 1$ so $|z^2 - dx^2| < d$ with $z := t - dy$. Note that $z^2 - dx^2 \equiv t^2 \bmod d$, so for fixed t there are just two values for $z^2 - dx^2$.

Case 1: $d \equiv 2 \bmod 4$.

Suppose we can find an odd t such that $2d < t^2 < 3d$. Then $z^2 - dx^2 = t^2 - md$, where $m = 2$ or 3. Therefore $z^2 - t^2 = d(x^2 - m)$. But $d \equiv 2$ or $6 \bmod 8$ and the quadratic residues modulo 8 are 0, 1 and 4, so $d(x^2 - m) \equiv 2$, 4 or $6 \bmod 8$ and $z^2 - t^2 \equiv 0$, 3 or $7 \bmod 8$ because t was chosen to be odd. Hence $z^2 - t^2 = d(x^2 - m)$ has no solutions for x and z, so $(0, t/d)$ does not belong to any unit neighborhood $U(x, y)$.

If there is no odd t such that $2d < t^2 < 3d$ then $(2u - 1)^2 < 2d < 3d \leqslant (2u + 1)^2$ must hold for some integer u, whence $2(2u + 1)^2 > 3(2u - 1)^2$. This fails for $u \geqslant 5$ because $2 \cdot 11^2 < 3 \cdot 9^2$. It follows that there is a suitable odd t if $3d > 9^2$, that is, if $d > 27$. Also $t = 5$ and 7 settle the cases $d = 10$ and 22, respectively.

When $d = 26$, take $t = 39$ and note that $58d < t^2 < 59d$. Slight modification of the previous argument leads to $z^2 - t^2 = d(x^2 - m)$, with $m = 58$ or 59. Because $m \equiv 2$ or $3 \bmod 8$, it follows as before that there are no solutions for x and z, so no unit neighborhood contains $(0, 39/26)$.

Finally, when $d = 14$ we consider $(1/2, 1/2)$. If $U(x, y)$ contains this point, then $|(2x - 1)^2 - 14(2y - 1)^2| < 4$. But $(2x - 1)^2 - 14(2y - 1)^2 \equiv 3 \bmod 8$, so necessarily $(2x - 1)^2 - 14(2y - 1)^2 = 3$. This has no solutions for x and y, since it implies $(2x - 1)^2 \equiv 3 \bmod 7$ but 3 is a quadratic nonresidue modulo 7. Thus $(1/2, 1/2)$ is not in any unit neighborhood.

Case 2. $d \equiv 3 \bmod 4$.

If there is an odd t such that $5d < t^2 < 6d$, minor modification of the earlier argument leads to $z^2 - t^2 = d(x^2 - m)$, with $m = 5$ or 6. But $d(x^2 - m) \equiv 1$, 2, 4, 5 or $6 \bmod 8$ and $z^2 - t^2 \equiv 0$, 3 or $7 \bmod 8$, so $z^2 - t^2 = d(x^2 - m)$ has no solutions for x and z, and $(0, t/d)$ does not belong to any unit neighborhood $U(x, y)$.

Since $5 \cdot 23^2 < 6 \cdot 21^2$, it follows as before that there is a suitable odd t if $6d > 21^2$, so if $d > 74$. Also, for $d = 15$, 23, 31, 39, 43, 51, 55, 67 and 71 there is an odd square between $5d$ and $6d$. Three discriminants remain to be settled, namely $d = 35$, 47 and 59.

When $d = 47$, take $t = 25$ and note that $13d < t^2 < 14d$, so we have $z^2 - t^2 = d(x^2 - m)$, with $m = 13$ or 14. Since $m \equiv 5$ or $6 \bmod 8$, it follows as before that there are no solutions for x and z, so no unit neighborhood contains $(0, 25/47)$.

When $d = 59$, take $t = 47$. Then $37d < t^2 < 38d$, so $z^2 - t^2 = d(x^2 - m)$, with $m = 37$ or 38. Note that $m \equiv 5$ or $6 \bmod 8$, so as before it follows that no unit neighborhood contains $(0, 47/59)$.

Finally, consider $d = 35$. If $(1/2, 1/2) \in U(x, y)$ then $|(2x - 1)^2 - 35(2y - 1)^2| < 4$. But $(2x - 1)^2 - 35(2y - 1)^2 \equiv 6 \bmod 8$, so we must have $(2x - 1)^2 - 35(2y - 1)^2 = -2$. Hence $(2x - 1)^2 \equiv 3 \bmod 5$. But 3 is a quadratic nonresidue modulo 5, so there are no solutions for x and y. Therefore $(1/2, 1/2)$ does not belong to any unit neighborhood. ∎

8. HISTORICAL NOTE. Fifty years ago, identifying all Euclidean quadratic fields was a major unsolved problem. The results we have demonstrated in this paper had been obtained, and it was known that cases with discriminant $d \equiv 1 \bmod 4$ seemed intrinsically more difficult to settle.

In 1938, Erdös and Ko [**7**] showed that there are only finitely many Euclidean quadratic fields. By 1948, an explicit upper bound of $d < 2^{14}$ for the discriminant of any Euclidean quadratic field had been obtained by Davenport. This followed from the demonstration that there is always a rational point (r, s) for which

$Nd(r, s) > \sqrt{d}/2^7$, where Nd is a quadratic form corresponding to the norm used in the present paper (though not identical to it). Davenport's remarkable result was not published until 1951 [**5**], by which time work involving Chatland [**3**], [**4**] had completed the search in the interval $100 < d < 2^{14}$. However, the priorities and credit for completing the search belong elsewhere. Hua and Min, in unpublished work carried out in the mid-1940s, had shown that there is no Euclidean quadratic field with $100 < d < 10^6$, apart from six unresolved cases in which $d \equiv 1 \bmod 24$ and $193 \leqslant d \leqslant 601$. Hua [**9**] pointed this out in his 1949 review of a 1947 paper of Inkeri [**10**], in which all cases with $100 < d < 5000$ were settled, including the six left open by Hua and Min.

Thus, although he was unaware of it at the time, Davenport's 1948 result finished the problem of identifying all Euclidean quadratic fields—finished, that is to say, apart from one remaining job. The final spike was driven in 1952 by Barnes and Swinnerton-Dyer [**1**], [**2**], who showed that $Q(\sqrt{97})$ is not Euclidean, correcting an earlier published claim of another. They present extensive work relevant to the behavior of the norm in quadratic fields which shows, for example, that $(0, 1/2)$ is the only point in the fundamental rectangular region which is not covered by a unit neighborhood when $d = 10$. They also give an extensive bibliography on the subject.

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Overview of Mathematical Social Sciences

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1. INTRODUCTION. Mathematical social sciences today are very well established. Mathematical economics, linguistics, social choice, and the theory of games involve elegant mathematical systems which have been developed by outstanding social scientists and mathematicians, including even Weil and von Neumann.

The objectives of mathematical social sciences include both ambitious and more modest goals. The ambitious goals are prediction and the ability to control large real social systems by design of structures which might eliminate such evils as depressions. More modest goals include mathematical indices like power indices and models of very specific social processes.

This paper provides a brief survey of some important models which use interesting mathematics in a variety of fields of social science. Specifically we discuss mathematical applications in demography, economics, management, political science, psychology, sociology, and other areas. Readers who are interested in a particular subject in more depth should consult an appropriate reference for a more thorough discussion and further results and examples.

Historically, one of the beginnings of mathematical social science may be Leibniz's idea of a universal calculus which would apply mathematics to all areas of learning. The work of the French physiocrats in economics and the study of voting theory, e.g., Condorcet's paradox in political science around the time of the French revolution, are some of the earliest useful work in mathematical social sciences.

In the 1800s Quetelet applied mathematics (statistics) in sociology as Galton and Fechner did in psychology. Malthus produced a mathematical theory of population, Ricardo gave elementary mathematical arguments supporting some of Adam Smith's economic work, and Cournot developed a mathematical theory of duopoly. Bentham invented the idea of calculating social welfare by adding utility, and Walras gave a detailed plan for mathematical economics.

In the twentieth century the pace accelerated with great advances in mathematical statistics, economics, game theory, and other areas. *A propos* our dedication, the existence of competitive equilibrium in economics was first proved by A. Wald about 1933 following a line of work started by K. Menger. Menger [**4**] was also one of the first to develop mathematical utility, a real-valued function giving the value of a bundle of goods to a person. Utility is basic to game theory and economics today. He also initiated the theory of fuzzy sets [**15**] in which the statement "x belongs to a set S" has a degree of truth from 0 to 1, expressing imprecise data. Zadeh [**20**] began the widespread use and development of fuzzy sets.

Dedicated to the memory of Professor Karl Menger (1902–1985) who was one of the pioneers of the mathematical social sciences.

For general mathematical social sciences, see [8] and [10]. Due to a lack of space we have kept references to a bare minimum.

2. DEMOGRAPHY. For simplicity, populations are often studied in terms of women, since the male population can be derived from birth ratios and survival rates. Let $n_i(t)$ denote the number of women whose ages are in interval i at time t (all intervals are in terms of a basic time unit). Let $n_0(t)$ denote the number of women born at time t. Let s_i denote the proportion of women in age group i who survive to age group $i + 1$. Let b_i denote the number of daughters born to an average woman in age group i. Then if we break down the population of women at time $t + 1$ into those just born and those surviving from time t, we have

$$n_0(t+1) = b_0 n_0(t) + \sum s_0 s_1 \cdots s_{i-1} n_0(t-i) b_i \tag{2.1}$$

where $n_i(t) = s_0 s_1 \cdots s_{i-1} n_0(t-i)$.

The fundamental theorem of demography asserts the existence of a stable growth rate.

Theorem 1. (Existence of a Stable Population Growth Rate). *In equation* (2.1), *assuming populations are positive, there exist constants* r, N_i *such that each* $n_i(t)$ *is asymptotically equal to* $N_i(1 + r)^t$.

The proof [5] involves writing the system in matrix form and using the Perron-Frobenius theorem which states that nonnegative matrices whose large powers are positive have a positive eigenvector which is unique up to positive multiples and which is dominant in that its eigenvalue exceeds the absolute value of all other eigenvalues.

3. ECONOMICS. Many economic models are variants of the following basic model of general equilibrium due to Arrow and Debreu. There exists a set of consumers, a set of firms, a set of goods, a utility function giving the set of preferences of each consumer, a vector of prices, a matrix of company ownership, a vector of goods held initially by each consumer (such vectors of goods are called *commodity vectors*), a *consumption vector* of goods held finally by each consumer, and a *production possibility set* Y_k for each firm. The latter is the set of *production vectors* y which a firm can produce, e.g., if it converts 2 tons of iron ore to 1 ton of pig iron Y_k could include vectors $(-2r, r)$ for any positive real number r.

An *equilibrium* consists of a set of prices at which supply equals demand, where the production vectors maximize each firm's profits within its production possibility set and where the consumption vectors maximize each consumer's utility subject to his or her budget constraint, stating that expenditures be no more than income.

Under convexity assumptions on the utility functions and production possibility sets, the fundamental theorem is that of existence of an equilibrium.

Theorem 2. (Existence of General Equilibrium). *The economy as described above has an equilibrium.*

The proof [16] involves a hypothetical means of adjusting prices p,

$$\theta_j(p, x) = \frac{p_j + \max(-x_j, 0)}{1 + \sum_k \max(-x_k, 0)}.$$

If we map (p, x) to the set of (θ, x^*) such that x^* is the excess supply commodity vector at price p, the hypotheses of Kakutani's fixed point theorem are satisfied, giving existence.

The extensive literature of models of this kind also uses matrix theory (stability conditions), differential topology (the generic number of equilibria), functional analysis, topology, measure theory, control theory, chaos theory, and the theory of automata and mathematical logic. More general issues of incentive compatibility, seeking to make what is socially desirable also maximizing for individual utilities, have also been extensively studied.

Game theory has also been applied in economics and is the subject of a number of excellent expositions, e.g., [**12**]. Noncooperative game theory serves as a general theory of behavior for economics and other social sciences, in particular the concept of Nash equilibrium. Its existence also follows from the Kakutani fixed point theorem. For general mathematical economics, see [**1**] and [**7**].

Example 1 (Forecasting Using Econometric Models). Most of mathematical social science is directed towards *structural analysis*, that is, towards understanding the mechanisms of social behavior in highly simplified and idealized settings (analogous to a hydrogen atom rather than an object in the macrocosm). In some settings, however, the further goals of science to predict and to control the variables of a social system have been realized. For example, the equations of Section 2 have been used to predict population levels, game theory approaches have been used in practice in questions of fair division, even in courts of law, and approval voting and other mathematical voting systems have been used in various organizations.

An extensive body of applications of mathematical social sciences to prediction exists in applied econometrics. This approach relies upon an estimated system of equations describing the macro economy or some sector or aspect of the economy, such as a region, an industry, a firm, a market, or an economic process. Econometric forecasting involves statistical extrapolation, using an estimated behavioral model of the economy. The model could treat a particular market, involving supply and demand relationships for a good or service. It could represent interacting markets, as in the general equilibrium model. It could involve overall macroeconomic variables for the national economy or for international economic transactions. The method uses data to estimate the parameters of the model and then employs the estimated model to make forecasts conditional on historical values of endogenous variables (those determined by the model), expected future values of exogenous variables (those determined outside the model), and add factors (adjustments of the forecasts to account for variables not included explicitly in the model). Such econometric forecasts combine the various elements of pure extrapolation, use of related variables, leading indicators, and expert judgment.

Example 2 (Computable General Equilibrium). An important recent application of mathematical social sciences is that of computable general equilibrium models for the purpose of policy evaluation in economics. The computation of equilibrium prices in a general equilibrium model builds on the Scarf algorithm to find fixed points of a transformation such as the price adjustment mechanism above. Once it is possible to compute the equilibrium of a general equilibrium model, however, it is possible to determine the effects of alternative policies affecting the economy. For example, a change in tax rates has not only direct effects but also manifold indirect effects, which can be determined by computing the equilibrium with and without the change in tax rates. The alternative equilibrium outcomes are *counter-*

factuals, representing what would have happened if a particular policy or set of policies were pursued. This technique can be used to study policies not only on taxes, but also on expenditures, tariffs, quotas, exchange rates, unemployment, insurance, etc., and it is used on a regular basis to shape economic programs required by major international economic organizations, such as the World Bank, as a condition for their loans.

4. MANAGEMENT SCIENCE. Management science involves many mathematical ideas of optimization and programming. One specific concept which has been developed is the Marschak and Radner [**13**] concept of *team*. This is basically the idea of a group of agents with common goals but with individually varying information as to who must coordinate their acts. There is a utility function $\mathfrak{u}$ giving the values resulting from a vector of actions $\mathfrak{a}\langle i\rangle$ of each team member i in a state of nature x. An *information structure* is a function on the set $\mathbf{X}$ of states of nature. The probability distribution on $\mathbf{X}$ is given.

Assume $\mathfrak{u}$ is strictly quasiconcave and differentiable as a function of the n-tuple of strategies. The fundamental theorem of Marschak and Radner is that an n-tuple of strategies is optimal if and only if for each agent, his strategy is a local optimum when the strategies of the others are held fixed, as in the Nash equilibrium.

Marschak and Radner also studied the questions of optimum organization structure. They computed the efficiency of management methods under different conditions, e.g., holding conferences in which the team is divided into n sets of m members each of whom meet regularly and share information, holding conferences of highly unequal size, holding conferences only of those who report exceptional observations, holding conferences of everyone when anyone reports an exceptional observation.

5. POLITICAL SCIENCE. The *Arrow impossibility theorem* could be claimed by both political science and economics, and it is a central result in voting theory and welfare theory. There is a set $\mathbf{N}$ of n voters, and a set $\mathbf{X}$ of alternatives from which they choose. The preferences of individual i are specified either by a utility function from $\mathbf{X}$ to the real numbers or by a binary relation. In the latter case let $(x, y) \in R\langle i\rangle$ if and only if person i considers x at least as good as y for $x, y \in \mathbf{X}$. To arise from a utility function this relation must be transitive and complete in that, for all $x, y \in \mathbf{X}$ either $(x, y) \in R\langle i\rangle$ or $(y, x) \in R\langle i\rangle$. Complete transitive binary relations are called *weak orders*.

A *social welfare function* on $\mathbf{N}$ is a function f from $\mathbf{W}^{\mathbf{N}}$ to $\mathbf{B}_{\mathbf{X}}$ where $\mathbf{W}$ is a specified set of weak orders, and $\mathbf{B}_{\mathbf{X}}$ is the set of all binary relations on $\mathbf{X}$. The value of f represents the *social choice*.

Example. *Majority rule* is the social welfare function such that $(x, y) \in f(R\langle 1\rangle, \dots, R\langle n\rangle)$ if and only if the set of i such that $(x, y) \in R\langle i\rangle$ has cardinality at least $n/2$.

Theorem 3. (Arrow's Impossibility Theorem). *There is no social welfare function* $\mathbf{X}$ *having at least* 3 *elements, satisfying*

(A1). *Independence of Irrelevant Alternatives: For all* $x, y \in \mathbf{X}$, *if* $R\langle i\rangle$ *and* $S\langle i\rangle$ *are equally restricted to the set* $\{x, y\}$ *then* $f(R\langle 1\rangle, \dots, R\langle n\rangle)$ *and* $f(S\langle 1\rangle, \dots, S\langle n\rangle)$ *are equal when they are restricted to the set* $\{x, y\}$.

(A2). *Pareto Optimality*: *If for all i*, $(x, y) \notin R\langle i\rangle$ (*meaning every person strictly prefers y to x*) *then* $(x, y) \notin f(R\langle 1\rangle, \ldots, R\langle n\rangle)$ (*meaning the group strictly prefers y to x*).

(A3). *Nondictatorship*: *One individual does not determine social choice.*

(A4). *Universal Domain*: $\mathbf{W}$ *is the set of all weak orders of* $\mathbf{X}$.

(A5). *Weak Order*: *The range of f lies in the set of complete, transitive binary relations.*

In particular, majority rule violates (A5) in yielding intransitivities in social choice; see Sen **[17]** for a detailed study of this. Some of the other types of work in mathematical political science includes McKelvey's **[14]** work on legislative processes; models of primary voting; game theory in relation to legislative coalitions; the question of rounding the fractions of population in assigning a whole number of legislative seats; power indices; and theories of bargaining and negotiation, as in Brams **[2]**.

6. PSYCHOLOGY. The problem of quantifying things which may be basically subjective or ordinal in nature is called *scaling*. For instance, if a subject reports one stimulus is stronger than another, is there a way to say how much stronger it is? One method is *additive conjoint measurement*. We are given ordinary data on three quantities x, y, and $z = F(x, y)$. We wish to choose scales given by real valued functions f, g, h such that $f(z) = g(x) + h(y)$. Our basic data is a weak order on $\mathbf{S} \times \mathbf{T}$, where $x \in \mathbf{S}$, $y \in \mathbf{T}$, and the overall weak order is that z values which should satisfy the axioms of

(1). *Independence*: *For all pairs* $a, b \in \mathbf{S}$, $p, q \in \mathbf{T}$, $(a, p) \geq (b, p)$ *if and only if* $(a, q) \geq (b, q)$ *and* $(a, p) \geq (a, q)$ *if and only if* $(b, p) \geq (b, q)$.

(2). *Double Cancellation*: *For all triples* $a, b, c \in \mathbf{S}$, $p, q, r \in \mathbf{T}$ *if* $(a, r) \geq (c, q)$ *and* $(c, p) \geq (b, r)$ *then* $(a, p) \geq (b, q)$.

(3). *Unrestricted Unique Solvability. Given any three of* $a, b \in \mathbf{S}$, $p, q \in \mathbf{T}$ *the fourth exists uniquely such that* (a, p) *is indifferent* (*equivalent in order*) *to* (b, q).

(4). *Archimedean Property*: *The induced linear order on the sets* $\{(b, q)$: (a, p) *is indifferent to* $(b, q)\}$ *is order equivalent to a closed subset of the real numbers.*

Theorem 4. (Existence of Scaling Functions). *If the above axioms* (1)–(4) *hold, then there exist functions* f_i: $\mathbf{S}(\mathbf{T})$ *to the real numbers such that* $(a, p) \geq (b, q)$ *if and only if* $f_1(a) + f_2(b) \geq f_1(p) + f_2(q)$. *The functions* f_i *are unique up to a linear transformation, i.e., multiplication by a positive constant and addition of a constant* (*Luce* **[11]**).

The proof involves defining a binary operation on the set of equivalence classes which are associative.

Learning theory, response theory, and factor analysis are some other areas of mathematical psychology.

7. SOCIOLOGY. A small group of human beings can be represented by a collection of binary relations (Boolean matrices) on the group specifying the significant relationships among members of the group, e.g., friendship, or which members have contact with which other members in certain settings.

We may then represent the binary relations by (0, 1)-*Boolean matrices* $A\langle i\rangle$ such that $A\langle i\rangle_{jk} = 1$ if individual j has relationship i to individual k and $A\langle i\rangle_{jk} = 0$

otherwise. One problem is to partition such a matrix or group of matrices into blocks such that two members of the same block have the same relationship to others, as discussed in White, Boorman, and Breiger [**19**]. Another approach is to partition the group subject to the condition that we have a semiring homomorphism on the semiring of Boolean matrices generated by the $A\langle i \rangle$.

8. OTHER AREAS. In anthropology, kinship systems are of interest. Many tribal societies have rules dividing them into clans and specifying which clans may marry which other clans. These rules have the effect of preventing incest. White [**18**] gives a set of eight axioms which imply that the set of clans has the structure of a finite group, such that the clan of a man's wife and the clan of his children both represent multiplication by generators of the group.

The grammatical structure of language is to a large extent mathematical. The set of grammatically correct sentences is represented by "derivations." For example, the sentence "the house is red" can be represented as an article, a noun; a verb; and a predicate adjective, which in turn is an expansion of subject, predicate.

We have a set **X** called an *alphabet* (really a set of words). Let $\mathbf{S}^*$ be the set of all finite sequences of elements of a set **S**. A *phrase structure grammar* [**6**] is a quadruple $(\mathscr{T}, \mathscr{N}, \mathscr{R}, \mathscr{S})$ where $\mathscr{T}, \mathscr{N}$ are called the set of *terminals* (e.g., "house" above) and *nonterminals* (e.g., predicate adjective), $\mathscr{P}$ is the set of *productions* (e.g., replace subject by article-noun), and $\mathscr{S}$ is the *starting symbol*, representing the entire sentence. It is required that $\mathscr{T}$ be nonempty, $\mathscr{N}$ and $\mathscr{T}$ be disjoint, $\mathscr{R}$ be contained in $((\mathscr{N} \cup \mathscr{T})^* \setminus \mathscr{T}^*) \times (\mathscr{N} \cup \mathscr{T})^*$, $\mathscr{S}$ be in $\mathscr{N}$, and $\mathscr{N}, \mathscr{T}, \mathscr{R}$ be finite sets. The resulting grammar is the set of sequences of words obtainable from $\mathscr{S}$ by applying $\mathscr{R}$. One important set of theorems in mathematical linguistics associates classes of languages to types of *automata* which can recognize them.

Diverse methods are used in general systems theory [**9**]. In the version of Mesarovic, a *system* is an n-ary relation, a subset of a Cartesian product $\tau_1 \times \cdots \times \tau_n$ giving the combinations of states $\tau_\mathbf{i}$ of element **i** which can occur.

A *general dynamical system* is very similar to the concept of a finite state machine. The mathematical theories of control and topological dynamics are relevant to the study of these systems.

9. FUTURE PROSPECTS. *Are prediction and regulation of social systems possible*? Chaos theory and the theory of algorithmic unsolvability have been proposed as theoretical limits to prediction and regulation of social systems. On the other hand, new types of mathematics, of which fuzzy sets [**20**] and inclines [**3**] provide examples, more powerful computers which could reasonably simulate societies, advances in artificial intelligence which could help explain individuals, and theoretical advances along present lines offer many possibilities for improved prediction and regulation of social systems.

10. UNSOLVED PROBLEMS. The main problem in mathematical social science is simply that of translating social science into mathematics, but some general classes of open problems of a mathematical nature can be stated.

1. What is the largest size of a set of linear preference orders for n alternatives such that majority voting is transitive when each voter chooses his preferences from this set?

2. The problem of incentive compatibility, i.e., making socially beneficial actions of self interest to individuals, is usually at the expense of efficiency. What is the trade-off between the two?

3. While many particular solution concepts exist for n-person cooperative games, what is a more comprehensive theory of such games?

4. Clustering algorithms divide objects into groups on the basis of a matrix giving similarity values between them. Many methods exist, perhaps hundreds. What general theorem can be stated as to when these methods converge?

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Ol' Abner Has Done It Again

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Could Abner Doubleday, the supposed inventor of baseball, have envisioned that one batter might outhit another over each of two seasons, and yet be outhit by the other when the two seasons are combined? While the possible occurrence of this instance of Simpson's paradox [2] has been noted [1], the following two examples [3, 4] show that this phenomenon has in fact taken place recently in major league baseball.

	Ken Oberkfell			Mike Scioscia		
	Hits	At-Bats	Batting Average	Hits	At-Bats	Batting Average
1983	143	488	.293	11	35	.314
1984	87	324	.269	93	341	.273
Combined (1983–84)	230	812	.283	104	376	.277

	Dave Justice			Andy Van Slyke		
	Hits	At-Bats	Batting Average	Hits	At-Bats	Batting Average
1989	12	51	.235	113	476	.237
1990	124	439	.282	140	493	.284
Combined (1989–90)	136	490	.278	253	969	.261

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Sequential Partitioning

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You have just agreed to repaint your parents' guest bedroom. In their garage sits a crusty old one gallon can of paint left over from the last time the room was painted. You try to pry the lid off with a screwdriver, loosening here and there at the edge of the lid, but the lid does not yield easily. The problem is a sticky one.

Soon you begin to wonder just how many places around the rim you will have to pry before the lid can be removed, and what pattern will produce the desired result most quickly. If it were known in advance exactly how many pryings would be required, it would clearly be best to pry at points equally spaced around the lid's circumference, knowing that the last of these actions would free the lid. Unfortunately, you are not able to anticipate this number so the above strategy cannot be used. The next best procedure would be one in which the prying locations are as evenly spaced as possible around the lid's rim for every potential stopping point of the process—but how to accomplish this?

In order to mount an analytical attack on this problem, it is necessary to adopt a criterion by which to gauge the degree of evenness of spacing that a collection of points scattered around a circle possesses. Clearly there are several possible measures of evenness which could be used. The standard we shall primarily use here is the size of the maximum gap (arc length) between any two consecutive points, not only due to its simplicity, but because if the largest spacing can be kept sufficiently small, this will necessarily impose considerable evenness among the other spacings as well. (It should also be noted that when one is opening a sticky can of paint, the size of the largest unloosened arc will probably be the primary determinant of whether the lid will be removable—and also the main factor in the tendency of the can to be secure from leaks when it is hammered shut after use.)

To bring the above problem into a more formal setting, consider a circle of circumference 1 obtained by joining together the ends of the interval $[0, 1]$. In the discussion and figures below, this circle will be traced out in a counterclockwise direction with 0 and 1 meeting at the top of the circle. A *cutting sequence* will refer to an infinite sequence of distinct points selected on this circle; any individual point belonging to this sequence will be termed a *cut*, inasmuch as (except for the first point) it subdivides an existing arc into two subarcs, thereby increasing the total number of arcs by one. We may assume that the location of the first cut is at $0 = 1$, thereby returning the circle to the original unit interval. Hence the problem under consideration is really one of sequentially partioning an interval evenly in the sense described above; however, there are advantages to working on a circle which we will see later.

The goal which we wish to achieve can be loosely stated as:

Keep the largest gap small at all stages of the cutting process. (1)

Thus we are taking a minimax-type approach to the cutting problem—we want to have a "good" partition regardless of when the process is terminated.

Clearly there are conflicts in trying to accomplish this goal. For instance, making the second cut at $1/2$, which is best for stopping after two cuts, offers the worst possible prospects for the maximum gap which will exist after *three* cuts, among all choices of the second cut. Thus sacrifices at particular stages are necessary in order to achieve consistently good performance.

LEAPFROG SEQUENCES. In a search for an optimal cutting scheme, a natural first step is to consider the order in which the cuts and the resulting intervals should be generated. Two rules can be developed: (i) It seems reasonable that each cut should be made in (one of) the largest existing interval(s) present at that stage. Only in this way can the size of the largest gap be reduced as soon as possible—in one step, unless there is a tie for the largest interval size. (ii) Secondly, suppose that n cuts have been made so far, and let the size of the *smallest* interval be S. Then the next $n - 1$ cuts can at best produce a partition in which the new *largest* interval has size S, and the only way that this can happen is if every interval which is larger than S after n cuts is divided into two subintervals each no larger than S.

Taking the above two considerations together yields the following paradigm: Each cut should divide any largest existing interval into two subintervals both no larger than the smallest existing interval. A great benefit of this paradigm is that the gaps generated by such a cutting sequence can be described by a single ordered sequence. Let x_1 represent the size of the initial interval, obtained by cutting at 0; thus $x_1 = 1$. Label the interval sizes obtained from the second cut as x_2 and x_3 with $x_2 \geqslant x_3$; thus $x_1 = x_2 + x_3$. The third cut divides the interval of length x_2 into subintervals of lengths x_4 and x_5, where we shall take $x_4 \geqslant x_5$; we then have three intervals having lengths $x_3 \geqslant x_4 \geqslant x_5$ satisfying $x_3 + x_4 + x_5 = 1$. Continuing in this way, the collection of intervals generated by such a cutting sequence satisfies the following three conditions:

$$x_1 = 1;$$
$$x_n = x_{2n} + x_{2n+1}, \qquad n = 1, 2, 3, \ldots;$$
$$x_1 \geqslant x_2 \geqslant x_3 \geqslant \cdots.$$

Any sequence satisfying the above three conditions shall be referred to as a *leapfrog sequence*. Note that after any number n of cuts there will be intervals with lengths $x_n \geqslant x_{n+1} \geqslant \cdots \geqslant x_{2n-1}$ summing to 1; the $(n + 1)$-st cut then causes the leftmost term, x_n, to "leapfrog" over the other interval sizes to form two new terms on the right. Our goal is to keep the maximum gap size x_n 'small' for all n.

There are an infinite number of leapfrog sequences. A simple case is the one which follows the rule of always bisecting a largest existing interval; this produces the sequence $\{x_n\} = \{1, 1/2, 1/2, 1/4, 1/4, 1/4, 1/4, 1/8, \ldots\}$. Figure 1 shows the

<table>
<tr><th>Cut #</th><th colspan="8">Intervals</th></tr>
<tr><td>1</td><td colspan="8">x_1</td></tr>
<tr><td>2</td><td colspan="4">x_2</td><td colspan="4">x_3</td></tr>
<tr><td>3</td><td colspan="2">x_4</td><td colspan="2">x_5</td><td colspan="4">x_3</td></tr>
<tr><td>4</td><td colspan="2">x_4</td><td colspan="2">x_5</td><td colspan="2">x_6</td><td colspan="2">x_7</td></tr>
<tr><td>5</td><td>x_8</td><td>x_9</td><td colspan="2">x_5</td><td colspan="2">x_6</td><td colspan="2">x_7</td></tr>
</table>

Figure 1

relationship of a leapfrog sequence x_n to the corresponding partitioning of the unit interval created by the first cut. The particular interval sizes shown represent the initial stages of the bisecting sequence.

THE OPTIMAL LEAPFROG SEQUENCE. It is easy to show that any leapfrog sequence tends to zero at the rate of $1/n$. Clearly $x_n \geqslant 1/n$ for each n, with equality possible only if the gap sizes are all equal for some particular n, as in the bisecting sequence above for $n = 1, 2, 4, 8, \ldots$. To obtain an upper bound on x_n note that $1 = x_n + x_{n+1} + \cdots + x_{2n-1} \geqslant n x_{2n}$ since $\{x_n\}$ is nonincreasing, hence $x_{2n} \leqslant 1/n$, i.e., $x_n \leqslant 2/n$ for n even; a similar argument justifies the same bound for odd values of n as well.

Let us therefore study the behavior of the *normalized* maximum gap $M_n = n x_n$, which is of stable order and remains between the values 1 and 2 for all n for any leapfrog sequence. A refined version of the objective given in (1) concerning the long run behavior of the maximum gap can now be formulated:

$$\text{Find } \{x_n\} \text{ such that } L = \limsup_{n \to \infty} M_n \text{ is minimized.} \tag{2}$$

For the bisecting sequence described above,

$$\{M_n\} = \{1, 1, 3/2, 1, 5/4, 3/2, 7/4, 1, \ldots\}.$$

Figure 2 shows the graph of $\{M_n\}$ for this sequence. When n is any power of 2, all gaps are of equal size and $M_n = 1$, the lowest possible value. However, for intermediate values of n the bisecting sequence can do very poorly indeed. In fact, this cutting strategy exhibits the worst possible value of L, 2, among all leapfrog sequences (recall that $x_n \leqslant 2/n$ for all n).

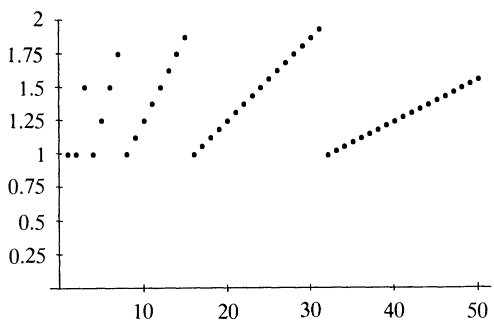

Figure 2

It seems likely that for an optimal leapfrog cutting sequence, the graph of M_n would not contain peaks and valleys such as those in Figure 2. Is it possible, then, to find leapfrog sequences for which M_n possesses a *limit*, and will this lead to a solution to (2)?

To answer these questions, the following ingredients are needed. First, note that since the x_n's are nonincreasing, $x_n = x_{2n} + x_{2n+1} \leqslant 2x_{2n}$, thus $M_n \leqslant M_{2n}$ for all n. Hence $L \geqslant M_n$ for all n since every M_n is a member of a nondecreasing infinite subsequence of M_n's.

Now let $S_n = L/n + L/(n+1) + \cdots + L/(2n-1)$. From the result just shown we have that for each n, $S_n \geqslant x_n + x_{n+1} + \cdots + x_{2n-1} = 1$. Furthermore, comparing the partial harmonic series S_n/L to $\int (1/x)\,dx$ shows that S_n approaches the limit $L \ln 2$ from above. Thus the best value of L that can be hoped for is $L = 1/\ln 2 \approx 1.44$.

This result shows that the minimum price which must be paid to achieve optimality in the sense of (2) is a 44% increase in maximum gap size (for large n) compared with the equal-spaced design which would be used if the total number of cuts to be made was specified in advance. It remains to show that a leapfrog sequence $\{x_n\}$ achieving this value of L exists.

To this end, define $y_n = \sum_{i=1}^{n-1} x_i$ for $n = 2, 3, \ldots$. For $n = 2^k$ we have

$$\begin{aligned} y_n &= x_1 + (x_2 + x_3) + (x_4 + x_5 + x_6 + x_7) + \cdots + (x_{2^{k-1}} + \cdots + x_{2^k - 1}) \\ &= k = \log_2 n. \end{aligned}$$

This suggests that to obtain a cutting sequence whose gap sizes decrease smoothly, we could set $y_n = \log_2 n$ for *all* n to determine values of x_n from the relationship $x_n = y_{n+1} - y_n$; we obtain from this the sequence

$$x_n = \log_2((n+1)/n), \qquad n = 1, 2, \ldots .$$

To check that $\{x_n\}$ is in fact a leapfrog sequence, note that

$$\begin{aligned} x_{2n} + x_{2n+1} &= \log_2\left(\frac{2n+1}{2n}\right) + \log_2\left(\frac{2n+2}{2n+1}\right) \\ &= \log_2\left(\frac{2n+2}{2n}\right) = \log_2\left(\frac{n+1}{n}\right) = x_n; \end{aligned}$$

the other two conditions for a leapfrog sequence are apparent at once. We shall refer to this sequence as the *logarithmic cutting sequence*. It is easy to see that for this sequence, M_n possesses a limit and that that limit is indeed $1/\ln 2$, hence the logarithmic cutting sequence is asymptotically optimal.

The graph of $\{M_n\}$ for the logarithmic cutting sequence is shown in Figure 3. Note that the curve approaches its limit from below, hence we have obtained as a bonus that the sequence performs particularly well for small values of n. Rather remarkably, although there are many partitioning schemes that yield a smaller maximum gap size for *some* values of n (such as the bisecting sequence), only the logarithmic sequence achieves criterion (2):

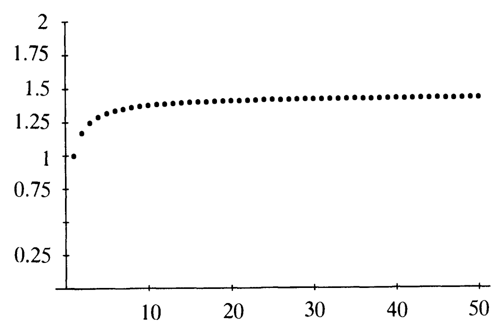

Figure 3

Theorem. *Let $\{x_n\}$ be the logarithmic sequence and let $\{z_n\}$ be any competing leapfrog sequence. Then $\limsup_{n\to\infty} nz_n > 1/\ln 2$.*

Proof: Write $z_n = x_n + \varepsilon_n$ for $n = 1, 2, 3, \ldots$ and let $\xi_n = n\varepsilon_n$. Since $nz_n = nx_n + \xi_n$, it suffices to show that $\limsup_{n\to\infty} \xi_n > 0$. Now the conditions of a leapfrog sequence give $\varepsilon_1 = 0$ and $\varepsilon_n = \varepsilon_{2n} + \varepsilon_{2n+1}$ for each n. Thus either all $\varepsilon_n = 0$ or

some $\varepsilon_n > 0$. If a particular $\varepsilon_n > 0$ then $\max(\varepsilon_{2n}, \varepsilon_{2n+1}) > \varepsilon_n/2$, which immediately yields $\max(\xi_{2n}, \xi_{2n+1}) > \xi_n$. Since this argument can be repeated indefinitely, the theorem follows.

We have shown that the logarithmic cutting sequence $x_n = \log_2((n+1)/n)$ is the *unique* optimal leapfrog cutting sequence with respect to the minimax criterion (2).

THE DUAL PROBLEM. The criterion given in (1) and more precisely in (2) is of course not the only standard which could be used to measure the evenness of a sequential partitioning algorithm. One obvious alternative is to concentrate instead on the *smallest* interval which exists at each stage rather than the largest. This leads to the following dual to the objective given in (2):

$$\textit{Find } \{x_n\} \textit{ such that } 1 = \liminf_{n\to\infty} m_n \textit{ is maximized}, \tag{3}$$

where $m_n = nx_{2n-1}$ is the normalized smallest gap which exists after n cuts of a leapfrog sequence.

One might naturally conjecture that, since making the larger intervals smaller must make the smaller intervals bigger because the sum of all the interval lengths is constrained at 1, the logarithmic cutting sequence is again the unique optimal solution to this new criterion. The following theorem verifies that this is indeed the case:

Theorem. *The logarithmic cutting sequence is uniquely optimal with respect to criterion* (3), *achieving a value of* $l = (1/2)\ln 2$.

Proof: The relationship $x_{2n-1} = x_{4n-2} + x_{4n-1}$ yields $x_{2n-1} \geqslant 2x_{4n-1}$; multiplying by n then gives $m_n \geqslant m_{2n}$ for all n. Thus $m_n \geqslant l$ for all n. Now

$$\begin{aligned}
1 &= x_{2n-1} + \cdots + x_{4n-3} \\
&\geqslant x_{2n-1} + 2(x_{2n+1} + x_{2n+3} + \cdots + x_{4n-3}) \\
&= m_n/n + 2[m_{n+1}/(n+1) + m_{n+2}/(n+2) + \cdots + m_{2n-1}/(2n-1)] \\
&\geqslant 2l[1/(n+1) + 1/(n+2) + \cdots 1/(2n-1) + 1/2n] \\
&\geqslant 2l\int_{n+1}^{2n+1} dx/x = 2l\ln[(2n+1)/(n+1)].
\end{aligned}$$

Taking $n \to \infty$ establishes the claimed maximal value of l. It is again easy to show (by expanding the logarithm function) that the logarithmic cutting sequence achieves this value.

To prove uniqueness, let $\{z_n\}$ be any leapfrog sequence which achieves the optimal value $l = 1/2\ln 2$. First we show that the lim inf can be extended from the odd terms z_{2n-1} to the even terms z_{2n}: using the fact that $\{z_n\}$ is nonincreasing gives $\liminf_{n\to\infty}(n+1/2)z_{2n} \geqslant \liminf_{n\to\infty}(n+1/2)z_{2n+1} = \liminf_{n\to\infty}(n+1)z_{2n+1} = \liminf_{n\to\infty} nz_{2n-1} = 1/2\ln 2$. Combining the two cases then yields $\liminf_{n\to\infty}((n+1)/2)z_n = (1/2)\ln 2$, i.e., $\liminf_{n\to\infty} nz_n = 1/\ln 2$. Writing $z_n = x_n - \varepsilon_n$, $n = 1, 2, 3, \cdots$ and reasoning as in the previous theorem, it follows that ε_n must be 0 for all n so that once again the logarithmic sequence alone is optimal.

Just as the size of the normalized maximum gap M_n *increases* to its limit for the optimal cutting sequence, the size of the normalized minimum gap *decreases* to its limiting value as $n \to \infty$, so the logarithmic sequence is especially good for small values of n under both optimization criteria.

An Application to Data Analysis: Sunflower Plots. When a large amount of data on two variables is displayed in a scatterplot, frequently several data values will have the same position on the graph; this may cause a distorted impression of the data set to be rendered to the observer. Cleveland and McGill [**3**] introduced a graphical device they called *sunflowers* to display such multiple observation points. A sunflower is simply a collection of equal-length spokes each representing one observation, emanating from a common center which represents the variable values that these data share.

Normally the data are completely compiled before the scatterplot is made, in which case the spokes of each sunflower are spaced perfectly evenly, with angles of $2\pi/n$ between adjacent spokes. However, frequently situations occur in which the data is processed "on line," or a file is updated when new data becomes available. In these situations it is not possible to anticipate the spacing which the spokes should ultimately have. Clearly, to maximize the resolution of distinct observations at a common site (precisely the problem that motivated the idea of sunflowers in the first place), one would want to keep the minimal angle between any two spokes as large as possible. This is exactly criterion (3); hence the logarithmic cutting sequence is a most appropriate technique for such a situation. Figure 4 illustrates a scatterplot of data from Chambers, Cleveland, Kleiner and Tukey [**2**] in which the sunflowers have been constructed according to this paradigm. The sunflowers are no longer symmetric but the overall appearance of the plot is still very similar to the original (see [**2**], p. 111). Even for values with as many as twelve coincident observations, the spokes are clearly resolvable.

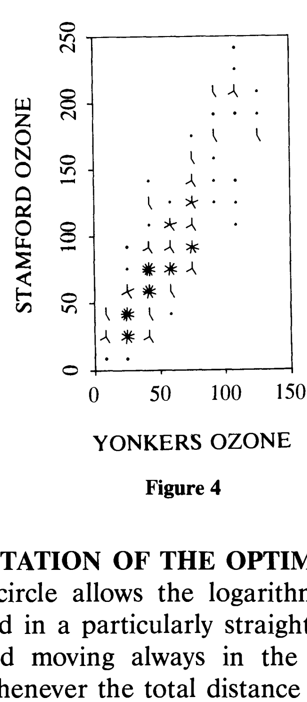

Figure 4

A SIMPLE IMPLEMENTATION OF THE OPTIMAL CUTTING PROCEDURE. The periodicity of the circle allows the logarithmic leapfrog sequence derived above to be implemented in a particularly straightforward manner: Beginning at the point labeled 0 and moving always in the same direction (clockwise or counterclockwise), cut whenever the total distance traveled (arc length) is a value of $\log_2 c$ for $c = 1, 2, 3, \ldots$. Figure 5 shows the locations of the cut points for counterclockwise winding. Note that whenever c is even, i.e., $c = 2k$ for integer k, the cut at that point will already have been made since $\log_2 c = \log_2 k + \log_2 2 = \log_2 k + 1$; thus these cuts can be eliminated. Every odd value of c on the other hand yields a new cut; taking $c = 2n + 1$ for integer n we have $\log_2 c = \log_2(n +$

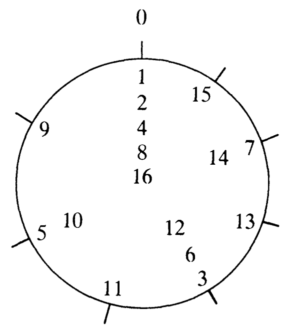

Figure 5

$1/2) + 1$, which shows that the cut divides the interval between the cuts made at $\log_2 n$ and $\log_2(n+1)$. This interval, which has length $\log_2(n+1) - \log_2 n = \log_2((n+1)/n)$, is therefore divided into subintervals of lengths $\log_2(n + 1/2) - \log_2 n = \log_2((2n+1)/2n)$ and $\log_2(n+1) - \log_2(n+1/2) = \log_2((2n+2)/(2n+1))$, which is precisely the requirement used above to derive the logarithmic leapfrog sequence.

To summarize this partitioning recipe, starting with a cut at 0, wind in a particular direction by cutting at each value of $\log_2 c$ for odd c. The procedure always jumps from the location of the cut just made, over the next existing cut, to the interior of the next interval which lies ahead in the direction of winding; this gives a second justification for the *leapfrog* adjective. One can easily imagine a machine programmed to carry out this operation very rapidly since no changes of direction are involved. Note that at any stage of the process, the intervals are ordered with respect to size.

OTHER PARTITIONING SCHEMES

Fixed Angle Cutting. Suppose that in the same fashion as described just above, we travel around the circle making each cut after a prescribed arc length has been traversed. Now suppose, however, that this arc length must be the same each time —what is possible in this case? Remarkably, it turns out that no matter what arc length (or equivalently, angle) is selected, there will never be more than three different gap sizes present at any given stage! (The reader may wish to experiment with this surprising result—angles that are simple fractions are the easiest computationally, however the statement holds for all irrational angles as well.) This result is known as the Three Gap Theorem; it was originally a conjecture of Steinhaus. A proof and references to this theorem may be found in a recent article by van Ravenstein [**8**], where it is also shown that using an angle equal to the golden ratio $\phi = (\sqrt{5} - 1)/2 = .61803\ldots$ is optimal among the class of fixed angle partitioning strategies in the sense that the minimum over n of the ratio of smallest to largest gap sizes is maximized. The author notes that virtually all plants that produce leaves sequentially grow essentially according to this pattern in order to reduce leaf overlap.

Fixed angle cutting schemes are not leapfrog sequences; however it can be shown (see [**8**]) that each new cut in such a scheme divides a largest existing gap and produces one new gap whose size is equal to that of the smallest existing gap (the other new gap may be larger, however). The values of M and m for the

golden ratio strategy are easily found to be $M = 1 + 2/\sqrt{5} = 1.89$ and $m = 1/\sqrt{5} = 0.45$. This represents significantly inferior results to those achieved for the optimal leapfrog sequence.

Random Partitioning. Suppose the partitioning is done by selecting each cutting point completely at random. How much worse is this method? Of course (with probability 1) we will not obtain a leapfrog sequence. It might be conjectured, however, that such a scheme will do fairly well asymptotically inasmuch as the cut points will eventually tend to become quite evenly spread out around the circle. In fact it can be shown that the maximum gap tends to shrink not at the rate of $1/n$ but at the somewhat slower rate of $\ln n/n$. Thus random cutting does infinitely worse asymptotically than any leapfrog sequence.

RELATED PHENOMENA

Benford's Law. There is an interesting connection between the optimal leapfrog sequence and the well known result known as Benford's Law [**1**], which concerns the remarkably consistent but non-uniform distribution of the decimal digits $1, 2, \ldots, 9$ which is observed among the first significant digits of naturally occurring numbers, such as those in tables of physical constants, tables in almanacs, etc. Benford's Law states that each digit d occurs with frequency $\log_{10}((d+1)/d)$ for $d = 1, \ldots, 9$.

Many arguments attempting to justify Benford's Law have been put forward since it first appeared in print in the paper of Newcomb [**4**], who preceded Benford by fifty-seven years; see [**7**] for a review. One of the most appealing of these, due to Pinkham [**5**], invokes the principle of invariance. The argument goes essentially as follows: Suppose that there is in fact such a law; i.e., each digit d occurs with frequency f_d throughout the great majority of natural tables. Then the law must certainly be independent of the units used; for example, the proportions of each digit's occurrence should not change measurably when a table using inches is retabulated in centimeters. The key observation is that regardless of the general magnitude of a number n, the first digit of n corresponds to a given range for the mantissa of the common logarithm of that number; for example, any number whose first digit is 1 has a mantissa between $\log_{10} 1 = .000$ and $\log_{10} 2 = .301$. Now rescaling the units by any factor F adds $\log_{10} F$ to the logarithms of each value in the table, which cycles the mantissas around by that amount (mod 1). In order for the distribution of first digits of a set of numbers, and hence the mantissas of their logs, to remain unchanged regardless of the scaling factor F, the distribution of the latter must be uniform. This directly yields Benford's Law.

Benford's Law extends easily to second and other leading digits in a straightforward way, using the uniform distribution of the mantissas. The law holds in any base b simply by using logs in that base. Thus, the optimal leapfrog sequence corresponds to Benford's Law for the distribution of leading digits in numbers represented in binary form. Of course for first digits alone, Benford's Law in this case is trivial—every number begins with 1. Suppose however that we look instead at the first k digits where k can be any positive integer. What proportion of naturally occurring numbers, then, have a binary representation which begins with a specific configuration of digits, say those which represent the integer n? Since only the mantissa of the base 2 logarithm of n is important, the answer is $\text{mantissa}(\log_2(n+1)) - \text{mantissa}(\log_2 n) \pmod 1 = \log_2(n+1) - \log_2 n =$

$\log_2((n + 1)/n)$, precisely the values generated by the optimal leapfrog cutting sequence.

Figure 6 is a base 2 version of Figure 5 which can be used to illustrate Benford's Law for binary numbers. Let the circle represent the mantissa scale for base 2 logarithms, winding counterclockwise from 0 to 1, which meet at the top of the circle. The circle is subdivided into eight intervals that correspond to the eight possible configurations of the first four digits of any binary number. A number whose base 2 mantissa falls in a given interval will have as its leading binary digits the string shown at the clockwise edge of the interval. Thus in a collection of binary data, we would expect more numbers to begin with 1000 than 1001, more to start with 1001 than 1010, and so forth, in proportion to the interval lengths shown in Figure 6. To obtain the relative frequencies of occurrence for binary numbers for which a smaller number of leading digits is specified, simply combine the appropriate neighboring intervals in Figure 6. To visualize Benford's Law as it applies to more than four leading digits, just continue the cutting process described for Figure 5. The law for a specific digit after the first can be illustrated by shading in alternating blocks of intervals beginning at the top of the circle. For instance, the frequency of 0 in the third digit of binary numbers according to Benford's Law is shown in Figure 6 by the total length of the arcs between 1000 and 1010 and between 1100 and 1110. Figure 6 can easily be modified to work for the original (base 10) version of Benford's Law or for any other base.

Circular Slide Rules. These now archaic devices have a close connection to the optimal leapfrog cutting sequence. Circular slide rules have two indicators similar to the hands of a clock, situated over a logarithmic scale which is wound several rotations and is numbered typically from 1 to 10. That is, if there are w windings to span the range 1 to 10, each revolution increases the value shown on the slide rule by $10^{1/w}$. The diagrams shown in Figures 5 and 6 have a logarithmic scale which increases by a factor of two per revolution.

A crude slide rule for binary calculations can be constructed by adding two hands to Figure 6 and converting these binary values to the range 1 to 2 by inserting "decimal points" (binary points?). To illustrate, consider the calculation of 9×5, which in base 2 is 1001×101. Placing the first hand of the slide rule at 1.001 and the second at 1.010, rotate the two hands together so that the first hand is now on 1.010 and read off the answer from the position of the second hand. If you try this on Figure 6 you will find that the result is located at a point just

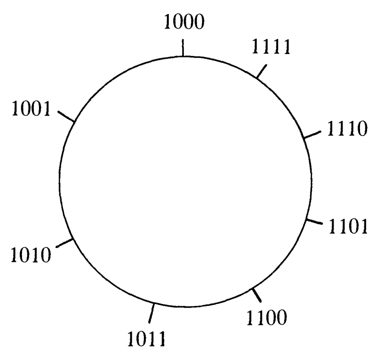

Figure 6

greater than (counterclockwise from) the cutting position 1.011 (labeled as 1011 in Figure 6). This yields the leading digits of the product, 45, which has a binary representation of 101101. Raimi [**6**] discusses the connection between Benford's Law and the circular slide rule for base 10 numbers in his article on the first digit problem.

ACKNOWLEDGMENT. The author is indebted to Ann Watkins for suggesting the application to sunflower plots.

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It is true that Fourier had the opinion that the principal object of mathematics was public use and the explanation of natural phenomena; but a philosopher like him ought to know that the sole object of the science is the honor of the human spirit and that under this view a problem of [the theory of] numbers is worth as much as a problem on the system of the world.

—*C. Jacobi*

Goldbach's Problem in the Ring $M_n(\mathbf{Z})$

Jun Wang

In [1] Vaserstein proved that given any integer p and any matrix A in $M_2(\mathbf{Z})$, there are x, y in $M_2(\mathbf{Z})$ such that $x + y = A$ and $\det(x) = \det(y) = p$. He also asked how about the analogous question for $M_3(\mathbf{Z})$? In this note we answer it for $M_n(\mathbf{Z})$, $n \geq 3$.

For $A = (a_{ij})$ in $M_n(\mathbf{Z})$ we define $d(A) = d = \gcd\{a_{ij}\}$. (Here, we allow zero to divide zero and set $\gcd\{0, 0\} = 0$.)

Lemma 1. (See [2, Ch. 3]). *For any A in $M_n(\mathbf{Z})$ there are U, V in $M_n(\mathbf{Z})$ with* $\det(U) = \det(V) = 1$, *such that* $UAV = \operatorname{diag}(d_1, \ldots, d_n)$, *where $d_1 = d(A)$ which divides each d_i.*

By Lemma 1, we can assume that the matrix $A = \operatorname{diag}(d_1, \ldots, d_n)$ is diagonal. As in [1], we write

$$\operatorname{diag}(a, b) = \begin{pmatrix} a & 1 \\ -p & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ p & b \end{pmatrix}.$$

From this it is easy to see by the use of matrices made up of 2-by-2 blocks that if n is even, then for any integer p and any A in $M_n(\mathbf{Z})$, there are x, y in $M_n(\mathbf{Z})$ such that $x + y = A$ and $\det(x) = \det(y) = p$.

Our main result is the following:

Theorem. *Let $n > 1$ be an odd integer and p a fixed integer. Then for any A in $M_n(\mathbf{Z})$ there are x, y in $M_n(\mathbf{Z})$ such that $x + y = A$ and* $\det(x) = \det(y) = p$ *if and only if $d(A)$ divides $2p$.*

Proof: Suppose $x + y = A$ and $\det(x) = \det(y) = p$. By the definition of $d = d(A)$ we have $\det(x) = \det(A - y) \equiv \det(-y) = -\det(y) \pmod{d}$, from which it follows that $2p \equiv 0 \pmod{d}$. Conversely, put $2p = kd$. From Lemma 1 and Vaserstein's result, it suffices to consider the case $n = 3$ and $A = \operatorname{diag}(d, a, b)$. Then we take

$$x = \begin{pmatrix} d & 1 & 0 \\ 0 & a & 1 \\ -p & -k & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ p & k & b \end{pmatrix}$$

and we see that $A = x + y$ and $\det(x) = \det(y) = p$. ■

From the preceding theorem we can obtain a corollary.

Corollary. *Let A be in $M_n(\mathbf{Z})$, where $n > 1$ is odd. Then for any integer p, there are x, y in $M_n(\mathbf{Z})$ such that $x + y = A$ and $\det(x) = \det(y) = p$ if and only if $d(A) = 1$ or 2.*

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One of the big misapprehensions about mathematics that we perpetrate in our classrooms is that the teacher always seems to know the answer to any problem that is discussed. This gives students the idea that there is a book somewhere with all the right answers to all of the interesting questions, and that teachers know those answers. And if one could get hold of the book, one would have everything settled. That's so unlike the true nature of mathematics.

—L. Henkin at ICME, 1980

A Complex Rolle's Theorem

J.-Cl. Evard and F. Jafari

1 INTRODUCTION. It is well known that many results of classical real analysis are consequences of the Rolle and Mean Value Theorems. In the general case of maps from a subset of a Banach space into another (see [**4**], [**5**] for example), the Mean Value Theorem is an inequality which may be adequate in many applications but falls short of establishing a Rolle's Theorem in the form of an equality as this theorem exists in one real variable. Recently, other variations and interesting applications of Rolle's Theorem have also appeared ([**1**], [**2**], [**3**], [**12**], [**14**]).

Concerning the complex case, Jean Dieudonné [**6**] in 1930 published a necessary and sufficient condition for the existence of a zero of $f'(z)$ in the interior of a circle with diameter ab when f is holomorphic and $f(a) = f(b) = 0$. M. Marden ([**10**], [**11**]) furnishes results about the relative locations of the zeros of a complex polynomial and the zeros of its derivative. I. J. Schoenberg [**13**] conjectures an analogue of Rolle's theorem for polynomials with real or complex coefficients.

It is well known that Rolle's Theorem is not valid for holomorphic functions of a complex variable as it is shown by the function $f(z) = e^z - 1$ which takes the value 0 at $z = 2k\pi i$ for every $k \in \mathbb{Z}$, but $f'(z) = e^z$ has no zeros in the complex plane. It is also easy to see that Rolle's Theorem is not valid for real harmonic functions. For example, the zeros of the partial derivatives of $u(x, y) = x^2 - y^2$ do not separate the zeros of u. Therefore, there is no hope to establish a Rolle's Theorem about the real part or about the imaginary part of a holomorphic function. To establish our Rolle's Theorem, we will need to use a combination of $\Re(f)$ and $\Im(f)$.

The aim of this paper is to present a generalization of Rolle's Theorem to holomorphic functions of a complex variable and to show how a Mean Value Theorem for holomorphic functions follows from this theorem. To emphasize the main ideas of our results we will give the simplest possible form of the theorems, and will refer to extensive generalizations and applications of these results which will be given elsewhere ([**8**], [**9**]). The basic nature and far reaching consequences of these theorems suggest that they should become standard results for holomorphic functions of a complex variable.

We begin by stating and proving the Complex Rolle's Theorem in Theorem 2.1. In Theorem 2.2 we apply Theorem 2.1 to prove a Complex Mean Value Theorem. In Corollary 2.3 we obtain a standard result in complex analysis as a Corollary of our Complex Mean Value Theorem. We conclude by providing several examples and remarks in 2.4. Throughout this paper, we will use the standard notation $z = x + iy$ for $z \in \mathbb{C}$, where $x = \Re(z)$ and $y = \Im(z)$. If a and b are distinct points in $\mathbb{C}$, we will denote by $]a, b[$ the open line segment joining a and b:

$$]a,b[= \{a + t(b - a) : t \in]0,1[\}.$$

2 RESULTS. The main idea of our complex version of Rolle's Theorem below is to consider the relation between the zeros of a holomorphic function f and the zeros of $\Re(f')$, or between f and $\Im(f')$, knowing that no Rolle's Theorem can be established about $\Re(f)$ only or about $\Im(f)$ only.

Theorem 2.1. (Complex Rolle's Theorem). *Let f be a holomorphic function defined on an open convex subset D_f of $\mathbb{C}$. Let $a, b \in D_f$ be such that $f(a) = f(b) = 0$ and $a \neq b$. Then there exists $z_1, z_2 \in]a, b[$ such that $\Re(f'(z_1)) = 0$ and $\Im(f'(z_2)) = 0$.*

Proof: Let $a_1 = \Re(a)$, $a_2 = \Im(a)$, $b_1 = \Re(b)$, $b_2 = \Im(b)$, $u(z) = \Re(f(z))$, $v(z) = \Im(f(z))$ for every $z \in D_f$. Let

$$\phi(t) = (b_1 - a_1)u(a + t(b - a)) + (b_2 - a_2)v(a + t(b - a))$$

for every $t \in [0, 1]$. Then $f(a) = f(b) = 0$ implies that $u(a) = u(b) = v(a) = v(b) = 0$. Consequently, $\phi(0) = 0$ and $\phi(1) = 0$. Therefore, by Rolle's Theorem, there exists $t_1 \in]0, 1[$ such that $\phi'(t_1) = 0$. Let $z_1 = a + t_1(b - a)$. Then

$$0 = \phi'(t_1) = (b_1 - a_1)\left[\frac{\partial u}{\partial x}(z_1)(b_1 - a_1) + \frac{\partial u}{\partial y}(z_1)(b_2 - a_2)\right]$$

$$+(b_2 - a_2)\left[\frac{\partial v}{\partial x}(z_1)(b_1 - a_1) + \frac{\partial v}{\partial y}(z_1)(b_2 - a_2)\right].$$

By the Cauchy-Riemann equations it follows that

$$0 = \frac{\partial u}{\partial x}(z_1)\left[(b_1 - a_1)^2 + (b_2 - a_2)^2\right].$$

Therefore,

$$\Re(f'(z_1)) = \frac{\partial u}{\partial x}(z_1) = 0.$$

By applying this first part of the theorem to the function $g = -if$ we obtain that there exists a $z_2 \in]a, b[$ such that

$$0 = \Re(g'(z_2)) = \frac{\partial v}{\partial x}(z_2) = -\frac{\partial u}{\partial y}(z_2) = \Im(f'(z_2)). \qquad \blacksquare$$

An important application of Theorem 2.1 is the generalization of the real Mean Value Theorem.

Theorem 2.2. (Complex Mean Value Theorem). *Let f be a holomorphic function defined on an open convex subset D_f of $\mathbb{C}$. Let a and b be two distinct points in D_f. Then there exist $z_1, z_2 \in]a, b[$ such that*

$$\Re(f'(z_1)) = \Re\left(\frac{f(b) - f(a)}{b - a}\right) \quad \text{and} \quad \Im(f'(z_2)) = \Im\left(\frac{f(b) - f(a)}{b - a}\right).$$

Proof: Let

$$g(z) = f(z) - f(a) - \frac{f(b) - f(a)}{b - a}(z - a) \tag{1}$$

for every $z \in D_f$. Clearly, $g(a) = g(b) = 0$. Therefore, by Theorem 2.1, there exist $z_1, z_2 \in]a, b[$ such that $\Re(g'(z_1)) = 0$ and $\Im(g'(z_2)) = 0$. But by (1)

$$g'(z) = f'(z) - \frac{f(b) - f(a)}{b - a}$$

for every $z \in D_f$. Therefore,

$$0 = \Re(g'(z_1)) = \Re(f'(z_1)) - \Re\left(\frac{f(b) - f(a)}{b - a}\right),$$

and

$$0 = \Im(g'(z_2)) = \Im(f'(z_2)) - \Im\left(\frac{f(b) - f(a)}{b - a}\right).$$ ■

Let us show that our Complex Mean Value Theorem (2.2) is strong enough to imply the following basic result in complex analysis.

Corollary 2.3. *Let f be a holomorphic function defined on an open connected subset D_f of $\mathbb{C}$ such that $f'(z) = 0$ for every $z \in D_f$. Then f is constant.*

Proof: By Lemma 2.1 in [7], or by the Analytic Continuation Theorem of complex analysis, it is sufficient to show f is locally constant. Let z_0 be arbitrary in D_f and let U_{z_0} be a convex neighborhood of z_0 contained in D_f. Let z be a point of U_{z_0}, $z \neq z_0$. By Theorem 2.2, there exist $z_1, z_2 \in]z_0, z[$ such that

$$\Re\left(\frac{f(z) - f(z_0)}{z - z_0}\right) = \Re(f'(z_1)) = 0,$$

and

$$\Im\left(\frac{f(z) - f(z_0)}{z - z_0}\right) = \Im(f'(z_2)) = 0.$$

Therefore, $f(z) = f(z_0)$. Thus f is constant in U_{z_0}. ■

We conclude this note by providing several examples. These examples shed light on the Complex Rolle's Theorem and illustrate the assertion that the zeros of the real and imaginary parts of the derivative of a holomorphic function separate the zeros of that holomorphic function.

Examples and Remarks 2.4. (i) Let $f(z) = e^z - 1$ and note that $f(z) = 0$ for $z = 2k\pi i$ for every integer k. Since $f'(z) = e^z = e^x \cos y + ie^x \sin y$, $\Re(f'(z)) = 0$ if $y = (2k + 1)\pi/2$, and $\Im(f'(z)) = 0$ if $y = k\pi$. Therefore the zeros of the real and imaginary parts of f' are straight lines both separating the zeros of f.

(ii) If $f(z) = (z - a)(z - b), a \neq b$, then $f(z) = 0$ when $z = a$ or $z = b$. Since $f'(z) = 2z - a - b$, $\Re(f'(z)) = 0$ if $x = \Re(a + b)/2$, $\Im(f'(z)) = 0$ if $y = \Im(a + b)/2$. So again the zeros of the real and imaginary parts of f' are lines both separating the zeros of f.

(iii) Note that in general the zero set of $\Re(f'(z))$ and $\Im(f'(z))$ need not be straight lines as it may be seen by considering $f(z) = z^3 + z^2 + z + 1$; the zero set of $\Re(f')$ is a hyperbola in this case.

We provide many extensions and applications of these theorems in a separate paper [**9**].

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Logic is the hygiene the mathematician practices to keep his ideas healthy and strong.

—H. Weyl

Picture Puzzle
(from the collection of Paul Halmos)

Un analyste noir.

(*See page* 884.)

Mathematical rigor is like clothing: in its style it ought to suit the occasion, and it diminished comfort and restricts freedom of movement if it is either too loose or too tight.

—*G. F. Simmons*

Underwood Dudley, Gerald A. Edgar, Michael A. Filaseta, Ira M. Gessel, Richard A. Gibbs, Douglas A. Hensley, John R. Isbell, Mourad E. H. Ismail, Murray Klamkin, Daniel J. Kleitman, Frederick W. Luttmann, Frank B. Miles, Richard Pfiefer, Stephen L. Portnoy, J. O. Shallit, John Henry Steelman, Kenneth B. Stolarsky, Douglas B. Tyler, Daniel Ullman, Edward T. H. Wang, and William E. Watkins.

Answer to Picture Puzzle:

The description was a feeble pun on this Frenchman's German name: it is Laurent Schwartz, approximately forty years ago.

It is a perennial problem for mathematicians to explain to the public at large what makes mathematics worthwhile if not its practicality. It is like explaining to someone who has never heard music what a lovely melody is... Do let us try to teach the general public more of the sort of mathematics that they can use in everyday life, but let us not allow them to think—and certainly let us not slip into thinking—that this is an essential quality of mathematics.

There is a great cultural tradition to be preserved and enhanced. Each generation must learn the tradition anew. Let us take care not to educate a generation that will be deaf to the melodies that are the substance of our great mathematical culture.

—*B. Chandler & H. M. Edwards*

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LETTERS

Definition of Chaos

In [**1**], Devaney defines a continuous function f on a metric space X to be *chaotic* if it satisfies properties (1), (2) and (3):

(1) f is transitive,
(2) the periodic points of f are dense in X, and
(3) f has sensitive dependence on initial conditions.

In [**2**], Banks et al. prove that (1) and (2) imply (3), but no mention is made of whether any two of the properties imply the third.

First we note that (1) and (3) do not imply (2). Let $X = S^1 \setminus \{e^{i2\pi p/q} | p, q \in \mathbf{Z}, q \neq 0\}$, equipped with the usual arclength metric d. Define the continuous function f: $X \to X$ by $f(e^{i\theta}) = e^{i2\theta}$. Then f has *no* periodic points (we've removed the $2^n - 1$ roots of unity for all n, which are shown in [**1**] to be the only possibilities). Yet f is transitive (because any nonempty open set in X is eventually expanded to cover X) and f has sensitive dependence on initial conditions (given any $e^{i\theta} \in X$ and any $e^{i\phi} \in X$ with $0 < |\theta - \phi| < \pi$, select n such that $2^n|\theta - \phi| \leqslant \pi < 2^{n+1}|\theta - \phi|$; then $d(f^n(e^{i\theta}), f^n(e^{i\phi})) > \pi/2$).

Now we show that (2) and (3) do not imply (1). Equip the unit circle S^1 and unit interval $[0,1]$ with their usual metrics, and consider the cylinder $Y = S^1 \times [0,1]$ with the induced "taxicab" metric. Define the continuous function g: $Y \to Y$ by $g(e^{i\theta}, t) = (e^{i2\theta}, t)$. Then g is not transitive (taking $U = S^1 \times [0, 1/2)$ and $V = S^1 \times (1/2, 1]$, we see that $g^n(U) \cap V = U \cap V = \varnothing$ for any n). Yet the periodic points of g are dense in Y (a point of the form $(e^{i\theta}, t)$ is periodic for g precisely when $e^{i\theta}$ is a $2^n - 1$ root of unity for some n) and g has sensitive dependence on initial conditions (the argument is like that in the first example).

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UNSOLVED PROBLEMS

Edited by: **Richard Guy**

In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Typescripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N 1N4.*

The Opaque Cube Problem

Kenneth A. Brakke

1. INTRODUCTION. The Opaque Square Problem has been floating around a long time:

What is the shortest length fence that can block any line of sight across a square plot of ground?

The best known solution is shown in Figure 1. It has straight fences from three corners meeting at a point at angles of $2\pi/3$ plus a fence from the fourth corner to the center. It has not been proved that this is in fact the best possible. For more on opaque plane regions, including opaque circles and polygons, see **[5]**.

Martin Gardner **[6]** has raised the Opaque Cube Problem:

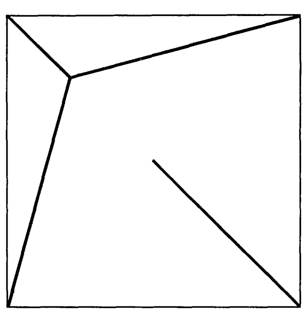

Figure 1. The best known opaque square solution.

What is the least area surface that can block all lines of sight through a cube?

In any dimension, one has the Opaque Region Problem:

What is the least measure hypersurface that intersects all lines that pass through a given region?

One can pose two versions of each problem: the restricted version, which permits fences only inside the region, and the unrestricted version, which also permits fences outside. Despite the simplicity of the statement of the problem, practically nothing has been proved for any region. No fence has been proved optimal for any region in dimension 2 or higher that does not lie in a hyperplane. Even the opaque equilateral triangle is unproved.

I will present a possible solution to the Opaque Cube Problem, and I will use the Opaque Sphere Problem to suggest that the optimal surface may not exist. To get a solution, it may be necessary to widen the type of object considered as a fence, to include varifolds, for example. In that case, the notion of "opaqueness" needs clarification. I use the term *fence* to refer to the object making the region opaque in order not to prejudge what type of object is proper.

2. THE OPAQUE CUBE PROBLEM. Let the region to be made opaque be a unit cube. An obvious way to make it opaque is with twelve triangles from the edges to the center, as shown in figure 2. It has area $3\sqrt{2} \approx 4.2426$. However, this cannot be the best because the central vertex is not one of the types allowed in minimal surfaces. The only types of singularities found in the interiors of minimal surfaces are three surfaces meeting along a curve at angles of 120° or six surfaces meeting at a point with tetrahedral angles [**9**].

If a cubical wire frame is dipped in soap solution, then the surface that forms is shown in figure 3. The central vertex of figure 2 has been replaced by a rounded square, and all the singularities are of the proper type. It has an area of

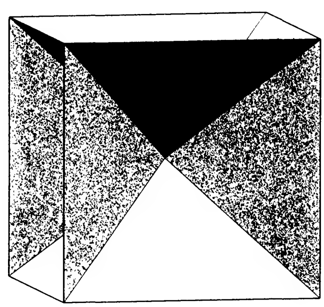

Figure 2. An opaque cube solution with twelve triangles meeting at the center. The area is $3\sqrt{2} \approx 4.2426$.

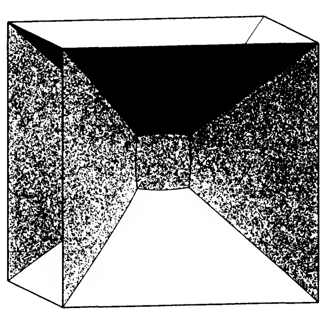

Figure 3. The soap film that forms on a cubical frame, with a central rounded square. Area ≈ 4.2396.

approximately 4.2398. (All the areas cited hereafter in this section were calculated with my program called the Surface Evolver [**2**].) But this is not the best possible solution.

A better solution (see figure 4) can be constructed as a three dimensional version of the Opaque Square solution. The best way to visualize this surface is to begin with a non-optimal fence made up of flat planes and then imagine it shrinking like a soap film to the final state shown in figure 4. Begin with a cubical frame. Add the top and bottom faces of the cube. Add four vertical rectangles between the top and bottom faces so that their horizontal cross-section is the opaque square solution. The central gap in the opaque square solution becomes a tunnel through the cube, but of course there is no line of sight through the tunnel. When this initial configuration is run through the Surface Evolver to minimize area, the top face gets pulled down and the bottom face gets pulled up. The area is approximately 4.2342. There is still a tunnel from the front face, through the middle, and out to the right side, but the vertical edge of the surface in the middle is constrained to stay on the vertical centerline, so one cannot see all the way through the tunnel. This surface could be made as a real soap film if a central vertical wire were added to a cubical frame, but it might be tricky to convince the soap film to take up this particular topology.

My best solution is shown in figure 5. It is similar to figure 4, except that twofold symmetry has been replaced by threefold symmetry. The area is approximately 4.2324. There are three entrances to the central tunnel, from the front, from the bottom, and from the right. Another way to describe the topology is to

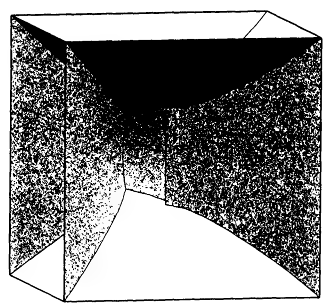

Figure 4. A better opaque cube solution. A horizontal slice through the middle looks like the opaque square solution. Area ≈ 4.2342.

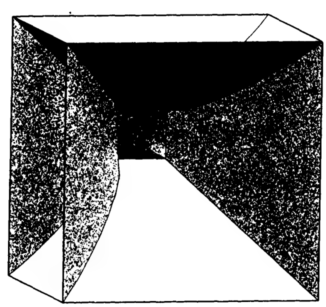

Figure 5. The best known opaque cube solution. It is like figure 4, but with threefold symmetry in place of twofold. Area ≈ 4.2324.

start with a soap film on a cubical frame with a cubical bubble in the middle, and then remove three adjacent faces of the bubble. A soap film version would need three wires coming out from the center at right angles to hold the edges of the surface.

A videotape showing these shapes is available in [**3**].

Martin Gardner [**7**] is offering a \$50 prize for "the best improvement" on figure 5.

3. THE OPAQUE SPHERE PROBLEM. This section will construct a sequence of fences for a unit sphere that converges to a set of larger area than the limit of the areas. The problem will be the unrestricted version, permitting fences outside the sphere. The first fence F_1 consists of the lower hemisphere plus a cylinder around the upper hemisphere, as shown in figure 6. (The top of the cylinder is not included.) The area is $A_1 = 4\pi$, which is the same as the area of the sphere. Each successive fence F_{n+1} is formed by slicing each cylinder of F_n in half horizontally and shrinking the top half until it hits the sphere. This sequence was first found by R. Laver, as cited in [**5**]. The areas A_n form a strictly decreasing sequence, and

$$A_\infty = \lim_{n\to\infty} A_n = 2\pi + \int_0^1 2\pi\sqrt{1-z^2}\,dz = 2\pi + \pi^2/2.$$

Note, however, that the limiting point set is the surface of the sphere, which has a larger area than A_∞. This suggests that the least-area hypersurface of the Opaque Region Problem may fail to exist. Can the Opaque Region Problem be reformulated so that a solution can always be proved to exist in some sense?

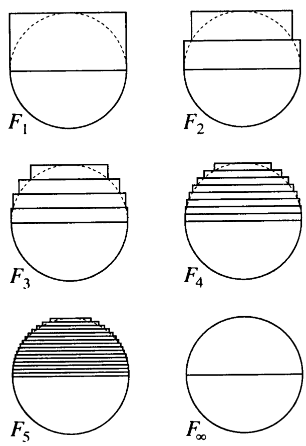

Figure 6. A sequence of opaque sphere solutions, with limiting varifold F_∞.

4. A VARIFOLD FORMULATION. A standard method of solving minimization problems is to find a minimizing sequence of objects, use compactness to guarantee the existence of a limit object, and show the limit object is a valid solution and absolutely minimizes the objective function. If the opaque sphere sequence F_n is in fact a minimizing sequence, then this strategy fails if the objects are point sets. This example shows that we need to reconsider the type of object that a fence is.

To make the compactness argument succeed, fences need to be from a topological space in which area is lower semicontinuous and the limit of an opaque

minimizing sequence is also opaque. Clearly the tangent planes of the F_n must be taken into account in the limit. Varifolds provide a setting in which area and tangent planes behave properly in the limit. A k-dimensional *varifold* in $\mathbf{R}^m$ is a measure on $\mathbf{R}^m \times G_k\mathbf{R}^m$, where $G_k\mathbf{R}^m$ is the Grassmannian manifold of unoriented k-planes through a point (see [**1**], [**8** p. 109]). In other words, the measure is on planes at points, not just points. The varifold area (or **mass**) is the total measure of $\mathbf{R}^m \times G_k\mathbf{R}^m$, and the space of varifolds is compact with area being lower semicontinuous. A smooth manifold naturally corresponds to a varifold in which the measure is on the geometric tangent plane at each point.

If the sphere fences F_n are regarded as varifolds, then the limit varifold F_∞ exists and behaves as desired. The upper hemisphere of F_∞ has all of its measure on vertical planes, a sort of infinitesimal venetian blind effect, and the area of F_∞ is $A_\infty = 2\pi + \pi^2/2$.

There remains to be stated a definition of opaqueness for varifolds. I propose the following. First, define a point P to be a *point of opacity* for a line L if the projection of the varifold in any neighborhood of P on the perpendicular hyperspace of L has at least unit density at the projection of P. Second, say that a varifold makes a region *opaque* or is a *fence* if almost every line that intersects the region has a point of opacity on it. "Almost every" is understood in the measure theoretic sense on the manifold of lines.

This definition says "almost every" because in some solutions that we want to keep (such as the opaque cube solutions with tunnels) there are lines that graze the edges of fences. The projected density locally along these lines is only $1/2$. These points cannot be counted as points of opacity, or else the density all over could be cut down. Another alternative to "almost every" would be to say that a line is blocked if some arbitrarily near line has a point of opacity. But then one could block all lines with an arbitrarily thin but dense dust of tiny varifolds, which again thwarts our purpose.

The limit sphere varifold F_∞ is opaque in this sense. In particular, every nonhorizontal line that intersects just the upper hemisphere has a point of opacity at its lower intersection with the hemisphere surface. Here the points of opacity of the limit are the limits of the points of opacity of the F_n since the support of the limit F_∞ is a manifold.

It is not clear in general that the limit of a minimizing sequence of opaque varifolds must be an opaque varifold. It is conceivable that the limit varifold may be smeared out so that there would be no points of opacity. On the other hand, perhaps the constraint of being a minimizing sequence is strong enough to force the limit to behave properly.

5. OPEN PROBLEMS. I conclude with a list of open problems and topics for research:

1. Prove that the opaque square solution in figure 1 is optimal.
2. Is the limit varifold of a minimizing sequence of opaque varifolds opaque?
3. Find an example where the solution is provably a varifold, or some other non-manifold.
4. Find a plane region whose optimal fence is plausibly a varifold, or prove that varifolds are not needed in two dimensions.
5. Find an example where restriction of the fence to the region is provably significant.

6. Find the maximum opaque volume for a given area. Is it a hemisphere?
7. For dimension four or greater, the cone over a hypercube (analogous to figure 2) does minimize area [**4**]. Is it also the optimal fence?

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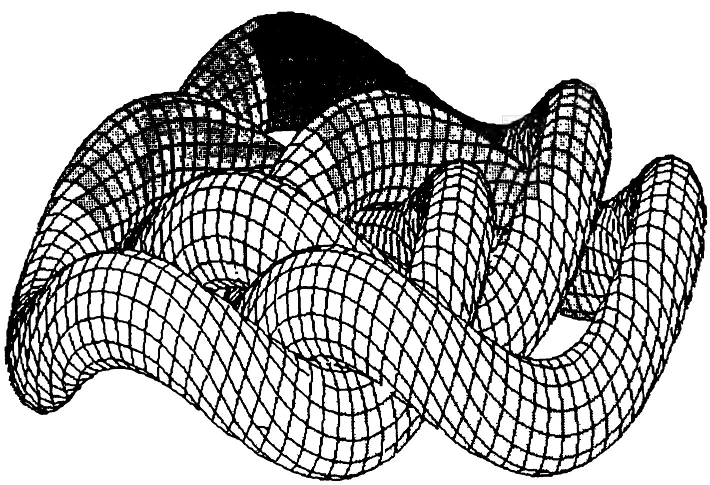

Borromean Rings
(drawn with Maple V software)

10264. *Proposed by L. W. Shapiro, Howard University, Washington, DC, and D. G. Rogers, Australian National University, Canberra, Australia.*

Let $C_n = 1/(n+1)\binom{2n}{n}$ for $n \in \mathbb{N}$ and form the generating function

$$C(x) = \sum_{n \geq 0} C_n x^n.$$

Establish the identities:

(a) $\sum_{n \geq 0} (n+1)x^n C(x)^{2n+2} = \sum_{m \geq 0} (4x)^m.$

(b) $\sum_{n \geq 0} (2n+1)x^n C(x)^{2n+1} = \sum_{m \geq 0} (4x)^m.$

NOTES

(10262) The Fibonacci numbers probably need no introduction, but the answer to this question depends on the initial conditions $F_1 = F_2 = 1$. The remaining numbers are then characterized by the recurrence $F_{n+2} = F_{n+1} + F_n$. **(10264)** The C_n are known as "Catalan Numbers." More information can be found in Graham, Knuth and Patashnik, *Concrete Mathematics*. Since the Catalan numbers arise in a variety of combinatorial problems (see William G. Brown, "Historical note on a recurrent combinatorial problem," this MONTHLY 72(1965), 973–977), one might hope for at least one combinatorial interpretation of the formulas given here.

SOLUTIONS

Digits Occurring with Exactly the Right Frequency

E3418 [1991, 55]. *Proposed by E. T. Parker, University of Illinois, Urbana, IL.*

For $k = 1, 2, \ldots, 9$ let S_k be the set of positive integers n such that the number of digits in the decimal expansion of n is a multiple of 10 and such that each of digits $1, \ldots, k$ occurs in exactly one-tenth of the places. For which value of k does

$$\sum_{n \in S_k} n^{-1}$$

converge?
Note: Professor Parker died on 31 December 1991 at the age of 65.

Composite solution by Kevin Ford (student), University of Illinois, Urbana, IL, and Richard Stong, University of California, Los Angeles, CA. The given sum converges for $k \geq 3$ and diverges for $k \leq 2$. To see this let $A_{m,k}$ be the number of elements of S_k which have exactly $10m$ digits. Any one of these elements of S_k

contributes between 10^{-10m} and $10^{-(10m-1)}$ to the sum, so that

$$\sum_{m=1}^{\infty} 10^{-10m} A_{m,k} \leq \sum_{n \in S_k} n^{-1} \leq 10 \sum_{m=1}^{\infty} 10^{-10m} A_{m,k}. \tag{1}$$

The number of strings of $10m$ decimal digits having the specified property (that each of the digits $1, \ldots, k$ occurs in exactly one-tenth of the places) is equal to

$$\binom{10m}{m}\binom{9m}{m} \cdots \binom{(11-k)m}{m}(10-k)^{(10-k)m}.$$

By symmetry exactly one-tenth of these strings begin with zero. Hence

$$A_{m,k} = \frac{9}{10} \frac{(10m)!}{(m!)^k[(10-k)m]!}(10-k)^{(10-k)m}.$$

Using Stirling's formula we readily obtain

$$10^{-10m} A_{m,k} \sim \frac{9}{10}\left(\frac{10}{10-k}\right)^{1/2} (2\pi m)^{-k/2} \quad (m \to \infty).$$

Thus the sums in (1) converge if and only if $k \geq 3$.

Editorial comment. Both R. High and the proposer observed that the above conclusion (divergence for $k \leq 2$ and convergence for $k \geq 3$) is valid for any base b, provided of course that $k < b$; if b is 2 or 3, there are no cases of convergence.

Solved also by D. Callan, R. High, O. P. Lossers (The Netherlands), A. Nijenhuis, A. Pedersen (Denmark), The Central Michigan University Problem Group, The National Security Agency Problems Group, and the proposer. One incorrect solution was also received.

Holomorphic Functions on a Square

E3420 [1991, 55]. *Proposed by S. G. Merzlyakov, Mathematical Institute, Ural Branch of the Academy of the USSR.*

Find all functions f holomorphic on the square $S = \{x + iy\colon -1 < x < 1, -1 < y < 1\}$ in the complex plane for which there exist real-valued functions α and β on $(-1, 1)$ satisfying

$$|f(x + iy)| = \alpha(x) + \beta(y)$$

throughout S.

Solution by the proposer. All such functions f are of the form (a) $f(z) = A(z - B)^2$, where A and B are complex, or of the form (b) $f(z) = (Ae^{Cz} + Be^{-Cz})^2$, where A, B, and C are complex and C^2 is real. If f is not identically zero, consider a disk $D \subset S$ on which f does not vanish. Letting g be an analytic square root of f on D, we have $g(z)\overline{g(z)} = \alpha(x) + \beta(y)$ for $z \in D$. Differentiating this with respect to x yields $2\Re(g'(z)\overline{g(z)}) = \alpha'(x)$ for $z \in D$. Differentiating this with respect to y yields $\Re(ig''(z)\overline{g(z)} + ig'(z)\overline{g'(z)}) = 0$ for $z \in D$. This implies $|g(z)|^2\Re(ig''(z)/g(z)) = 0$ for $z \in D$, and hence $\Im(g''(z)/g(z)) = 0$ for $z \in D$.

From the Open Mapping Theorem and the Identity Theorem, we now deduce $g''(z) = C^2 g(z)$ for $z \in S$, for some real constant C^2. If $C = 0$, then g is linear, and either f is of form (a) or is constant and of form (b). If $C \neq 0$, then $g(z) = Ae^{Cz} + Be^{-Cz}$ for some complex A and B, and f is of form (b).

Editorial comment. The proposer observed that this problem generalizes problem 6533[1987,81; 1988,669]. S. Haruki pointed out that this problem is essentially solved in H. Haruki, "Studies on certain functional equations from the standpoint of analytic function theory", *Sci. Rep. Osaka Univ.* 14(1965), 1–40, and in J. Aczel and H. Haruki, *Commentary to Einar Hille's collected works*, MIT Press, 1975, 651–658, because the given functional equation reduced to the equation $|g(x + iy)| = |g(x)| + |g(iy)|$ studied there, via the normalization $|g(x)| = |f(x)| - \alpha(0) - \beta(0)$.

Solved also by J. Anglesio (France), R. B. Israel (Canada), O. P. Lossers (The Netherlands), T. McCoy, and R. Stong. Partially solved by S.-J. Bang (Korea), D. Brown (Canada), and T. McDonald. One incorrect solution was received.

Functions preserving a thrice-punctured sphere

6648 [1991, 63]. *Proposed by Walter Rudin, University of Wisconsin, Madison, WI.*

Let Ω be the region obtained by removing the points $0, 1, \infty$ from the Riemann sphere. Find all nonconstant holomorphic functions defined on Ω which map Ω into itself.

Solution by the proposer. There are precisely 6 such functions. They form a group G of linear fractional transformations, taking z to

$$z, \frac{1}{z}, 1 - z, \frac{1}{1 - z}, \frac{z - 1}{z}, \frac{z}{z - 1}.$$

These permute the set $E = \{0, 1, \infty\}$.

To prove this, let f be as in the statement of the problem, then $f(\Omega)$ does not contain 0, 1, or ∞. The big Picard theorem shows therefore that no point of E is an essential singularity of f. Thus f extends to a rational function on the Riemann sphere $\mathbf{S}$, and therefore $f(\mathbf{S}) = \mathbf{S}$. Every $q \in E$ is therefore $f(p)$ for some $p \in E$. The restriction of f to E is therefore a permutation of E, and there is a $\phi \in G$ such that $\phi = f$ on E. Put $g = \phi^{-1} \circ f$. Then g fixes 0, 1, and ∞, and $g(\Omega) \subset \Omega$. Consideration of 0 and ∞ shows that $g(z) = cz^m$, for some $c \neq 0$ and some positive integer m. If $m > 1$, then $g(z) = 1$ has roots outside E. Thus $m = 1$, and now $g(1) = 1$ forces $c = 1$, hence $g(z) = z$. Since g is the identity, $f = \phi \in G$.

Editorial comment. Sharad Kanetkar used a similar argument to prove the following.

Theorem. *Let E be any finite subset of the Riemann sphere containing* $0, \infty$, *and at least one other point. Let Γ be the group of all Möbius transformations that map E onto itself. Suppose that for every e_1 and e_2 is E, there is a function f in Γ such that $f(e_1) = e_2$. Then Γ is precisely the set of all nonconstant holomorphic functions satisfying $f(\Omega) \subset \Omega$, where Ω is the complement of E.*

He also noted that if $E = \{0, \infty\}$, the function $f(z) = e^{1/z}$ satisfies the conditions of the problem but f is obviously not a Möbius transformation.

All solvers used the big Picard theorem. The proposer had hoped for a more elementary proof.

Solved also by F. Brulois, R. J. Chapman (U.K.), K. Ford (student), R. B. Israel (Canada), S. Kanetkar, R. Mortini (Germany), A. Riese & J. T. Kirk, R. M. Robinson, R. Rupp (Germany), H. Solbrig (student), L. A. Tristán Vega (Spain), and the Western Maryland College Problems group.

A Common Least Multiple

E3431 [1991, 264]. *Proposed by Jeffrey Shallit, Dartmouth College, Hanover, NH.*

If n is a positive integer, let $f(n)$ denote the least common multiple of $1, 2, \dots, n$ and let $g(n)$ denote the least common multiple of

$$\binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n}.$$

Prove that

$$g(n) = f(n+1)/(n+1).$$

Editorial comment. David Callan, Allan Pedersen, David Singmaster, and Michael Vowe remarked that this problem also appeared as MONTHLY problem E2686 [1977, 820; 1979, 131]. All solutions were similar to the published solution of that problem. Briefly, that requires identifying the largest power of each prime which can divide $f(n+1)$ or $(n+1)g(n)$, and relating these through the identity $(n+1)\binom{n}{k} = (k+1)\binom{n+1}{k+1}$. In addition, a later communication from Olivier Ramare via the proposer observed that a form of the result appears as Theorem 3 of M. Nair, "On Chebyshev-type inequalities for primes," this MONTHLY 89 (1982), 126–129.

Solved by R. Betts (student), D. Callan, R. J. Chapman (U.K.), J. Christopher, M. Dindos (Czechoslovakia), J. Duemmel, E. C. Greenspan & S. A. Greenspan, R. J. Hendel, R. High, S. Kanetkar, D. W. Koster, M. E. Kuczma (Poland), O. P. Lossers (The Netherlands), J. Manoharmayum (India), H. M. Marston, J. B. Muskat (Israel), A. Pedersen (Denmark), B. Ravikumar, D. Singmaster (U.K.), R. Stong, G. W. Teck (student, U.K.), M. Vowe (Switzerland), C. Wildhagen (The Netherlands), M. Woltermann, and the proposer.

Asymptotic Linearity

6652 [1991, 272]. *Proposed by D. M. Bloom, Brooklyn College of the City University of New York.*

For x a positive integer put

$$E(x) = \sum_{0 \le i \le x} \frac{(-1)^i}{i!}(x-i)^i e^{x-i}.$$

Evaluate

$$\lim_{x \to \infty} \{E(x) - 2x\}.$$

Solution I by WMC Problems Group, Western Maryland College, Westminster, MD. The limit is $2/3$. To see this, let D_{-1}^k denote the operation of taking k derivatives and evaluating at $z = -1$. Then we have

$$E(n) = \sum_{k=0}^{n} \frac{1}{k!} D_{-1}^k e^{(k-n)z}.$$

We can write this as a sum of contour integrals around a loop Γ which surrounds the point $z = -1$:

$$E(n) = \sum_{k=0}^{n} \frac{1}{2\pi i} \int_\Gamma \frac{e^{(k-n)z}}{(z+1)^{k+1}}\, dz.$$

Next, we interchange summation and integration and sum the finite geometric series to obtain:

$$E(n) = \frac{1}{2\pi i}\int_\Gamma e^{-nz}\frac{\left(\dfrac{e^z}{z+1}\right)^{n+1} - 1}{\left(\dfrac{e^z}{z+1}\right) - 1}\cdot\frac{1}{z+1}\,dz.$$

Now, if Γ does not include zero, we can ignore the part of the integrand analytic near $z = -1$ and write:

$$E(n) = \frac{1}{2\pi i}\int_\Gamma e^{-nz}\frac{e^z}{e^z-(z+1)}\cdot\frac{1}{(z+1)^{n+1}}\,dz.$$

A Laurent expansion shows that the function $(2(z+1)/z^2)/(z+1)^{n+1}$ has residue $2n$ at the pole at $z = -1$, so we can incorporate the $-2n$ term in the integral:

$$E(n) - 2n = \frac{1}{2\pi i}\int_\Gamma\left(\frac{e^z}{e^z-(z+1)} - 2\frac{z+1}{z^2}\right)\frac{1}{(z+1)^{n+1}}\,dz.$$

Now, suppose that we replace Γ by a contour Γ' which starts at a real point of Γ to the left of -1, goes once around Γ in the counter-clockwise direction, then runs along the negative real axis to -3, then around the circle $|z| = 3$ in the clockwise direction, and then back to its starting point along the negative real axis. The first part of the integrand is bounded on Γ', while the term $1/(z+1)^{n+1}$ goes to zero as $n \to \infty$ on the circle $|z| = 3$ (and the line on the real axis is traversed once in each direction), so the limit is not altered by this change of contour.

The only pole of the integrand contributing to the integral around Γ' is a simple pole at the origin, which is counted with weight -1 since we are now circling the origin in a clockwise sense. A series computation shows that the residue of this pole is $-2/3$, which completes the proof.

Solution II by Richard Holzsager, The American University, Washington, DC. Let

$$E(x) = \sum_{0\le j\le x}\frac{(-1)^j(x-j)^j}{j!}e^{x-j},$$

for all real $x > 0$. Note that the function $E(x)$ so defined extends to a continuous function on $[0,\infty)$, that the restriction of $E(x)$ to $(1,\infty)$ is continuously differentiable, and that $E(x)$ satisfies the functional equation

$$f'(t) = f(t) - f(t-1) \qquad (*)$$

for all $t \ge 1$. The equation $(*)$ gives a very special example of a "linear, autonomous functional differential equation." The theory of such equations is now well-developed: see [**1**] or [**2**]. The so-called characteristic equation of $(*)$—obtained by seeking a solution of the form $f(x) = e^{\lambda x}$—is

$$\lambda = 1 - e^{-\lambda}.$$

It is well-known (and easy to prove) that this equation has a double root at $\lambda = 0$ and that all other roots λ satisfy $\Re(\lambda) < 0$. It follows from the general theory of linear functional differential equations (see [**2**, chapter 7]) that there exist constants a and b with

$$\lim_{x\to\infty} E(x) - ax - b = 0. \qquad (**)$$

The theorem below gives an elementary proof of this which is independent of the general theory of linear functional differential equations.

Assuming for the moment the existence of a and b in $(**)$, we now determine their values.

Integrating $(*)$ shows that

$$f(n+1) - \int_n^{n+1} f(t)\,dt$$

is independent of n. Multiplying $(*)$ by t and integrating shows that

$$\int_n^{n+1} (t+1) f(t)\,dt - (n+1) f(n+1)$$

is also independent of n.

From the claimed asymptotic equivalence of $E(x)$ with $ax + b$, it follows that we get the same values when we substitute these two functions into either of the above invariants, so

$$a/2 = a + b - \int_0^1 (at+b)\,dt = e - \int_0^1 e^t\,dt = 1$$

and

$$-a/6 + b/2 = \int_0^1 (t+1)(at+b)\,dt - (a+b) = \int_0^1 (t+1)e^t\,dt - e = 0.$$

Solving these gives $a = 2$ and $b = 2/3$, so that $E(x) \approx 2x + 2/3$.

Theorem. *If f is an averaging function, i.e., satisfies $(*)$, and on the interval $[n-1, n]$ the maximum and minimum values of f' differ by d, then the corresponding difference on $[n, n+1]$ is at most $(\sqrt{e} - 1)d < .649d$. Furthermore,*

$$\{f'(x) \colon n \le x \le n+1\} \subset \{f'(x) \colon n-1 \le x \le n\}$$

for all $n \ge 2$.

Proof: If, for any $n \ge 2$, there exist constants a and b such that $f(x) = ax + b$ for $n - 1 \le x \le n$, then it is easy to derive from $(*)$ that $f(x) = ax + b$ for all $x \ge n$. Thus, we may assume that $\max_{n-1 \le x \le n} f'(x) \ne \min_{n-1 \le x \le n} f'(x)$. By replacing $f(x)$ by $g(x) = d^{-1}(f(x) - \beta x) - \gamma$ with $\beta = \min_{n-1 \le x \le n} f'(x)$, $d = \max_{n-1 \le x \le n} f'(x) - \min_{n-1 \le x \le n} f'(x)$ and $\gamma = d^{-1}(f(n-1) - \beta(n-1))$, we get another solution to $(*)$. This allows us to assume that $d = 1$, $0 \le f'(x) \le 1$ on $[n-1, n]$ and $f(n-1) = 0$. Denote $f(n)$ by c, so that $0 < c < 1$. For $x \in [n-1, n]$, the assumptions that $f(n-1) = 0$, $f(n) = c$, and $0 \le f'(x) \le 1$ imply that: (1) $f(x) \le \min(x - n + 1, c)$ and (2) $f(x) \ge \max(0, x - n + c)$. It follows from inequality (1) and from $(*)$ that, for $n \le t \le n + c$, $f'(t) \ge f(t) - t + n$, from which we find by multiplying by $e^{-(t-n)}$ that $(d/dt)(e^{-(t-n)}f(t)) \ge -(t-n)e^{-(t-n)}$. Integrating the latter inequality from n to x gives $f(x) \ge (c-1)e^{x-n} + x - n + 1$. For $n + c \le x \le n + 1$, we have $f'(x) \ge f(x) - c$ and the same kind of argument (using the estimate $f(n+c) \ge (c-1)e^c + c + 1$) gives $f(x) \ge (c-1)e^{x-n} + e^{x-n-c} + c$. Combining these lower bounds for f on $[n, n+1]$ with the upper bounds on $[n-1, n]$, we get $f'(x) = f(x) - f(x-1) \ge (c-1)e^{x-n} + 1$ for $n \le x \le n + c$ and $f'(x) \ge (c-1)e^{x-n} + e^{x-n-c}$ for $n + c \le x \le n + 1$. The overall minimum given by these inequalities is $f'(x) \ge (c-1)e^c + 1 \ge 0$. Applying the same sort of reasoning to $f(x) \ge \max(0, x - n + c)$ gives $f'(x) \le$

$ce^{1-c} \leq 1$. Since $ce^{1-c} - (c-1)e^c - 1$ takes a maximum of $\sqrt{e} - 1$ at $c = 1/2$, the result follows. Explicit inequalities establishing the inclusion of ranges of $f'(x)$ were obtained in the proof. This inclusion also follows from the fact that $(*)$ may be interpreted, via the mean value theorem, as implying that for each t there is a u with $t - 1 < u < t$ such that $f'(t) = f'(u)$.

Corollary. *An averaging function is asymptotically linear.*

Proof: By the theorem, the intervals of values of f' on the successive intervals $[n, n+1]$ form a nested sequence of intervals whose lengths approach zero. They therefore have a unique intersection point a. Furthermore, $f(x) - ax$ has derivative converging geometrically to zero. By the Cauchy convergence criterion, $f(x) - ax$ has a limit b as x goes to infinity.

Editorial comment. A related proposal was submitted independently by Richard Parris, Phillips Exeter Academy, Exeter, NH, while this problem was in press. Both proposers pointed out the probabilistic origin of the problem: when real numbers are drawn at random from the interval $[0, 1]$ until their sum is at least x, the expected number of drawings is $E(x)$.

Seung-Jin Bang observed that this problem appeared as problem 1190 in Crux Mathematicorum. The solution appears in vol. 14 (1988), no. 2, pp. 53–55.

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2. Jack K. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.

Solved also by D. Borwein (Canada) and R. Richberg (Germany).

Special Sequences of Real Numbers

E 3433 [1991, 365]. *Proposed by Tatsuhiko Aoyagi, Ōhori High School, Fukuoka, Japan.*

Find all sequences $\{a_n\}_{n=1}^{\infty}$ of real numbers satisfying the two conditions

$$\{1 - (n+1)a_{n+1}\} \prod_{j=1}^{\infty} (2 - ja_j) = a_1 \quad \text{for} \quad n = 1, 2, \ldots, \tag{1}$$

and

$$\sum_{n=1}^{\infty} a_n a_{n+1} = a_1. \tag{2}$$

Solution by Robin J. Chapman, University of Exeter, Exeter, U.K. Each such sequence has the form $a_1 = 2/(c+2)$ and $a_n = (c + n - 2)/n(c + n - 1)$, for $n \geq 2$, where c is a solution of $\Sigma_{n=1}^{\infty}[(c+n)n(n+1)]^{-1} = 1/4$. The equation for c has infinitely many solutions. One of these is $c = 2$, yielding $a_n = 1/(n+1)$. There is also, for each positive integer m, a solution c_m with $-m - 1 < c_m < -m$. These are the only solutions.

For convenience, put $b_n = na_n$. If $a_1 = 0$, then applying (1) repeatedly gives $b_n = 1$ for all $n \geq 2$, which implies $\Sigma_n a_n a_{n+1} > 0$ and contradicts (2). Hence we may assume $a_1 \neq 0$. It follows that $b_n \neq 2$ for all n and that $b_n \neq 1$ for $n \geq 2$.

Comparison of consecutive instances of (1) yields $(1 - b_{n+2})(2 - b_{n+1}) = 1 - b_{n+1}$ for all n, or

$$\frac{1}{1 - b_{n+2}} = 1 + \frac{1}{1 - b_{n+1}}.$$

This implies that there is a fixed constant $c = 1/(1 - b_2) - 1$ such that $1/(1 - b_n) = c + n - 1$ for all $n \geq 2$. Clearly c is not a negative integer, and $b_n = (n + c - 2)/(n + c - 1)$, as claimed. Also, (1) requires $(1 - b_2)(2 - b_1) = b_1$, which implies $b_1 = 2/(c + 2)$.

From (2), we now obtain

$$\frac{2}{c+2} = \frac{c}{(c+1)(c+2)} + \sum_{n=2}^{\infty} \frac{c + n - 2}{(c+n)n(n+1)}.$$

Since $\Sigma 1/[n(n+1)]$ "telescopes," a careful rearrangement yields

$$\frac{2}{c+2} = \frac{2}{(c+2)} + \frac{1}{2} - 2\sum_{n=1}^{\infty} \frac{1}{(c+n)n(n+1)},$$

which implies the claimed condition $\Sigma_{n=1}^{\infty}[(c+n)n(n+1)]^{-1} = 1/4$. The desired sequences are those generated by solutions c to this equation.

Note that $f(c) = \Sigma_{n=1}^{\infty}[(c+n)n(n+1)]^{-1}$ is a strictly decreasing function of c on every interval on which it is defined. Also, $f(c)$ tends to $+\infty$ as c approaches a negative integer from above, and $f(c)$ tends to $-\infty$ as c approaches a negative integer from below. By evaluating the telescoping sum $\Sigma_{n=1}^{\infty}[n(n+1)(n+2)]^{-1}$, we see that $f(c) = 1/4$ has the solution $c = 2$ on the interval $(-1, \infty)$. By the divergence of f as c approaches negative integers, it is clear that for each positive integer m there is a unique $c_m \in (-m-1, -m)$ such that $f(c_m) = 1/4$. Putting $c = c_m$ gives a solution where a_{m+2} is negative. Hence $a_n = 1/(n+1)$ is the only solution in positive reals.

Editorial comment. Most of the incomplete solutions found only the solution for which all terms are positive. The original proposal contained the additional condition that the sequence is monotone, which eliminates all but this solution. The incomplete solutions made various assumptions that have the same effect.

Solved also by J. Anglesio (France), S.-J. Bang (Korea), D. Callan, I. I. Kotlarski, O. P. Lossers (The Netherlands), and the Western Maryland College Problems group. Six incomplete solutions were received.

Theater Patrons in a Row

E 3435 [1991, 365]. *Proposed by Charles Vanden Eynden, Illinois State University, Normal, IL.*

An usher seats n patrons, one at a time, in the first row of a theater with n very narrow chairs. Whenever a new patron is seated, anyone in a chair adjacent to his must briefly stand, as well as those in chairs adjacent to those who stand, and so on. For example if $n = 5$ the usher might start by seating people in chairs 1, 3, 5. If he then fills chair 2, the patrons in chairs 1 and 3 must arise and sit down again. The last patron must be assigned chair 4, and the four previous patrons will have to arise and sit down again. The usher would like to seat people so as to minimize the total number of times someone sits down, which in our example is 1 + 1 +

$1 + 3 + 5 = 11$. Let $f(n)$ be the minimum total number of times someone sits down in filling the row. For example, $f(4) = 8$ and $f(5) = 11$. Find $f(100)$.

Solution by Gerry Myerson, Macquarie University, Sydney, NSW, Australia. We will show that $f(100) = 580$ and more generally that $f(n) = (n+1)k - 2^k + 1$ for all n, where $k = k(n)$ is the smallest integer such that 2^k exceeds n.

Considering the situation as the last patron arrives, we see that

$$f(n) = n + \min_{0 \le r < n} [f(r) + f(n-1-r)].$$

Together with the initial value $f(0) = 0$, this recurrence determines f. It thus suffices to show that the function $g(n) = (n+1)k - 2^k + 1$ satisfies the recurrence. The piecewise linear function agreeing with g on the whole numbers is convex and, thus, lies below its chords. Therefore

$$\min_{0 \le r < n} (g(r) + g(n-1-r)) = \begin{cases} g\left(\dfrac{n-1}{2}\right) & \text{if } n \text{ is odd;} \\ g\left(\dfrac{n}{2}\right) + g(-1)\left(\dfrac{n-2}{2}\right) & \text{if } n \text{ is even.} \end{cases}$$

The proof that $g(n)$ satisfies the recurrence follows in a straightforward manner from this formula. The case when n is a power of 2 should be distinguished from other even values of n.

Editorial comment. Robert High also studied two variants on this problem. In one, the usher seats patrons at random, and a similar recurrence may be used to show that the expected number of seatings is $O(n \log n)$. In the other, there are two competing ushers, "Minnie," who seeks to minimize the number of seatings, and "Max" who seeks to maximize the number. In this model, if Max is allowed to seat a positive fraction k of the patrons, the number of seatings will be greater than $C(k)n^2$. The solution to the recurrences arising in such problems has been studied in connection with "divide-and-conquer" algorithms in computer science. In particular, the solution of the recurrence of problem E3435 appears on page 539 of M. L. Fredman and D. E. Knuth, "Recurrence relations based on minimization," *J. Math. Anal. Appl.* 48 (1974), 534–559. A related recurrence relation is discussed in D. H. Greene and D. E. Knuth, *Mathematics for the analysis of algorithms*, Birkhäuser, (1982), section 2.2.1.

Solved by 37 readers. Two incorrect solutions were received.

Periodic Recursive Sequences

E 3437 [1991, 366]. *Proposed by Michael Golomb, Purdue University, W. Lafayette, IN.*

Given an integer ν greater than 1 and a monotonic finite sequence $\{a_0, a_1, \ldots, a_{\nu-1}\}$ of real numbers, define an infinite sequence $S = \{a_n\}_{n=0}^{\infty}$ by the recursion

$$a_{n+\nu} = \max\{a_{n+1}, a_{n+2}, \ldots, a_{n+\nu-1}, 0\} - a_n$$

for $n = 0, 1, 2, \ldots$. Prove that S is periodic and determine its period.

Solution by David Callan, University of Wisconsin, Madison, WI. The period is $3\nu - 1$, unless all terms are 0. Since the given recurrence defines a sequence forward and backward in symmetric fashion, we may assume without loss of generality that $a_0 \geq \cdots \geq a_{\nu-1}$. Suppose k of the given numbers are positive. Then $a_k, \ldots, a_{k+\nu-1}$ are nonpositive, because iteratively for $0 \leq i \leq k-1$, we have $a_{\nu+i} = \max\{a_{i+1}, 0\} - a_i \leq 0$. Let these nonpositive terms be $-b_1, \ldots, -b_\nu$. Since these are nonpositive, the next ν terms are the running sums $b_1, b_1 + b_2, \ldots, \Sigma b_i$. Now Σb_i is the maximum of these ν terms, and the next $\nu - 1$ terms remove the summands, from the beginning, yielding $\Sigma_{i=2}^{\nu} b_i, \ldots, b_\nu$. At this point, the maximum of these is $\Sigma_{i=2}^{\nu} b_i$, and we subtract $\Sigma_{i=1}^{\nu} b_i$ to obtain $-b_1$. The next $\nu - 1$ terms arise from similar subtractions, generating $-b_2, \ldots, -b_\nu$ and establishing the periodicity. From $-b_1$ to the next guaranteed appearance of $-b_1$ we saw ν nonpositive terms followed by $2\nu - 1$ nonnegative terms. Hence the only possibility for a period less than $3\nu - 1$ is that all the terms vanish.

Solved also by S.-J. Bang (Korea), R. S. Booth (Australia), R. J. Chapman (U.K.), J. Christopher, H. Lipman, O. P. Lossers (The Netherlands), S. Matz, M. D. Meyerson, A. Pedersen (Denmark), P. J. Zweir, the Central Michigan University Problem group, the National Security Agency Problems Group, the Western Maryland College Problems group, and the proposer.

When $a^3 + b^3 = c^3$

E 3438 [1991, 366]. *Proposed by Herbert Gülicher, Munster, Germany.*

Let $\Delta P_1P_2P_3$ have the longest side P_1P_2. For each of the six permutations $\begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$, let P_{ij} be the point on the ray $\overrightarrow{P_iP_j}$ such that $\angle P_kP_iP_{ij} = \angle P_iP_jP_k$. Let p_{ij} be the length of P_kP_{ij} and let p_i be the length of P_jP_k. Prove that

(i) $p_1^2 + p_2^2 = p_3^2$ if and only if $p_{12}/p_{13} + p_{21}/p_{23} = 1$;

(ii) $p_1^3 + p_2^3 = p_3^3$ if and only if $p_{31}/p_{13} + p_{32}/p_{23} = 1$.

Solution by László Zsilinszky, Nitra, Czechoslovakia. We show that for each permutation, $\Delta P_{ij}P_iP_k$ is similar to $\Delta P_iP_jP_k$. Indeed, $\angle P_kP_iP_{ij} = \angle P_iP_jP_k$, and $\angle P_iP_kP_j = \angle P_iP_kP_{ij}$ since P_{ij} is on the ray $\overrightarrow{P_kP_j}$. It follows that $P_kP_{ij}/P_kP_i = P_kP_i/P_kP_j$. That is, $p_{ij} = p_j^2/p_i$. Thus,

$$\frac{p_{12}}{p_{13}} + \frac{p_{21}}{p_{23}} = \frac{p_2^2/p_1}{p_3^2/p_1} + \frac{p_1^2/p_2}{p_3^2/p_2} = \frac{p_2^2 + p_1^2}{p_3^2}$$

and

$$\frac{p_{31}}{p_{13}} + \frac{p_{32}}{p_{23}} = \frac{p_1^2/p_3}{p_3^2/p_1} + \frac{p_2^2/p_3}{p_3^2/p_2} = \frac{p_1^3 + p_2^3}{p_3^3},$$

which yield the desired results.

Solved also by M. Dindos (Czechoslovakia), J. Fukuta (Japan), H. Lipman, O. P. Lossers (The Netherlands), G. W. Teck (student, U.K.), and the proposer.

Collaborating editors: *David F. Appleyard, Paul T. Bateman, Bruce C. Berndt, Duane M. Broline, Barry W. Brunson, Frank S. Cater, Gulbank D. Chakerian,*

Underwood Dudley, Gerald A. Edgar, Michael A. Filaseta, Ira M. Gessel, Richard A. Gibbs, Douglas A. Hensley, John R. Isbell, Mourad E. H. Ismail, Murray Klamkin, Daniel J. Kleitman, Frederick W. Luttmann, Frank B. Miles, Richard Pfiefer, Stephen L. Portnoy, J. O. Shallit, John Henry Steelman, Kenneth B. Stolarsky, Douglas B. Tyler, Daniel Ullman, Edward T. H. Wang, and William E. Watkins.

Answer to Picture Puzzle:

The description was a feeble pun on this Frenchman's German name: it is Laurent Schwartz, approximately forty years ago.

It is a perennial problem for mathematicians to explain to the public at large what makes mathematics worthwhile if not its practicality. It is like explaining to someone who has never heard music what a lovely melody is... Do let us try to teach the general public more of the sort of mathematics that they can use in everyday life, but let us not allow them to think—and certainly let us not slip into thinking—that this is an essential quality of mathematics.

There is a great cultural tradition to be preserved and enhanced. Each generation must learn the tradition anew. Let us take care not to educate a generation that will be deaf to the melodies that are the substance of our great mathematical culture.

—B. Chandler & H. M. Edwards

REVIEWS

Edited by **Darrell Haile**
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Old and New Unsolved Problems in Plane Geometry and Number Theory. By Victor Klee and Stan Wagon, MAA, 1991, xvi + 333 pp.

Reviewed by **P. R. Halmos**

Does every simple closed curve in the plane contain all four vertices of some square? Is there a box with integer sides such that the three face diagonals and the main diagonal all have integer lengths? Is π/e rational? What is the maximum possible area of a convex hexagon of diameter 1?

These four problems are typical of the 24 numbered problems that the book discusses—but there are many more than 24 problems altogether. Problem 11, for instance, is soon generalized to Problem 11.1, and there are also problems bearing numbers from 11.3 to 11.9. (I hunted, but I couldn't find Problem 11.2.)

"Problem" is possibly the most widespread slogan of the mathematical world nowadays (or the scientific world?, or all the world?). The motto of teachers who want to be with it is "don't tell 'em—ask 'em." Problem courses are burgeoning in colleges and problem books are sprouting in bookstores. I own two dozen problem books and they are nowhere near enough to give me a view over the field.

Most problem books are for learners—they teach by asking questions (and then often go on to cheat slightly by offering explanations and answers). One of the first, best, and most famous is Pólya and Szegő's *Aufgaben und Lehrsätze aus der Analysis*. It is an old book, but it is still alive and exciting and inspiring—it ought to be on every mathematician's desk (and, of course, at the top of every problemist's desk).

It's fun to search for, collect, or even to try to make up problems to give to students. It's much harder to find and much more dangerous to publish problems for the professionals—live research problems to which the answers are not known. To find a good student problem requires good taste—a rare quality, but we all think we have it. To find a good research problem requires creativity—an unusual quality that most of us are quite appropriately more modest about. Sure, any of us can modify an extant research problem and obtain a new one, and any of us can drop or modify some of the hypotheses from a theorem and ask whether or not the resulting statement is true—but it takes a rare kind of imagination and judgment to be led in that way to a problem of value. There are two big dangers in getting new problems from old questions and old statements: they can turn out to be either trivial or undoable.

Even great mathematicians can fail for a while to recognize a trivial problem as such, but the mathematical community as a whole is likely to be more perceptive than its individual members. If I published a problem and you solved it the next day, if I have overlooked the applicability of an algebraic theorem to an analytic question, but you didn't, and applying it you get a two-line solution, then I would

feel foolish for having posed a trivial problem, and I would wish I had kept it a secret.

By "undoable" I do not necessarily mean "unsolvable" in the technical sense of logicians. If I thought up a research problem that turned out later to be unsolvable in that sense, I wouldn't feel too bad. It is my religious belief that all "unsolvable" problems can be solved—possibly by using "illegal" techniques, possibly by refocusing the question, or possibly by reformulating what "answer" means. But even if a problem is not unsolvable in the logicians' sense, it can happen that it is not decently doable. It can happen that it has a solution that is—that must be—a mess, that the problem wants to split into 23 exhausting and exasperating subproblems, that the question is one whose answer doesn't add to our mathematical insight. When that happens, we have asked the wrong question—better we shouldn't have asked it at all.

The Klee-Wagon book is about unsolved problems, research problems. I don't know many books of that kind. The authors refer to three, and mention three others that are so far only fetal. The subjects of the problems are restricted to plane geometry and number theory, and the book is split into three chapters: *Two-dimensional Geometry*, *Number Theory*, and *Interesting Real Numbers*. (Exercise for the reader: to which of these chapters do the four sample problems belong?)

The authors begin by reminding us that problems can be simple and sophisticated. The simple ones (like Fermat's last theorem) ask questions such as "yes or no?", or "how many?"; the sophisticated ones might ask "How can such and such a theory or argument be extended so as to apply to a certain more general class of objects?" This book, they say, is "devoted exclusively to problems of the simple sort—ones whose statements are short and easy to understand." Yes, that's true, and, indeed, the first sentence of each of the 24 sections is one of the principal problems—and each of those 24 initial sentences ends with a question mark. But only a few of the problems are like the samples above—most of them require a sentence or two or a page or two of definitions, explanations, and discussion. The official prerequisites for reading the book are almost vanishingly small; words and phrases such as tiling, dense, collinear, Mersenne prime, and normal number are defined when they first occur, and complicated problems are formulated without the complicated technical terminology that usually accompanies them. So, for instance, the Riemann hypothesis is stated in terms of the integral $li(x)$, so that, in principle, every student who got a B or better in calculus can understand its statement.

Yes, the Riemann hypothesis—it is one of the 24 problems, and so are squaring the circle, Fermat's last theorem, and the existence of odd perfect numbers. These are the most famous problems in the book, the ones that all professional mathematicians and most students have at least heard about. There are, as the title promises, other old problems in the book too, and there are new ones, sometimes more esoteric, indicating the personal interests of the authors. The authors are helpful by being definite: they always state clearly which problems have not yet been solved and which subproblems have. The discussions of many of the problems are followed by theorems and exercises—this book can be read as well as used.

To give the reader a more detailed idea than the four initial samples can provide of the contents and the flavor of the book, I proceed to mention explicitly a few other problems.

Problem 3: when congruent disks are pushed closer together, can the area of their union increase?

That's not my line of country, and my reaction was that of an untutored foreigner—surprise, shock, worry, and the certainty that I must have completely misunderstood something. But no, Problem 3 asks exactly what it seems to ask, and it is followed one paragraph later by Problem 3.1: when congruent disks are pushed closer together, can the area of their intersection decrease?

The discussion of the problem begins, quite properly, by asking just what does it mean to speak of disks being pushed closer together. Does it mean mere repositioning of the centers in such a way that the distance between the centers of any two disks after the push is less than or equal to their distance before—or does it mean a continuous shrinking during which distances between centers never increase? Both interpretations are of interest, the authors tell us, but, we are warned, they may yield different answers. At the end of the section one of the exercises asks for an example of repositioning that cannot be obtained by continuous shrinking. Also, we are told that the problem is generalizable to balls in Euclidean spaces of all dimensions. The section has three theorems that say that the same answers are true (unions cannot increase and intersections cannot decrease) under certain restrictive conditions on the dimension of the space and on the number and size of the balls. I find that fascinating: even simple things can be complicated.

Problem 9 (squaring the circle): can a circle be decomposed into finitely many sets that can be rearranged to form a square?

This is, of course, not the classical Greek problem of squaring the circle; it is a modern measure-theoretic version, proposed by Tarski in 1925. One of the dangers of publishing research problems became dramatically realized for the authors in connection with this problem: they were going to describe it as unsolved, but, as the book was going to press, Laczkovich's solution appeared. (The answer is yes.) The section contains a pleasant discussion of matters related to the Banach-Tarski paradox, and I am glad it's there—and I extend my condolences to the authors.

Problem 13 (Fermat's last theorem): do there exist positive integers x, y, and z and an integer $n \geqq 3$ such that $x^n + y^n = z^n$?

Yes, every high school student knows about Fermat's last theorem, and so do many amateurs, but the section about it in this book is an interesting one (and ought to be compulsory reading for all those amateurs who continue to send us solutions). We learn that if x, y, and z form a counterexample to Fermat's last theorem with exponent n, then x^n has at least 10^{13} digits, and we learn (Problem 13.1) that this question is still unanswered: can five sixth powers sum to a sixth power? I am not sure I want to know the answer—but, as with many unsolved problems, what is exasperating is not that we don't know but that we don't know why we don't know. We also don't know (and we don't want to know) the millionth digit from the left of the decimal representation of $\pi^{1,000,000}$, but in principle we could find out, and, since we know why we don't know, the question doesn't bother us.

Problem 19: Is every positive integer eventually taken to the value 1 by the $3n + 1$ function?

This is a middling famous one—not like squaring the circle or the Riemann hypothesis, but it has made the rounds for several decades and has annoyed $3n + 1$ people for a large value of n. The domain of the function f in question is the set of positive integers; it maps n onto $n/2$ if n is even and onto $3n + 1$ if n is odd. Keep doing it—iterate the function—and ask whether you must always (no matter which n you started with) reach the number 1 sooner or later. If you do,

then you're stuck: if $n = 1$, then you get, one after another, $4, 2, 1, 4, 2, 1, \ldots$. The problem has annoyed people because they (we) think it is beautiful and interesting and, surely, we say to ourselves, it can't be that hard—but it has resisted all attempts at a general solution so far. The great authority Erdős is always quoted on the subject: "Mathematics is not yet ready for such problems." Annoying.

Problem 24: Is $1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \cdots$ irrational?

That's the last problem to be reported here—it is the last problem in the book. The answer to questions of this sort is known for even exponents (in place of 5); the answer is that the sum of the series for the exponent $2n$ is a rational multiple of π^{2n}. For $n = 2$, for instance, the problem can be solved by a bright calculus student; the well known answer is that the sum is equal to $\frac{\pi^2}{6}$. Odd exponents are harder. R. Apéry proved in 1978 that for the exponent 3 the sum is irrational, with a proof that made the experts unhappy—it was called "a mixture of miracles and mysteries." But, they seem to agree, the proof was a proof, and the statement is true.

Well, there it is. That should tell you something about a charming, friendly, interesting, and valuable contribution to your problem book shelf. I admire and applaud the authors' courage in undertaking to write it, and I congratulate them on their accomplishment.

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Problems for Mathematicians Young and Old. By Paul R. Halmos, Dolciani Mathematical Expositions No. 12, Mathematical Association of America, Washington, D.C., 1991, xviii + 318 pp., paperback.

Reviewed by **Stan Wagon**

Many people have written and spoken about the value of problems in a mathematics curriculum. And many have criticized problem-solving contests such as the venerable Putnam competition, arguing that the time limit and the focus on well-posed problems with well-defined answers give an unrealistic view of mathematics. Indeed, I have argued both sides, usually taking the second point of view after having spent a frustrating day alongside my students taking the Putnam. My view of the value of problem-solving took a big shift to the positive side recently when I took over Macalester's Problem of the Week, an extra-curricular tradition started by Joe Konhauser 25 years ago. I began the series with carefully chosen problems and was rewarded by a great amount of student interest. It is simply a fact that many students are intrigued by easily stated problems that they look upon as a challenge. They work on the problems, discuss them with other students and faculty, and feel satisfied when they solve them. Any faculty member should consider posting problems regularly, with some sort of reward for student solutions. Even a prize that consists only of prominent mention of solvers' names will

satisfy the students (but money helps, too). There are now many, many sources[1] for excellent problems. The book under review is a welcome addition to the area; it contains 165 problems with hints and complete solutions—problems that the author has found memorable for their shock value, their pedagogical value, or simply their inherent beauty.

What makes a problem interesting? Its statement should be simple, not requiring excessive explanations, and the solution should be readily understandable by the intended audience. If the result is a surprising one, so much the better. Thus, let me turn to Halmos's book by giving some of his problems that meet these criteria admirably and were new to me.

> At a party of five couples, no one shakes his or her own hand or the hand of his or her spouse. If the question, "How many hands did you shake?" elicits nine distinct integers among the ten answers, what is the missing number? (Problem 1H)
>
> Suppose people numbered $1, 2, \ldots, 1{,}000$, are seated in some order in chairs bearing the numbers from 1 to 1,000. Can they be reseated so as to preserve their circular order and with no person's number being the same as that of his or her chair? (1I)
>
> For which positive real numbers a is it true that $a^x \geqslant 1 + x$ for all real values of x? (2H)
>
> Which positive integers are sums of three or more consecutive positive integers? (3G)
>
> What is the shortest curve that bisects the area of an equilateral triangle? (5I)
>
> Are two triangles of the same area necessarily Cavalieri congruent? (5J)
>
> Is it possible to load a pair of dice so that the probability of the occurrence of each sum from 2 to 12 is the same as for honest dice? (7B)
>
> Is $\mathbb{R}^3$ a disjoint union of circles? (12G)

Several of these problems have surprising answers and I won't spoil your fun too much. But to whet your appetite for the book, here is the answer to problem 3G: all integers except for the primes and the powers of 2 (surprising, and surprisingly easy to prove). And the answer to the last question is YES.

Halmos has won several writing awards, and the reader won't be disappointed in the prose with which he wraps the problems. For example, after a detailed explanation of the solution to the aforementioned Cavalieri problem, we find:

> This is an astonishing result that seems to have gone unnoticed until its relatively recent discovery by Howard Eves. The proof here presented may appear verbose, and, indeed, proofs of the result can be given in many fewer words—but the result is subtle and, surely, the boredom that a few possibly unnecessary words induce is outweighed by the clarity they can achieve.

Who among us has not wished that other authors were as generous with their explanations?

[1]See, for example, *The Wohascum County Problem Book*, by George Gilbert, Mark Krusemeyer, and Loren Larson (to be published in the MAA's Dolciani series), which is an excellent source for undergraduate problems. If you would like to be added to the e-mail distribution list for my weekly problems contact me at *wagon@macalstr.edu*.

The book lives up to its title, which promises problems for both young and old. But is it the young or the old who are more likely to be impressed by the pretty and elegant elementary problems? I'm not sure. In any event, for the experienced mathematician or beginning graduate student seeking meatier fare the book contains several chapters with problems for the more mature reader. Examples: Can $\mathbb{R}$ be partitioned into four subsemigroups? Is $[0,1]$ a nontrivial Cartesian product? Is there a connected topological group in which every element is of order 2? Is there a finite group with an automorphism that maps exactly $4/5$ of the elements onto their own inverses? And an old chestnut that so impressed me in graduate school: Are the real numbers and the complex numbers isomorphic as additive groups? (This is given in slightly different form as problem 11D.)

Problem 8N asks whether the series of prime reciprocals diverges. One might argue that its inclusion is inappropriate because (a) it is too well known, and (b) an undergraduate student would not be able to solve it. But Halmos gives a most remarkable proof of divergence that is simpler than the well-known elementary proof using the series representation of $\log(1+x)$ (as presented in the classic book by Hardy and Wright, for example). The simple proof in Halmos's book is a clever and concise derivation based on the divergence of the harmonic series.

The book is not without its flaws. There are no references, and very few attributions. I have no complaint about the latter. It is often difficult to trace down the originator of a problem that has become folklore and, as Halmos states in his preface, "The beauty of the mathematics speaks for itself." But I do question the lack of references. Some of the problems are Putnam problems (4K and 10N, for example) or have appeared in Olympiad competitions. Some readers would find that information useful. Occasionally references are made to additional results, as on page 236: "the computation that proves this answer to be indeed optimal is rather cumbersome." Where can the interested reader pursue this? Here's another example for which I must admit to not being a disinterested observer. In problem 6K Halmos presents the notorious double-integral solution to a problem about tiling a rectangle: If a rectangle is tiled with rectangles, each of which has an integral side, then does the large rectangle necessarily have an integral side? The solution given is of historical interest, but it is by no means the best solution. Halmos observes that "ingenious as it may be, the [double-integral] solution is far from the only one." But shouldn't the reader have been directed to the paper (this MONTHLY, 94 (1987) 601–17) where a dozen other proofs may be found?

Here are some minor quibbles: There is no index, which is an inconvenience, although the Table of Contents is an adequate substitute. Some problems seem out of place, such as problem 8P, which asks whether $\Sigma(\pm 1/n)$ can add up to e. This result and its proof are well known to calculus students and are thus inappropriate in a chapter that deals with entire functions and Césaro continuity. And "Is the plane a union of countably many lines?" seems unsatisfactory since its solution depends too heavily on how much one knows. But these points are indeed minor. The book has already given me many hours of enjoyment, and I look forward to posting some of these problems over the next few years so that my students too can benefit from Paul Halmos's good taste and lucid explanations.

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TELEGRAPHIC REVIEWS

Edited by
Arnold Ostebee and Paul Zorn
with the assistance of
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Books and software submitted for review should be sent to *Book Reviews Editor, American Mathematical Monthly, St. Olaf College, Northfield, Minnesota 55057.*

Algebra, P. *Representations of Finite Dimensional Algebras.* Eds: H. Tachikawa, V. Dlab. Canadian Math. Soc. Conf. Proc., V. 11. AMS, 1991, xxii + 322 pp, $92 (P). [ISBN: 0-8218-6016-X] Partial proceedings of the Fifth International Conference on representations of finite dimensional algebras, this one held in Japan in 1990. SG

Real Analysis, T*(14-15: 2), L. *A First Course in Real Analysis, Second Edition.* Murray H. Protter, Charles B. Morrey, Jr. Undergrad. Texts in Math. Springer-Verlag, 1991, xviii + 534 pp, $39.95. [ISBN: 0-387-97437-7] A revision of the popular and successful 1977 text (TR, August-September 1977). Includes many new exercises, sharper figures, and smoother exposition. An excellent book just got better. TAV

Complex Analysis, T(18), S, P, L. *Harmonic Analysis.* Henry Helson. Wadsworth, 1991, xii + 190 pp, $24.95 (P). [ISBN: 0-534-15570-7] Reprint of the 1983 Addison-Wesley edition. Condensed presentation of the fundamentals of classical harmonic analysis. Brisk, insightful exposition hits high points, leaves some details to problem sections at end of each chapter. Small but useful bibliography. BH

Partial Differential Equations, P. *Nonlinear Methods in Riemannian and Kählerian Geometry, Revised 2nd Edition.* Jürgen Jost. DMV Seminar, Band 10. Birkhäuser, 1991, 154 pp, $34.50 (P). [ISBN: 0-8176-2685-9] Lies in the intersection of geometry and partial differential equations. After a brief presentation of some necessary background in geometry and analysis, this monograph offers a thorough study of harmonic maps and Yang–Mills equations. Concludes with geometric applications of harmonic maps. OJ

Dynamical Systems, P. *Fractal Geometry and Analysis.* Eds: Jacques Bélair, Serge Dubuc. NATO ASI Ser. C, V. 346. Kluwer Academic, 1991, xv + 472 pp, $129. [ISBN: 0-7923-1399-2] Ten papers, eight in English, two in French, presented at the meeting in Montréal in July 1989. SP

Dynamical Systems, T(16-17), L. *Dynamics and Bifurcations.* Jack K. Hale, Hüseyin Koçak. Texts in Appl. Math., V. 3. Springer-Verlag, 1991, xiv + 568 pp, $49. [ISBN: 0-387-97141-6] Geometry of dynamics and bifurcations of ordinary differential and difference equations. Text and exercises contain equations of theoretical and practical interest. MLR

Dynamical Systems, T(18), P. *Viability Theory.* Jean-Pierre Aubin. Syst. & Control: Found. & Applic. Birkhäuser, 1991, xxv + 543 pp, $94.50. [ISBN: 0-8176-3571-8] Mathematical theory providing metaphors of the dynamic evolution of complex systems (e.g., nonlinear control systems, biological and social systems). Stress on three main features: nondeterministic evolution, viability constraints the systems must obey to "live," inertia princi-

ple (under which the controls of the system are constant as long as the viability of the system is not at stake). Uses theory of differential inclusions, set-valued analysis, Lyapunov functions, differential games. RM

Dynamical Systems, P. *Foundations of Synergetics II: Complex Patterns.* A.S. Mikhailov, A. Yu. Loskutov. Ser. in Synergetics, V. 52. Springer-Verlag, 1991, viii + 210 pp, $79. [ISBN: 0-387-53448-2] Informal treatment of chaotic patterns that arise in distributed active systems. Topics include strange attractors, fractals, discrete maps, and spatio-temporal chaos. SP

Operator Theory, T(18), S, P. *One-Dimensional Linear Singular Integral Equations, I: Introduction.* Israel Gohberg, Naum Krupnik. Oper. Theory: Adv. & Applic., V. 53. Birkhäuser, 1992, 266 pp, $95. [ISBN: 0-8176-2584-4] English translation of the text which appeared in 1973 in Russian and in 1979 in German. Contains many "changes and addenda" from the original. Topics include "boundedness of singular integral operators in different function spaces, invertibility of such operators and methods for their inversion, and the Noether–Fredholm theory." Exercises at end of each chapter. BH

Analysis, P. *Lecture Notes in Mathematics-1438: Les Ondelettes en 1989.* Ed: P.G. Lemarié. Springer-Verlag, 1990, 212 pp, $22 (P). [ISBN: 0-387-52932-2] "Summaries [in French] of nine conferences [on wavelet theory] held at the Séminaire d'Analyse Harmonique d'Orsay in early 1989." Topics include general introduction to wavelet theory and wavelet orthonormal bases; applications to operator theory, computer vision, signal processing, and fractals. Short English summaries of each conference at end of volume. BH

Differential Geometry, T(18: 2), L. *Modern Geometry—Methods and Applications, Part I: The Geometry of Surfaces, Transformation Groups, and Fields, Second Edition.* B.A. Dubrovin, A.T. Fomenko, S.P. Novikov. Transl: Robert G. Burns. Grad. Texts in Math., V. 93. Springer-Verlag, 1992, xv + 468 pp, $59.80. [ISBN: 0-387-97663-9] Intended to "serve as a basic text from which the essentials for a course in modern geometry may be easily extracted." This edition differs minimally from the *First Edition* (TR, January 1985). JO

Differential Geometry, P. *Models for Smooth Infinitesimal Analysis.* Ieke Moerdijk, Gonzalo E. Reyes. Springer-Verlag, 1991, x + 399 pp, $79. [ISBN: 0-387-97489-X] A development of synthetic differential geometry relating it to classical differential geometry using a categorical sheaf-theoretic treatment of infinitesimals. JAS

Differential Geometry, T(17-18: 1), S.** *Riemannian Geometry.* Manfredo Perdigão do Carmo. Transl: Francis Flaherty. Math.: Theory & Applic. Birkhäuser, 1992, xiii + 300 pp, $39.50. [ISBN: 0-8176-3490-8] Well-written text for a first graduate course in differential geometry. First half covers basic concepts: metrics, connections, geodesics, and curvature. Second half emphasizes calculus of variations techniques to study global questions. Each chapter begins with historical background and an overview. Avoids differential forms. Exercises, bibliography. OJ

Control Theory, T(17), P. *Controllability of Dynamical Systems.* Jerzy Klamka. Math. & Its Applic., V. 48. Kluwer Academic, 1991, xvi + 248 pp, $114. [ISBN: 0-7923-0822-0] Treatment of different kinds of controllability for linear dynamical systems: finite-dimensional continuous-time, finite-dimensional discrete-time, and infinite-dimensional continuous-time systems and dynamical systems with delays. SP

Statistical Methods, P, L. *Truncated and Censored Samples: Theory and Applications.* A. Clifford Cohen. Stat.: Textbooks & Mono., V. 119. Marcel Dekker, 1991, xiv + 312 pp, $99.75. [ISBN: 0-8247-8447-2] Intended as a handbook for practitioners who need simple and efficient methods for the analysis of incomplete (truncated, censored) data. Mostly concerns estimation (and associated sampling error) for a wide variety of continuous and discrete univariate probability models, as well as a handful of sampling and truncation/censoring schemes. Numerous, though brief, examples and lots of results with little or no derivations. MK

Statistical Methods, S(17), P*. *Biopharmaceutical Statistics for Drug Development.* Ed: Karl E. Peace. Stat.: Textbooks & Mono., V. 86. Marcel Dekker, 1988, xii + 640 pp, $137. [ISBN: 0-8247-7798-0] Coherent presentation by practitioners and other experts of the statistical aspects of the entire process of pharmaceutical human drug development: discovery

of the new chemical; *in vitro* and animal studies; clinical trials on humans; assessment of safety; manufacturing and quality control. RSK

Statistical Methods, P. *ANOVA: Repeated Measures.* Ellen R. Girden. Quantit. Applic. in the Soc. Sci., V. 84. Sage Publ, 1992, vi + 77 pp, $8.50 (P). [ISBN: 0-8039-4257-5] One-, two-, and three-factor analysis of variance (ANOVA) in which subjects may undergo more than one treatment. Discusses situations appropriate for such repeated measures. Gives sum of squares partitions and the associated (quasi) F ratio distributions with careful attention to underlying assumptions. Illustrated with studies in the social sciences. RWJ

Statistics, T(17-18: 2), P. *Model Assisted Survey Sampling.* Carl-Erick Särndal, Bengt Swensson, Jan Wretman. Ser. in Stat. Springer-Verlag, 1992, xv + 694 pp, $49. [ISBN: 0-387-97528-4] Develops survey sampling ideas from perspective of unequal probability sampling. Model-assisted approach clarifies use of auxiliary information. Includes recent developments in survey data analysis, domain estimation, variance estimation, nonresponse methods, and measurement error models. Text assumes background in statistical inference and linear models. RWJ

Computer Systems, P, L. *The Simple Book: An Introduction to Management of TCP/IP-based Internets.* Marshall T. Rose. Ser. in Innovative Tech. Prentice Hall, 1991, xxix + 347 pp. [ISBN: 0-13-812611-9] A readable survey of ideas and opinions about management of internets built on the Internet suite of protocols. JAS

Computer Systems, T(17-18), P, L. *Internetworking With TCP/IP, Volume I: Principles, Protocols, and Architecture, Second Edition.* Douglas E. Comer. Prentice Hall, 1991, xxiii + 547 pp. [ISBN: 0-13-468505-9] A thorough treatment "for the uninitiated" of the TCP/IP protocols and the management of interconnected networks. JAS

Computer Systems, T(17-18), P. *Object-Oriented Analysis, Second Edition.* Peter Coad, Edward Yourdon. Comput. Ser. Yourdon Pr (US Distr: Prentice Hall), 1991, xiv + 233 pp. [ISBN: 0-13-629981-4] A study of tools available for object-oriented programming including both methods of object-oriented analysis and management and CASE tools for dealing with object-oriented design. JAS

Computer Graphics, T(15-16: 1, 2), L. *The Geometry of Computer Graphics.* Walter F. Taylor. Wadsworth, 1992, xvi + 451 pp, $58.95. [ISBN: 0-534-17100-1] Text provides a marvelous way to solidify students' understanding of linear algebra, introduce projective and analytic geometry, and promote exploration of the usually under-taught geometric aspects of linear algebra. Most appropriate for a computer graphics course, but also appropriate for a geometry course for ambitious, computer-literate mathematics students. JO

Applications (Engineering), S(17), P. *Nonlinear Stability and Bifurcation Theory: An Introduction for Engineers and Applied Scientists.* Hans Troger, Alois Steindl. Springer-Verlag, 1991, xi + 407 pp, $89 (P). [ISBN: 0-387-82292-5] Introduction to the bifurcation theory approach to the loss of stability of nonlinear systems. SP

Applications (Engineering), P. *Mathematical Models in Electrical Circuits: Theory and Applications.* C.A. Marinov, P. Neittaanmäki. Math. & Its Applic., V. 66. Kluwer Academic, 1991, x + 160 pp, $66.50. [ISBN: 0-7923-1155-8] The author's research on models for electrical and electronic circuits. Results on nonlinear circuits with lumped parameters, bipolar transistors, and MOS circuits, focused on asymptotics and delay time. SK

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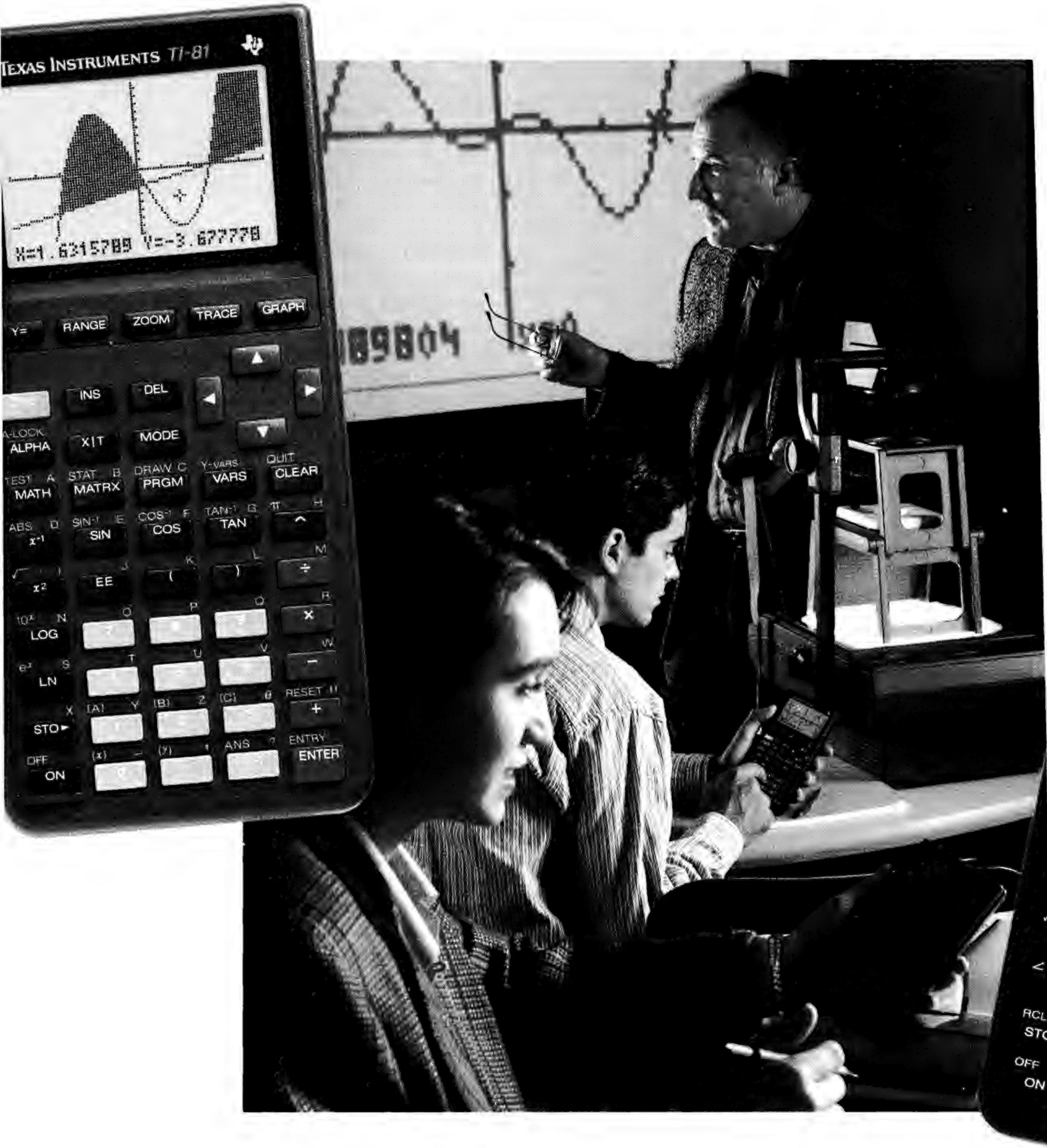

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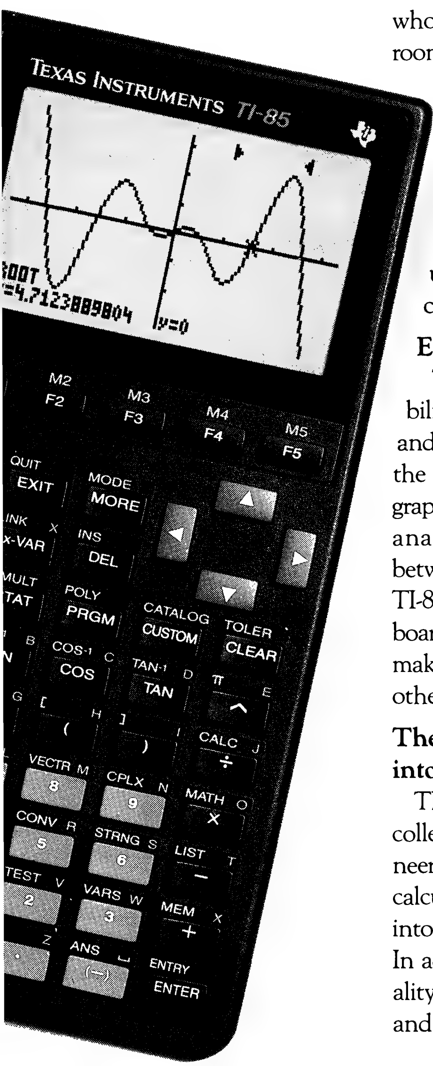

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exhaustive searches take too long

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Robert D. Grisso
University of Nebraska

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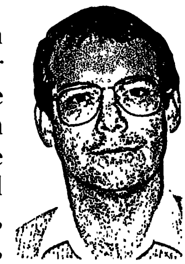

David D. Jones
University of Nebraska

Grisso and Jones developed the technique on a drive from Nebraska to a conference in Chicago. They were finishing their presentation on a Zenith Data Systems SupersPort 286 laptop.

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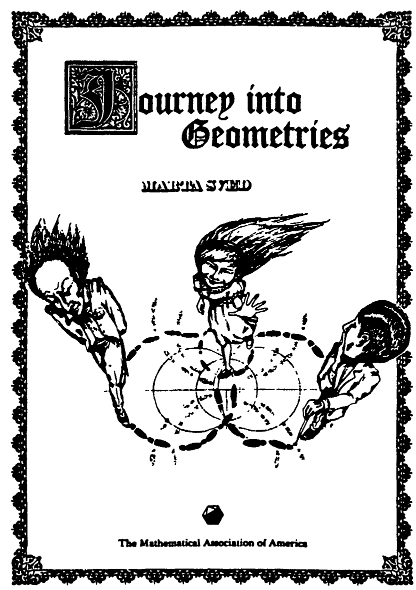

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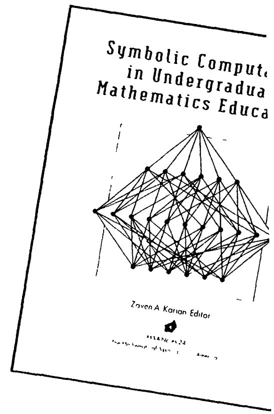

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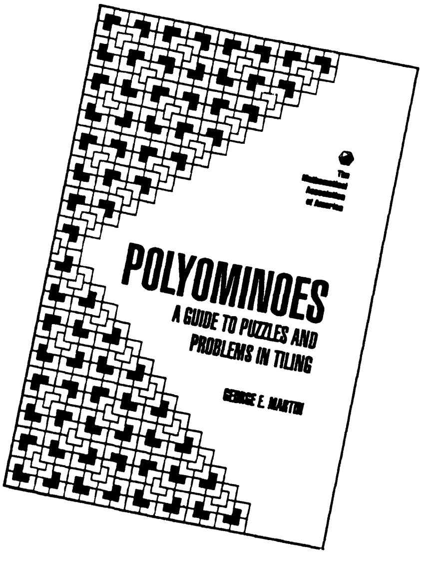

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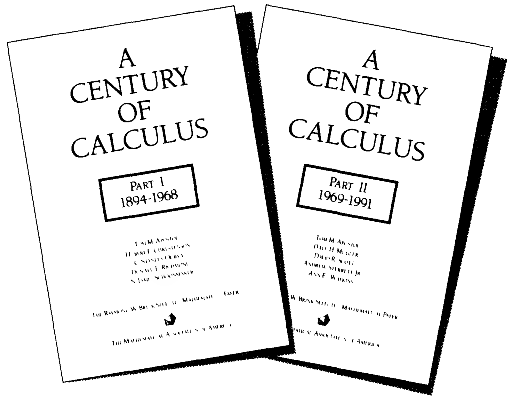

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REVIEWS BY AUTHOR

Names of authors are in ordinary type; those of reviewers in capitals.

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PROBLEMS PROPOSED

PROBLEMS SOLVED

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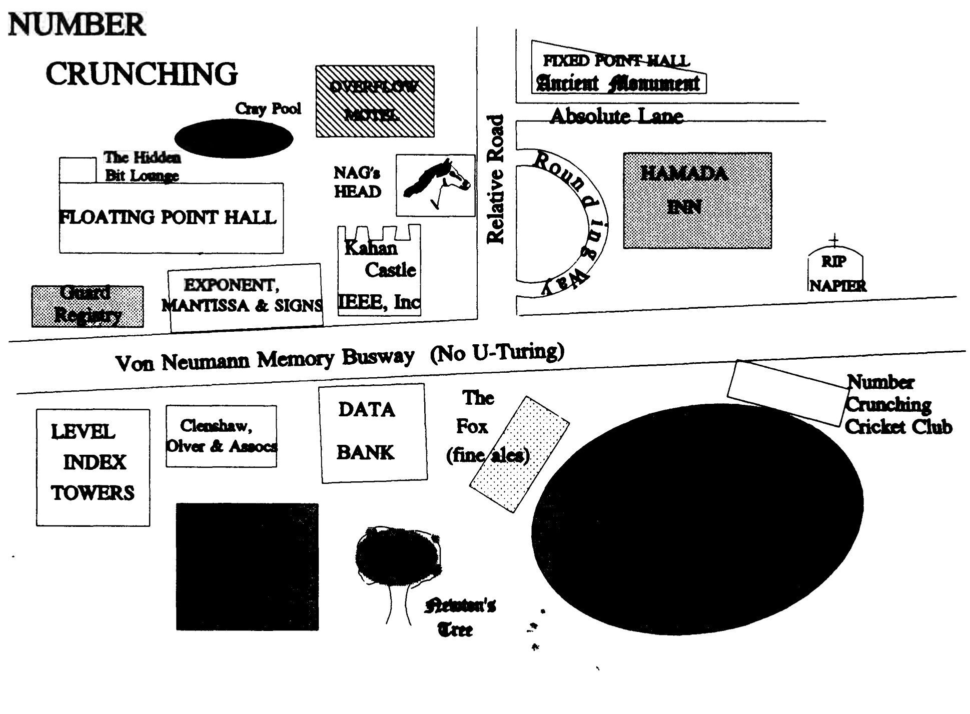
Volume 99, Number 10 / DECEMBER 1992



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The American
Mathematical Monthly

Volume 99, Number 10 / DECEMBER 1992
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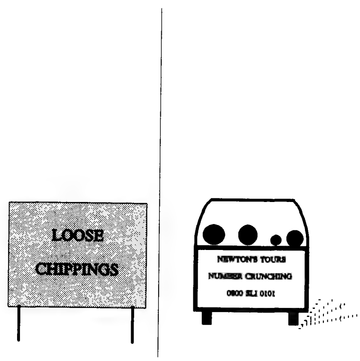
It's no fun turning down nearly 1000 papers each year, many of which have a great deal of merit. Most of those papers are not wrong, nor are they badly written, nor are they uninteresting. They simply do not fit into the *Monthly*, given our space limitations and the need to balance material for style and content. Reaching that conclusion is painful, but making tough decisions is what editors (responsible ones) have to do.

—John Ewing

A History of the Lords of Number–Crunching

Peter R. Turner

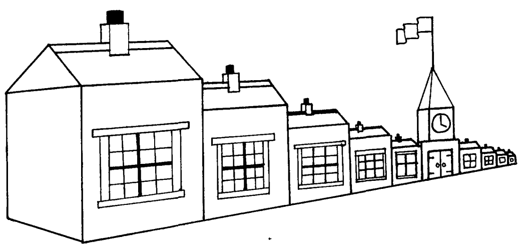
Some years ago I heard the tale of the American visitor to England who, on his travels through the countryside, had passed through the villages of Chipping Sudbury and Chipping Norton. Shortly afterwards, it being the season for road repairs, he and his native guide had passed a sign reading Loose Chippings at which the colonial had commented on the quaintness of the village names in that part of the country. At that very moment he was, would he had known, within a mere stone's throw of the ancient settlement of Number Crunching.



It is within this particular hamlet that the Lords of that manor still reside. Amongst the relics to be found there are the bones of Napier; while on the village green there still grows an apple tree germinated from the seeds of the very apple which struck Isaac Newton's head as he lay snoozing one summer afternoon. He was a frequent visitor to the ancestral home of the Lords of Number Crunching and is even believed to be the father of one of the most flourishing branches of the family. Since those times most British scientists (and many from overseas) have joined the almost continual stream of pilgrims visiting the hallowed halls in and around the village. Many have even found helpful assistants among the local population of number crunchers as the residents are affectionately known.

One of the fascinating aspects of the history of the Lords of Number Crunching has been in the architectural developments of their dwellings. Their Lordships have been particularly successful at adapting themselves to changes in society and to the growing need for suitable accommodation for all the branches of the family and their servants from the most significant down to the those who have only bit-parts in the drama of life. As we shall see, communication considerations also had major effects on both the designs and the life-styles of the residents.

In primeval days, the Lords of Number Crunching, their families and servants lived in long narrow single-storey houses with just one doorway. One of the best surviving examples of these dwellings is Fixed-Point Hall which is illustrated below. The name of the hall is derived from the fact that all the artisans and laborers employed by their lordships in those far-off days belonged to the ancient tribe known as Fixed-Point Man. Not all the fixed-point houses were of the same dimensions, nor did they all have the doorway—coincidentally the *fixed-point* from which all the rooms were reached—in the same place. Much depended on the wealth of the resident family and on the number of guests they wished to be able to accommodate. What was consistent throughout this period of Number Crunching architecture was that the rooms to the left of the entrance were larger than those to the right and indeed they became significantly greater as you moved further and further toward the left-hand end of the building. It is similarly true that as you walk down to the right in Fixed-Point Hall you feel that the occupants of each new room must have been less significant than those of the preceding ones.



Fixed-Point Hall

Archaeologists have found that in most of these buildings rooms were approximately half the size of their left-hand neighbor (and, therefore, twice that of the right-hand one). Interestingly though, there is evidence that some of the families used a system other than this *binary architecture* to distinguish between the relative importance of the rooms. Certainly there was a *Decimal School of Architecture* in Number Crunching for some years. Their designs were characterized by a ratio of one tenth between the successive rooms. This particular school had a liking for many classical features, as is evidenced by the remains of Gothic arches which have been found among their ruins.

As the years passed, many of the families' needs changed. Some fell on hard times, no longer could afford the servants and found the design of the house very inconvenient. The simplest of meals had to be prepared way down in the kitchen maid's quarters which were, naturally, well-removed from the main family apartments. Consequently it became a common sight to see something of no great significance in itself being carried right through the building up to the master of the house in the grand apartments at the left-hand end.

But the difficulty caused by the frequency of this carry operation was as nothing by comparison with the problems faced by their Lordships' family over the generations. It seemed that no sooner was the young master old enough to be out scrumping the apples from Newton's tree and taking the occasional byte of the fruit of the tree of numbers than there was another branch of the family to be accommodated in Fixed-Point Hall.

Frequently the family became so large as to overflow the available accommodation. Many an extension was built; but still with just one door and the same long

Floating-Point Arithmetic

In the *binary* floating-point system, a real number X is represented in the computer by

$$X = \pm f \times 2^E \tag{1}$$

where f is called the *fraction*, *mantissa* or *significand* and the integer E is called the *exponent* or *characteristic*. (The details of the electronic representation are not important here.)

The representation (1) is called *normalized* if the fraction f lies in the interval $[1/2, 1)$. (Other normalizations may use $f \in [1, 2)$.) In normalized binary floating-point representations, the first bit of the fraction is necessarily a 1 and so need not be stored explicitly. This is the so-called *hidden* or *implicit* bit. See [**5**] for more details.

The principal failings of the floating-point system arise from its finite range. There is a maximum value, $E_{\max}$ for the exponent. Multiplication of two floating-point numbers will result in *overflow* whenever the exponent of the result exceeds $E_{\max}$ which will certainly result if the sum of the exponents of the factors exceeds $E_{\max} + 1$. Similarly, there is a minimum exponent and a corresponding smallest positive representable number. Arithmetic operations which result in smaller values are said to underflow. In the IEEE standard, unnormalized representations are permitted with the minimum exponent so that underflow is *gradual*. While this alleviates the problem it does not eliminate it.

Underflow and overflow are serious problems for scientific computing and frequently result in the need to scale the variables to avoid this problem. Of course, such rescaling can only be performed if the problem is recognized. The more serious situation arises from computations which produce reasonable-looking-but-wrong output which may not even be identified as suspect. This can happen, for example, as a result of underflow in contour plotting where there will often be no simple scaling that could be used satisfactorily—even if the problem is identified.

Another problem is *catastrophic cancellation*. This results when two nearly equal numbers are subtracted. The normalized result has a significantly smaller exponent than either of the operands so that a large normalization shift is needed before storing the result. However, there is no useful information which can be shifted into the lower order bits of this result which thus has less precision. Loss of precision due to normalization (or pre-arithmetic alignment) shifts is alleviated by including one (or more) *guard bits* in the accumulator.

linear design being preserved. On other occasions the Lord decided to rescale the accommodation so that it could cope with the much larger numbers that were now to be housed. This was a major undertaking since it required a reduction in the size of every room in order to preserve the correct proportional importance for the various members of the household. The effect of this on the servants' quarters were severe. So much so that the senior of them, the butler Juan Halfsmith by name, was eventually obliged to complain to his master when it reached the point at which the least significant scullery maid could neither lie nor stand straight in her tiny bit of a room. Even the housekeeper, the formidable Mrs. Quarterstaff, had difficulty in arranging her (not inconsiderable) person within the confines of her room. The whole reputation and integrity of the Lords of Number Crunching were threatened if this state of affairs were to be revealed outside the village. Something had to be done.

Things had reached this pretty pass despite the fact that, years earlier, many of the second sons and daughters of successive Lords of the Manor decided that Fixed-Point Hall had been rescaled too much for comfort. Several of them had even moved into neighboring villages. This turned out to be the salvation of the Lords of Number Crunching.

One of the young Number Crunchers had in fact taken for his wife a sweet girl from the next county. In that part of the land architects were of a different race whose origins lay in a violent separation of two branches of the old tribe of fixed-point man. This people is still known as Floating-Point Man. Towards the end of the primeval age, the overcrowding in Fixed-Point Hall—which by then was forever overflowing—enabled a particularly virulent strain of influenza to spread so rapidly and successfully throughout the population of the Hall and thence the village that the title passed out of the village to this young distant cousin in the next village. He returned to Number Crunching to assume his Lordly duties and was appalled at the state of the hall and all he found. The old hall was preserved for posterity but the new Lord of Number Crunching determined that his family would not suffer its antiquated living conditions. To honor the tribal origins of the architects, *Exponent*, *Mantissa and Signs*, the new hall was named Floating-Point Hall.

This much grander edifice is still in use today although there are rumblings of discontent from some branches of the family that the accommodation is again inadequate for some of the enormous accumulations which take place there from time to time. Of course there was vehement opposition to the design of Floating-Point Hall in the early days from traditionalists who believed that the old ways were still the right ones. And, indeed, similarly reactionary forces are at work today trying to defend the "floating-point" design from the inevitable. They have heard the folktales of the rescaling that went on in Fixed-Point Hall and advocate similar solutions for the preservation of the present structure in order to overcome the occasional but persistent overflow problems. But there is much to recount before we reach the current family feuding.

The architects' plans for one of the early designs of Floating-Point Hall were recently uncovered. The front elevation is reproduced below—with the kind permission of Exponent, Mantissa and Signs—and shows all the essential features of the modern design. The important apartments for His Lordships' family are on the upper storey while the lesser rooms are at ground level.

The state rooms were of what can only be described as exponential splendor as befitted a family of such great and growing importance. The rooms downstairs were a mere fraction of the size and were much more modestly appointed. These

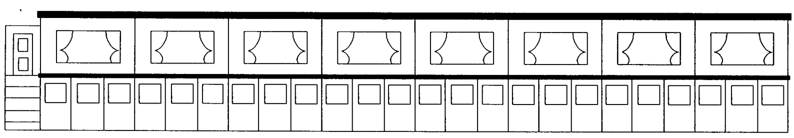
Modifications of Floating-Point

One simple expedient that has been proposed to avoid overflow involves the addition of an extra byte to the basic floating-point data format which would count the number of times the exponent has "wrapped around." However, unless we know, *in advance*, which quantities are likely to overflow such an extension would need to be added to every floating-point variable. This is precisely equivalent to simply adding a further eight bits to the exponent of floating-point representations. It is not a solution to the basic problem.

Modifications to the usual floating-point forms have been suggested to improve precision for numbers close to unity and extend the range of representable numbers. The basic idea was presented in Matsui and Iri **[6]** and followed up in Hamada **[4]** where a practical implementation is proposed. The basic idea is that the numbers of bits which are allocated to the exponent and mantissa can be varied. Their proposal is that a small number of bits in a computer word be used to indicate the number of bits which are used in that word for the exponent and therefore the number available for the mantissa. The rest of the representation is then normalized floating-point binary. (Hamada's implementation used the "indicator bits" to store part of the exponent information as well.)

By using the minimum possible number of bits for the exponent, more precision is available for numbers close to unity while allowing the number of exponent bits to grow means that significantly larger quantities can be represented. Of course, any such representation has variable relative precision in both the representation and the arithmetic. Like the level-index systems described later, these modified floating-point systems need a new and different error analysis. These modified schemes also require significantly larger accumulators than are needed for conventional floating-point arithmetic and their arithmetic would be much slower than standard floating-point.

The number of bits needed for the indicator (or the Hamada equivalent) is such that no wordlength shorter than 64 bits has been proposed for either of these systems. By far the most important drawback is that although they do enhance the range of the floating-point representation greatly, the principal difficulties of overflow and underflow remain.



Floating-Point Hall
South Elevation

were normally the servants' quarters although by now their Lordships had abandoned the idea of housing them all in the Hall itself and so only the most significant were allocated space within Floating-Point Hall. One of the great virtues of this design lies in the fact that if many more members of the family or their guests must be accommodated for a period then the number living in the exponential rooms above can be increased and the servants can be shifted any number of places to the right to allow figures of greater significance to occupy some of the lower rooms. After Floating-Point Hall had been occupied for just a short time it was appreciated that some additional space was needed for the displaced servants; some extra rooms were added to the guard house at the entrance to the estate.

The need for this became glaringly apparent one Christmas. A very large family gathering assembled this particular year—though only for a short time—and many of the servants were pushed out of their rooms. The result was catastrophic as many of them had nowhere to go and were lost forever to the bitter winter weather. When the guests all departed it became clear that there were no longer any servants left to shift back into those rooms. They lay empty and some of the most significant figures in the hall were lost.

Even with the benefit of this extra accommodation in the *Guard Register*, Floating-Point Hall had its shortcomings. During the residency of the fourth Lord of Number Crunching a scandal was uncovered in the village. Like many of the aristocracy, his father, the third Lord, had taken a mistress from among the ordinary folk of the village. At the time he had managed to keep the existence of his "bit-on-the-side" completely secret by providing for her special quarters along a secret passage. This secret room housed the *hidden bit* as she became known when the affair finally came to light. It was in fact located just to the left of the servants' rooms the first of which was occupied by 'Arf Quarterstaff, the son of the previous housekeeper. After the death of the third Lord and the discovery of this room it was appropriated by 'Arf as the most significant member of the household; he enjoyed the privacy thus afforded him during his all-too-rare moments off duty. This of course enabled one more of the less significant servants to be kept within the hall.

With all the ingenuity they could call upon however the architects could not prevent the hall overflowing from time to time with so many branches of the family —which were continually multiplying and dividing—trying to squeeze into the upper rooms for the big festivals around Newton Day at the beginning of the fall apple season.

Over the years many of the best architects and consultants have offered their plans for solving these problems. Apparently good ideas have been found to be impracticable; others were discarded because of the builders' obsession with speed. Some of them were much more concerned with how many buildings could be

Level-Index, LI, Arithmetic

The basic idea of the representation was proposed in Clenshaw and Olver [1]: a real number X is to be represented in the machine by the binary representation of x where

$$|x| = l + f \tag{2}$$

l is a nonnegative integer and f a fraction in $[0, 1)$ given by

$$f = \underbrace{\ln(\ln(\cdots(\ln|X|)\cdots))}_{l\ \text{times}}. \tag{3}$$

That is, l is the number of times the natural logarithm of $|X|$ must be taken to obtain a number in $[0, 1)$. The arithmetic operations for this system, and for the *symmetric level-index*, SLI, system in which small quantities are represented by the LI image of their reciprocals, work directly with the LI images. Details of these algorithms and possible implementations can be found in [2] and [3].

A maximum of three bits is necessary for the storage of the *level*, l since working to no more than 5,500,000 bits in the *index*, f, we see that if $x \geq 6$, $y < x$, then

$$\phi(x) \pm \phi(y) = \phi(x)$$

so that the system is closed under addition and subtraction. Its logarithmic nature immediately yields closure under multiplication and division (other than by zero) with levels 0 through 7.

Since the LI and SLI systems are closed, they eliminate overflow/underflow as a result of arithmetic operations. The arithmetic is likely to be somewhat slower than conventional floating-point but will allow simpler algorithms and shorter programs to be used.

Like the modified floating-point systems, computation using these arithmetics would require a new error analysis. However the smooth nature of the representation function should make this task more straightforward within the appropriate error measure, *generalized precision*, which corresponds to absolute precision in the index.

There are now several papers discussing various aspects of the LI and SLI systems: analytic, computational experience, software engineering implications, software and potential hardware implementations. A useful introduction is provided in [2].

constructed in a given time, others with the speed at which more people could be added to the register of guests. Communication between the rooms was also being improved constantly. I think it was the fifth Lord who even installed a miniaturized bus system along a special pipeline in order to shift guests in and out as efficiently as possible. But none of these improvements had any real impact on the overflow problem.

The situation was alleviated to some extent with a major remodelling of Floating-Point Hall in which the overall length of the accommodation was doubled. There were several plans drawn up as a result of an open competition for the design of this extended version which became known as Double-Length Hall. Detailed plans are not included here as the principal ideas are just the same as for the original Floating-Point Hall. The winning design came from an American multinational corporation IEEE Design Inc. who had long since taken a majority share-holding in Exponent, Mantissa and Signs. This design incorporated the "hidden bit" design and added more rooms of even greater importance to the upper storey as well as more than doubling the servants' accommodation. This design was indeed adequate to cope with some very large accumulations at Double-Length Hall and for several years it was felt to be sufficient for all conceivable situations. Indeed some Number Crunchers even suggested that should any gathering be too numerous to be housed within Double-Length Hall then it was almost by definition a mistake and the guest list should be reconsidered, amended and resubmitted to the Number Cruncher as his Lordship was known. (Incidentally, there are many mutually compatible outposts of Number Crunching in which imitations of Double-Length Hall have been constructed to slightly different designs. Some of these have an additional wing attached and have an even greater capacity than the IEEE design.)

During this period of self-satisfied contentment there were still some serious-minded residents of Number Crunching who were worried by the possibility of overflow in the Hall. These people spent much time discussing with some of the more avant-garde architects feasible designs for a new manorial hall which would be free of this troublesome feature. This is certainly not the only problem which is of concern to the gentlefolk of the village but it is one of the more pressing.

Among the suggestions are the construction of an Overflow Motel on the grounds of the main hall. In much the same way as some of the displaced servants can be temporarily housed in the Guard Register, it is intended to house the most important guests of all in this new accommodation. There are even to be royal apartments in Overflow Motel. Of course one of the servants would be charged with maintaining a register of just who is using the motel at any one time. One of the difficulties with this proposal is that some of the bit-part characters will need to move through enormous distances as guests arrive and depart.

There are also some ingenious Japanese designs which allow for the use of individual rooms to vary according to the numbers to be housed at any one time. Different parts of the building can be partitioned off almost arbitrarily to house guests and servants as efficiently as possible. The unfortunate part of this design is that a number of servants are needed merely to keep track of how the building is currently partitioned. Their role is sufficiently important to the smooth-running of the system that they are allocated permanent quarters within the main building. The problem of adding to or subtracting from the current list of residents is tolerably well organized for this system but it is still much slower than is the case at Double-Length Hall. The biggest problem with both these designs however is that they do not achieve the major objective of eliminating the overflow problem from

Number Crunching. It is the case that both systems render it much less likely but that is not enough to satisfy this very demanding family.

Perhaps the only proposed design which actually achieves this aim has been developed in the offices of *Clenshaw, Olver, and Associates*. The two senior partners have lived in, or close to, Number Crunching for most of their professional lives and have witnessed many of the important developments. Their firm is working on a revolutionary design, Level-Index Towers, which could be described as the first luxury high-rise development to be proposed within Number Crunching and as such is attracting some reactionary criticism from the local populace.

There are two designs under consideration. The first is for an eight storey edifice whose ground floor would be Level 0 with the next being Level 1 and so on up to the penthouse apartments on Level 7. (The ground floor being numbered zero is perhaps a reflection of the Anglo-Saxon origins of the senior partners.) The particularly inventive aspect of the design is that each floor can accommodate a very much larger number than the one below it. So much so that however many are added there will always be sufficient room for them all. This remains true even if each of them were to invite a very large number of guests thereby multiplying the number to be housed by some large factor.

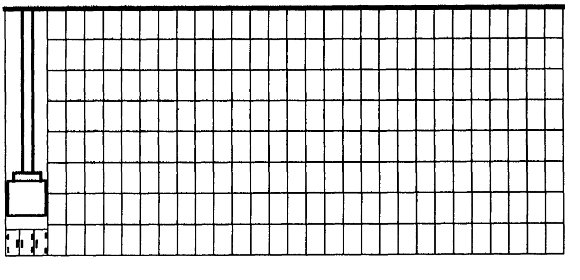
The actual relation between the numbers that can be housed on the different floors is that each subsequent level can accommodate e, the base of natural logarithms, raised to the number that can be housed on the one beneath it. That is,

$$N(k+1) = \exp(N(k)),$$

where $N(k)$ is the maximum number that can be accommodated on levels 0 through k. Of course the space between neighbors is reduced as one travels up through the levels and eventually the point is reached where guests can no longer be housed one to a room. There are, to be fair, other disadvantages: most notably, in the eyes of some of the reactionaries, the fact that the addition of some new occupants can be time-consuming, reorganizing them into their proper order and accommodation is not straightforward.

The other side of this coin is that because there is never any need to worry about whether even the largest numbers can be accommodated, the planning stages of any large operation are very much simplified. The overall time and effort expended on such details is thus likely to be no greater than hitherto—but with no risk of needing to resubmit the plans with a reduced guest-list or program of events.

One major architectural improvement incorporated into Level-Index Towers is the elevator, or *level-indicator*, at the front of the building. This contraption takes



Level-Index Towers

any number of visitors to the appropriate level and organizes their accommodations appropriately. It also provides an excellent view over the surrounding countryside on the way up.

The second design is for a building with as many floors below ground as there are above. This *symmetric level-index* design allows for servants to accommodate as large a household as may be necessary. It also enables the accurate storage and addition of fractional quantities like a half-firkin of strong ale. The rapid movement among the levels provided by the elevator is even more of a necessity for this design so that wine from the correct cellar level may be delivered to their Lordships' table in time for dinner. To save the new butler, Warton, from too much running hither and thither much of this fetching and carrying has been computerized and only at the final output of the wine to the glasses is he personally involved. The end-user of his services will notice little difference since the output of kitchen and cellar would be in familiar forms.

The great advantage that the level-index designs have is that they free the programmer of future events in Number Crunching from any worries about the scale of the operations leaving him free to concentrate on the important matters at hand.

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It is not the essence of mathematics to be conversant with the ideas of number and quantity.

—*Boole (1854)*

The Length of the Day

Richard S. Bassein

The following natural phenomenom appears to be little known and even less understood, despite the ease with which it may be observed and the elementary nature of the mathematics required for a reasonably accurate explanation: since the daylight is shortest at the winter solstice, one (especially one who must rise early!) would expect sunrise to occur earlier each day following the solstice. Nevertheless, as the data [1] in Table 1 show, the sunrise continues to occur *later* each day (in fact, for about two weeks) after the winter solstice.

TABLE 1. Sunrise at 40° lattitude in 1991

Date	Sunrise
22 December	7:19 am
25 December	7:20 am
27 December	7:21 am
31 December	7:22 am

This anomaly is caused by the fact that although our clocks mark each day of the year by a constant 24 hours, the length of the day defined by the position of the sun in the sky varies throughout the year, as shown by the data [1] in Table 2, reaching a maximum of about 24 hours and 30 seconds at the winter solstice.

TABLE 2. Noon in 1991

Date	Sun at highest point in sky	Length of day
22 March	12:07:02 pm	24 hr. − 18 sec.
23 March	12:06:44 pm	
...	...	
21 June	12:01:38 pm	24 hr. + 13 sec.
22 June	12:01:51 pm	
...	...	
23 September	11:52:32 pm	24 hr. − 21 sec.
24 September	11:52:11 pm	
...	...	
22 December	11:58:23 pm	24 hr. + 30 sec.
23 December	11:58:53 pm	

Thus, although the sunrise on the day after the winter solstice does precede the noon by just a little bit more time than on the solstice, it appears about 30 seconds later according to the clock. I have successfully used this topic to motivate trigonometric functions and analytic geometry in precalculus; the analysis could also serve as an example of using approximations in mathematical modeling or for

illustrating numerical techniques; for a perspective using elementary notions of calculus and focusing on the major cause of the effect, see [**5**].

To be precise, we define *noon* on a given day to be the moment at which the sun reaches its highest point in the sky and the *length of the day* as the time between one noon and the next. We will see that the two most important causes of the variation in the length of the day are (1) the relationship between the rotation of the earth on its tilted axis and the revolution of the earth around the sun, accounting for a 20 second lengthening of the day at the solstices and a similar shortening at the equinoxes, and (2) the variation in the earth's angular velocity around the sun resulting from the variation in its distance from the sun, resulting in an additional 8 second lengthening near the winter solstice and a similar shortening near the summer solstice.

Although it would not be difficult to give an exact analytic treatment of causes (1) and (2) together, it will simplify the computations and give a better understanding of their effects to sacrifice a small amount of accuracy and treat them separately and approximately; in what follows, physical measurements are accurate to the number of places shown. To study (1) alone, we treat the earth's orbit as circular, ignoring the $\pm 1.7\%$ variation of the distance between the earth and the sun. Establish a fixed coordinate system with origin at the sun and positive z-axis parallel to the axis of the earth's rotation and pointing north, as shown in Figure 1. In this coordinate system, the plane of the earth's orbit will be tilted at an angle θ of 0.41 radians and the "highest point" will mark the winter solstice. We place the positive x-axis directly under the winter solstice, which makes the y-axis pass through the points where the earth's orbit intersects the xy-plane.

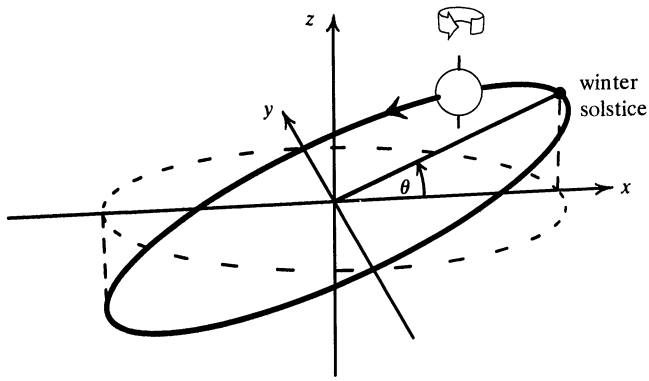

Figure 1. The coordinate system and the earth's orbit.

To examine the relationship between the rotation and revolution of the earth, we interpret the angles describing both motions relative to the z-axis by projecting those motions onto the xy-plane; for an approach appealing to trigonometry on the sphere, see [**5**]. For convenience, we choose the unit of length to be the radius of the earth's orbit and the unit of time t to be a year, that is, the 8766 hours, 9 minutes and 9.5 seconds it takes the earth to complete one circuit around the sun. If we set $t = 0$ at the winter solstice, then the projection of the earth's position onto the xy-plane is

$$(x, y) = (\cos(\theta)\cos(2\pi t), \quad \sin(2\pi t)). \tag{1}$$

Let $a(t)$ be the angle the earth rotates on its axis in time t. Since the earth rotates once on its axis each 23 hours, 56 minutes, and 4.091 seconds, which is $1/366.256$ of a year, letting $Y = 366.256$, gives

$$a(t) = 2\pi Y t. \tag{2}$$

Let $s(t)$ be the angle that the projection of the earth onto the xy-plane makes with the positive x-axis at time t. From Equation (1) we have

$$s(t) = \arctan\left(\frac{\sin(2\pi t)}{\cos(\theta)\cos(2\pi t)}\right) = \arctan\left(\frac{\tan(2\pi t)}{\cos(\theta)}\right), \tag{3}$$

for the proper choice of the branch of the arctan for each range of values of t.

Let P be a point on the earth's surface for which it is noon at time $t = 0$ and let t_k be the time at which the kth noon after $t = 0$ occurs for P. As Figure 2 illustrates, t_k is the solution to

$$a(t) = 2k\pi + s(t). \tag{4}$$

If the earth's axis were perpendicular to the plane of its orbit, then θ would equal 0, $\cos(\theta)$ would equal 1, $s(t)$ would be $2\pi t$, and every day would have length

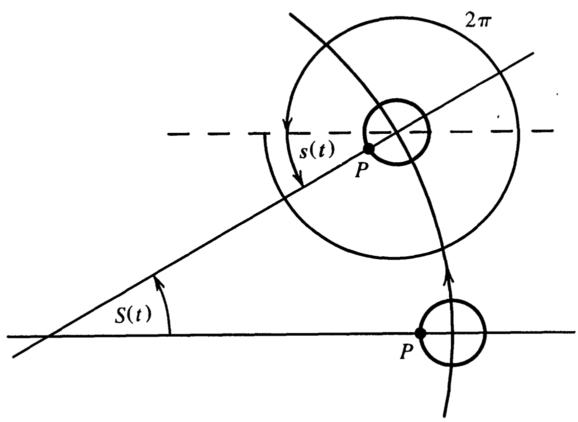

Figure 2. The occurrence of noon.

$t_{k+1} - t_k = 1/(Y - 1)$, which, in terms of hours, equals 24. Thus it is the tilt θ which, when we project the earth's position onto the xy-plane, expands the angles of revolution near the solstices and contracts the angles of revolution near the equinoxes, and thereby lengthens the days near solstices and shortens them near the equinoxes, according to Equation (4). Figure 3 shows how equal angles on a circle project when the circle is tilted.

Substituting Equations (2) and (3) into Equation (4) gives

$$2\pi Y t = 2k\pi + \arctan\left(\frac{\tan(2\pi t)}{\cos(\theta)}\right),$$

to which we could find approximate solutions by using Newton's method [**2**] with initial approximation $t_k = k/(Y - 1)$. On the other hand, to determine the length

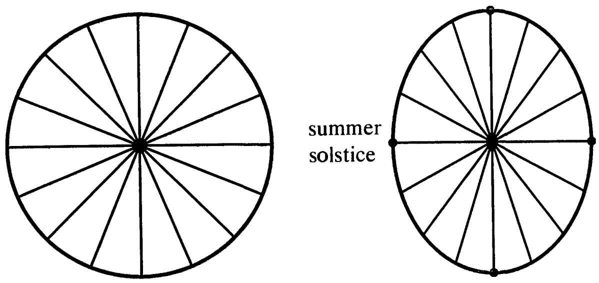

Figure 3. Projecting a tilted circle.

of the day starting at the winter solstice, for example, we can use the fact that for small values of their arguments, both the tangent and arctangent are very close to the identity function and instead solve the equation

$$2\pi Yt = 2\pi + 2\pi t/\cos(\theta) \tag{5}$$

to get

$$t = \frac{1}{Y - 1/\cos(\theta)},$$

which is 1.000248 as big as $1/(Y-1)$ and therefore corresponds to a day of length 24 hours and 21.4 seconds. The symmetry of Figure 1 shows that the result would be the same at the summer solstice. If we set $t = 0$ at an equinox instead, a similar analysis finds the length of the day starting at that equinox to be

$$t = \frac{1}{Y - \cos(\theta)},$$

which gives a day of length 19.6 seconds shorter than 24 hours. (Using Newton's method only affects the second decimal place of these results.)

Now we turn to cause (2), the variation in the angular velocity of the earth around the sun, which further modifies $s(t)$ and the solutions to Equation (4). According to Kepler's laws of planetary motion [3], the earth's orbit is an ellipse, with one focus at the sun, and the line from the sun to the earth sweeps out equal areas in equal amounts of time, as shown (with an exaggerated ellipse) in Figure 4. It follows that when the earth is closest to the sun, at a distance $r = 91.4$ million miles, its angular velocity is highest and when the earth is furthest from the sun, at a distance $R = 94.6$ million miles, its angular velocity is lowest. If, for convenience, we take the former to occur at the winter solstice (in fact, it is about two weeks later), then it follows from the geometry of the ellipse [4] that latter occurs at the summer solstice.

Since in one unit of time the line from the sun to the earth sweeps out the entire ellipse, whose area is

$$A = \pi\frac{R+r}{2}\sqrt{Rr},$$

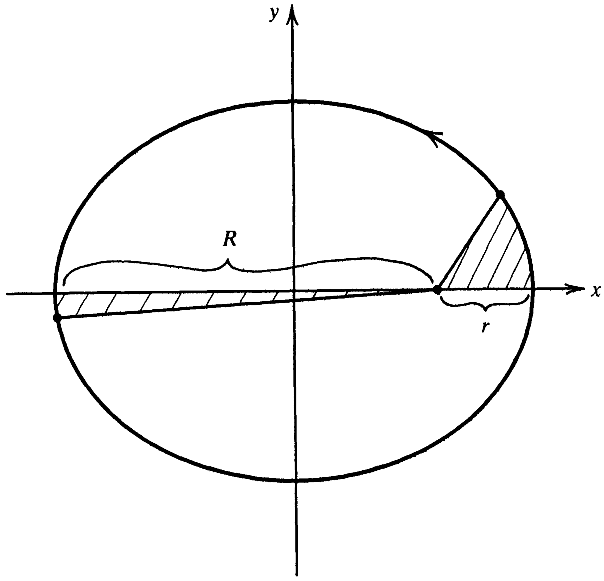

Figure 4. Kepler's laws.

it sweeps out an area of At in time t. Let ϕ be the angle of the sector swept out in time t following the winter solstice, when the radius is r. Approximating the area of that sector by the triangular area $r^2\phi/2$, we obtain

$$\phi = \frac{\pi(R + r)\sqrt{Rr}}{r^2}t = \left(1 + \frac{R}{r}\right)\sqrt{\frac{R}{r}}\,\pi t,$$

which is larger than the angle $2\pi t$ swept out on the circle in time t by a factor of 1.035. We can get a reasonable approximation to the effect this has on the day starting with the winter solstice by modifying Equation (5) to read

$$2\pi Yt = 2\pi + 2\pi(1.035)t/\cos(\theta).$$

The solution

$$t = \frac{1}{Y - 1.035/\cos(\theta)}$$

yields a day of length 24 hours and 30.4 seconds. The analogous computation for the summer solstice gives a day of length 24 hours and 12.6 seconds. Since the distance from the earth to the sun is about its average at the equinoxes, the adjustment in the length of the day at those times is only about 1.5 seconds.

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The Kelly Criterion and the Stock Market

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The purpose of this expository note is to describe the Kelly criterion, a theory of optimal resource apportionment during favorable gambling games, with special attention to an application in the U.S. stock market.

By a "favorable game" we mean one in which there exists a strategy such that $\Pr(\lim_{n\to\infty} X_n = +\infty) > 0$, where X_n is the player's capital after n trials. We shall first discuss the case of discrete *binomial* gambling games and then extend the discussion to *continuous* gambling games.

BINOMIAL GAMES

COIN TOSSING. Imagine that we are faced with an infinitely wealthy opponent who will wager even money bets made on repeatedly independent trials of a biased coin. Further, suppose that on each trial our win probability is $p > 1/2$ and the probability of losing is $q = 1 - p$. At the outset our initial capital is X_0 and the primary problem is that of deciding what amount B_i to bet on the ith trial.

A classical criterion is to choose B_i for each i so that the expected value $E(X_n)$ is a maximum after n trials. Letting $T_k = 1$ if the kth trial is a win and $T_k = -1$ if it is a loss, then $X_k = X_{k-1} + T_k B_k$ for $k = 1, 2, 3, \dots$, and $X_n = X_0 + \sum_{k=1}^{n} T_k B_k$. Then

$$E(X_n) = X_0 + \sum_{k=1}^{n} E(B_k T_k) = X_0 + \sum_{k=1}^{n} (p-q) E(B_k).$$

Since the game has a positive expectation, i.e., $p - q > 0$ in this even payoff situation, then in order to maximize $E(X_n)$ we would want to maximize $E(B_k)$ at each trial. Thus, to maximize expected gain we should bet *all of our resources* at each trial. Thus $B_1 = X_0$ and if we win the first bet, $B_2 = 2X_0$, etc. However, the probability of ruin is given by $1 - p^n$ and with $1/2 < p < 1$, $\lim_{n\to\infty}[1 - p^n] = 1$ so ruin is almost sure. Thus the criterion of betting to maximize expected gain is a fundamentally undesirable strategy.

Likewise, if we play to minimize the probability of eventual ruin (i.e., "ruin" occurs if $X_k = 0$ on the kth outcome) the well-known gambler's ruin formula in **[1]** can be used to show that we minimize ruin by making a *minimum* bet on each trial; but this has the unfortunate concomitant that it also minimizes the expected average gain. Thus "timid betting" is also unattractive.

Some intermediate strategy is required which is somewhere between maximizing $E(X_n)$ (and assuring ruin) and minimizing the probability of ruin (and minimizing $E(X_n)$). An asymptotically optimal strategy was first proposed by J. L. Kelly in **[2]**. Much credit for this note goes to L. Breiman who developed the theoretical underpinnings for the validity of the Kelly system. E. O. Thorp applied the Kelly

criterion to Casino Blackjack in [**3**], to other gambling games in [**4**], and to modern portfolio theory in [**5**].

In the coin-tossing game just described, since the gambling probability and the payoff at each bet are the same, it seems intuitively clear that an "optimal" strategy will involve always wagering the same fraction f of your bankroll. To make this possible we shall assume from here on that capital is infinitely divisible. "Ruin" shall henceforth be reinterpreted to mean that for arbitrarily small positive ε, $\lim_{n\to\infty}[\Pr(X_n \le \varepsilon)] = 1$. Even in this sense, as we shall see, ruin *can* occur under certain circumstances.

If we bet according to $B_i = fX_{i-1}$, where $0 \le f \le 1$, this is sometimes called "fixed fractional" betting in which we are always wagering the same percentage of our current resources. Where S and F are the number of successes and failures, respectively, in n trials, then our capital after n trials is given by $X_n = X_0(1+f)^S(1-f)^F$, where $S + F = n$. With f in the interval $0 < f < 1$, $\Pr(X_n = 0) = 0$. Thus "ruin" in the technical sense of the gambler's ruin problem cannot ever occur.

We note that since

$$e^{n \log\left[\frac{X_n}{X_0}\right]^{1/n}} = \frac{X_n}{X_0},$$

the quantity

$$\log\left[\frac{X_n}{X_0}\right]^{1/n} = \frac{S}{n}\log(1+f) + \frac{F}{n}\log(1-f)$$

measures the exponential rate of increase per trial. Kelly chose to maximize the expected value of the growth rate coefficient $G(f)$, where

$$\begin{aligned} G(f) &= E\left\{\log\left[\frac{X_n}{X_0}\right]^{1/n}\right\} = E\left\{\frac{S}{n}\log(1+f) + \frac{F}{n}\log(1-f)\right\} \\ &= p\log(1+f) + q\log(1-f). \end{aligned}$$

Note that $G(f) = (1/n)E(\log X_n) - (1/n)\log X_0$ so for n fixed, maximizing $G(f)$ is the same as maximizing $E\log X_n$. We usually will talk about maximizing $G(f)$ in the discussion below. Note that

$$G'(f) = \frac{p}{1+f} - \frac{q}{1-f} = \frac{p-q-f}{(1+f)(1-f)} = 0$$

when $f = f^* = p - q$.

Calculation shows that

$$G''(f) = \frac{-f^2 + 2f(p-q) - 1}{(1-f^2)^2} < 0$$

so that $G'(f)$ is monotone strictly decreasing on $[0,1)$. Also, $G'(0) = p - q > 0$ and $\lim_{f\to 1^-} G'(f) = -\infty$. Therefore by the continuity of $G'(f)$, $G(f)$ has a unique maximum at $f = f^*$, where $G(f^*) = p\log p + q\log q + \log 2 > 0$. Moreover, $G(0) = 0$ and $\lim_{f\to 1^-} G(f) = -\infty$ so there is a unique number $f_c > 0$, where $0 < f^* < f_c < 1$, such that $G(f_c) = 0$. The nature of the function $G(f)$ is now apparent and a graph of $G(f)$ versus f appears as shown in Figure 1.

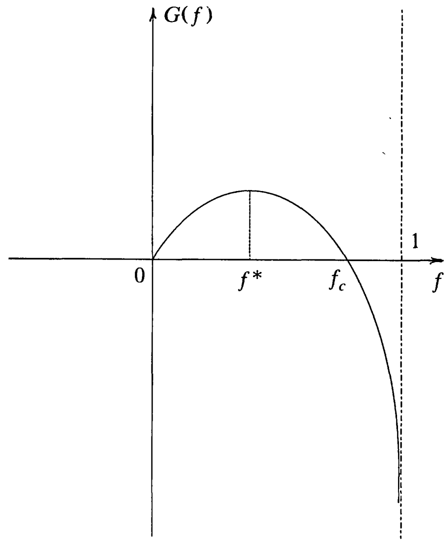

Figure 1

The following theorem recounts the important advantages of maximizing $G(f)$. The details are omitted here but proofs of (i), (ii), (iii), and (vi) for the simple binomial case can be found in [**4**]; more general proofs of these and of (iv) and (v) are in [**6**].

Theorem 1. (i) *If* $G(f) > 0$, *then* $\lim_{n\to\infty} X_n = \infty$ *almost surely*, *i.e.*, *for each* M, $\Pr[\liminf_{n\to\infty} X_n > M] = 1$;

(ii) *If* $G(f) < 0$, *then* $\lim_{n\to\infty} X_n = 0$ *almost surely*; *i.e.*, *for each* $\varepsilon > 0$, $\Pr[\limsup_{n\to\infty} X_n < \varepsilon] = 1$;

(iii) *If* $G(f) = 0$, *then* $\limsup_{n\to\infty} X_n = \infty$ a.s. *and* $\liminf_{n\to\infty} X_n = 0$ a.s.

(iv) *Given a strategy* Φ^* *which maximizes* $E \log X_n$ *and any other "essentially different" strategy* Φ (*not necessarily a fixed fractional betting strategy*), *then* $\lim_{n\to\infty} X_n(\Phi^*)/X_n(\Phi) = \infty$ a.s.

(v) *The expected time for the "running capital"* X_n *to reach any fixed preassigned goal* $\boldsymbol{X}$ *is, asymptotically, least with a strategy which maximizes* $E \log X_n$.

(vi) *Suppose the return on one unit bet on the* ith *trial is the binomial random variable* U_i; *further, suppose that the probability of success is* p_i, *where* $(1/2) < p_i < 1$. *Then* $E \log X_n$ *is maximized by choosing on each trial the fraction* $f_i^* = p_i - q_i$ *which maximizes* $E \log(1 + f_i U_i)$.

Part (i) shows that, except for a finite number of terms, the player's fortune X_n will exceed any fixed bound M when f is chosen in the interval $(0, f_c)$. But, if $f > f_c$, part (ii) shows that ruin is almost sure. Part (iii) demonstrates that if $f = f_c$, X_n will (almost surely) oscillate randomly between 0 and $+\infty$. Parts (iv) and (v) show that the Kelly strategy of maximizing $E \log X_n$ is asymptotically optimal by two important criteria. Part (vi) establishes the validity of utilizing the Kelly method of choosing f_i^* on each trial (even if the probabilities change from one trial to the next) in order to maximize $E \log X_n$.

Example 1. Player A plays against an infinitely wealthy adversary. Player A wins even money on successive independent flips of a biased coin with a win probability of $p = .53$ (no ties). Player A has an initial capital of X_0 and *capital is infinitely divisible*. Applying Theorem 1(vi), $f^* = p - q = .53 - .47 = .06$. Thus 6% of current capital should be wagered on each play in order to cause X_n to grow at the fastest rate possible consistent with exactly zero probability of ever going broke. If Player A continually bets a fraction smaller than 6%, X_n will also grow to infinity but the rate will be slower.

If Player A repeatedly bets a fraction larger than 6%, up to the value f_c, the same thing applies. Solving the equation $G(f) = .53\log(1+f) + .47\log(1-f) = 0$ numerically on a computer yields $f_c = .11973^-$. So, if the fraction wagered is above approximately 12% (up to 1), then even though Player A may temporarily experience the pleasure of a faster win rate, eventual downward fluctuations will occur that will inexorably drive the values of X_n toward zero. Calculation yields a growth coefficient of $G(f^*) = G(.06) = 0.016566^+$ so that after n successive bets the log of Player A's average bankroll will tend to $.016566n$ times as much money as he started with.

The Kelly criterion can easily be extended to uneven payoff games. Suppose player A wins b units for every unit wager. Further, suppose that on each trial the win probability is $p > 0$ and $pb - q > 0$ so the game is advantageous to player A. Methods similar to those already described can be used to maximize

$$G(f) = E\log(X_n/X_0) = p\log(1+bf) + q\log(1-f).$$

Arguments using calculus yield $f^* = (bp - q)/b$, the optimal fraction of current capital which should be wagered on each play in order to maximize the growth coefficient $G(f)$.

A criticism sometimes applied to the Kelly strategy is that capital is not, in fact, infinitely divisible. For any gambling game in the real world, no one ever uses fractional amounts of money (for example) smaller than \$0.01. Since bets are always necessarily quantized, "ruin" in the sense we defined it, is possible. It is not difficult to show, however, (see [7]) that if the minimum bet is small relative to the gambler's initial capital, then the probability of ruin is "negligible" and the theory herein described is a useful approximation.

CONTINUOUS GAMBLING GAMES

Each investment in a succession of stock market "gambles" only has a finite number of outcomes. But it is mathematically convenient to approximate a finite distribution using a continuous distribution model. The added refinements and hypotheses required are in one sense artificial generalizations of the discrete case described thus far; the continuous model results must preserve the conclusions of the discrete case. We, therefore, work to maximize $E\log(X_n/X_0)$ as before.

Example 2. An investor purchases a stock for \$100 per share now, while the anticipated price of the stock in one year is uniformly distributed on the interval $[30, 200]$. Inflation, broker's fees, and tax considerations are omitted from this discussion. The outcome per unit bet is described by $dF(s) = U_A(s)\,ds$, where $A = [-7/10, 1]$ and F is the associated probability distribution. We observe that $U_A(s) = 10/17$ for $s \in A$ and $U_A(s) = 0$ for $s \notin A$ as shown in Figure 2.

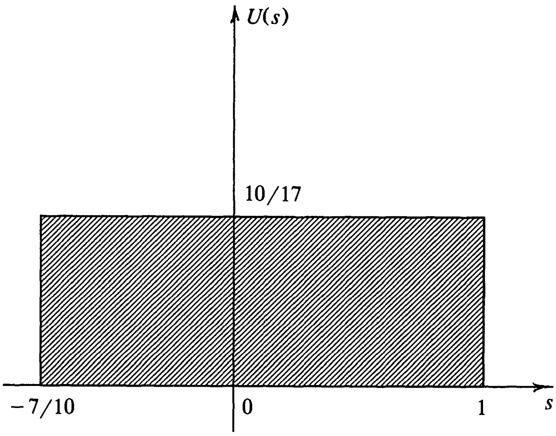

Figure 2

Observe that the mean $\mu = \int_{-7/10}^{1}(10/17)s\,ds = +0.15$. We now compute f^* and $G(f^*)$ assuming the stock is sold in one year. Note that we want to maximize the integral

$$G(f) = \int_{-7/10}^{1}[\log(1+fs)]\left(\frac{10}{17}\right)ds. \tag{1}$$

This can be accomplished explicitly by solving $G'(f) = 0$, where

$$G'(f) = \frac{10}{17}\int_{-7/10}^{1}\frac{s\,ds}{1+fs} = \left(\frac{10}{17f}\right)\int_{-7/10}^{1}\frac{fs\,ds}{1+fs}$$
$$= \left(\frac{10}{17f}\right)\left[\int_{-7/10}^{1}ds - \int_{-7/10}^{1}\frac{ds}{1+fs}\right].$$

Setting $G'(s) = 0$ reduces to solving

$$\frac{17}{10} = \frac{1}{f}\log\left(\frac{1+f}{1-\dfrac{7f}{10}}\right).$$

Calculation yields $f^* = 0.63^+$. Thus, consistent with our ability to continue to make similarly advantageous bets in the future, we should wager 63% of current capital. Integration of (1) yields $G(f^*) = 0.0472$. Ruin is inevitable for $f > 1.17$.

Under certain conditions it is possible that the maximum value of $G(f)$ will occur when $f = f^* > 1$. For the same present stock price of \$100 and without further calculation, we see at once that if $[30, 200] \to [65, 150]$, a scale change from the interval $-7/10 \le s \le 1$ to the interval $-7/20 \le s \le 1/2$, then $f^* = 2(.63^+) > 1$ and the value of $G(f^*)$ remains 0.0472 as before.

But suppose, instead, that the stock price in one year was uniformly distributed on the interval $[70, 150]$, with the current price \$100 as before, then $dF(s) = U_A(s)\,ds$, where $U_A(s) = 10/8$ for $s \in A = [-3/10, 5/10]$ and 0 for $s \notin A$. Then the maximum value of the integral $G(f) = (10/8)\int_{-3/10}^{5/10}\log(1+fs)\,ds$ occurs when $f^* = 1.95^-$; calculation yields a growth coefficient of $G(f^*) = 0.0956$. Note that the mean $\mu = +0.10$. Therefore, in this case we should be willing to buy on

margin and wager up to 1.95 times current capital, consistent with our ability to endure risk and our financial ability to cover later. Thus we have the interesting finding that under certain conditions the mean of investment A may be higher than the mean of investment B, but if the variability of investment B is sufficiently small, then it may turn out that $G(f_B^*) > G(f_A^*)$. The Kelly criterion would then choose investment B as the superior gamble.

In the previous example, we need the following theorem in order to guarantee that the integral $G(f) = \int_a^\infty \log(1 + fs)\,dF(s)$ has a unique maximum at $f = f^*$. With $-\infty < a < 0$, we define $a = \sup\{s\colon F(-\infty, s) = 0\}$.

Theorem 2. *If the mean* $\mu = \int_a^\infty s\,dF(s) > 0$, *then the function*

$$G(f) = \int_a^\infty \log(1 + fs)\,dF(s)$$

attains a unique maximum value $G(f^*)$ *where* $f^* \in (0, -1/a)$ *iff*

$$\lim_{f \to (-1/a)^-} G'(f) < 0.$$

Proof: First note that if $1 + fa > 0$, the integral $G(f) = \int_a^\infty \log(1 + fs)\,dF(s)$ is defined. Also,

$$G''(f) = \int_a^\infty \frac{-sf}{(1 + fs)^2}\,dF(s) < 0$$

so that

$$G'(f) = \int_a^\infty \frac{s}{1 + fs}\,dF(s)$$

is monotone strictly decreasing on $[0, -1/a)$. Observe that $G(0) = 0$. Also $G'(0) = \int_a^\infty s\,dF(s) = \mu > 0$ and $\lim_{f \to (-1/a)^-} G'(f) < 0$ by hypothesis. From the monotonicity and continuity of $G'(f)$ on $[0, -1/a)$ it follows that $G'(f)$ takes on all values on the interval $[G'(0), \lim_{f \to (1/a)^-} G'(f))$ exactly once and thus $G(f)$ has a unique maximum at $f = f^*$, where $0 < f^* < -1/a$.

Comment. Observe that if $a \to -\infty$, then $f^* \to 0$ so that the Kelly criterion applied to continuous distribution models will yield non-trivial results only if the lower limit of the integral $\int_a^\infty \log(1 + fs)\,dF(s)$ is *finite*.

AN APPLICATION TO THE U.S. STOCK MARKET

Investing in the stock market may be viewed as a continuous gambling game with a positive, one-year expected return equal to the average of the historical annual returns over a sufficiently long time span. Admittedly it is argumentative to suggest that only stationary processes are involved. To a reasonable first approximation, however, there is evidence to suggest that price changes in speculative markets behave like independent, identically distributed random variables with finite variances (see **[8]**). From the Central Limit Theorem, it would then follow that price changes in U.S. stocks are approximately normal (actually the lognormal distribution would provide a superior fit, but the computations are much more cumbersome to discuss here).

To an investor (i.e., "gambler"), what constitutes a profit over an extended period of time is complicated by the time-varying purchasing power of money and other factors such as brokerage commissions and taxes, as well as the perceived risk that may be involved. Since time is very important, an actual annual percent-

age return in the stock market has little meaning unless compared with the inflation rate or some proxy such as T-bill rates or money-market rates.

Historical annual *excess returns* (annual total returns on common stock in excess of Treasury bill returns) have been found to be relatively stable and thus the normal distribution is a reasonable approximation.

For the 59 year period from 1926 to 1984, the distribution of annual excess total returns on S&P 500 "blue chip" stocks had a calculated mean $\mu = 0.058$ and standard deviation $\sigma = 0.2160$. (See [**9**].) Each "return" in the calculation was expressed as the natural logarithm of one plus the annual excess return ER_i in formulas (2) and (3) below.

$$\mu = \frac{1}{59} \sum_{i=1}^{59} \log(1 + ER_i) = \log\left[\prod_{i=1}^{59} (1 + ER_i)\right]^{1/59}; \tag{2}$$

$$\sigma^2 = \frac{\sum_{i=1}^{59} [\log(1 + ER_i) - \mu]}{58}. \tag{3}$$

(Note that expressing returns in this fashion has the advantage that the mean of the natural logs is the continuously compounded geometric mean return.)

Various interesting probability calculations are possible if we assume that annual excess returns are independently distributed. It would then follow, for example, that the mean and standard deviation of an n-year forecast of annual excess returns would be $\bar{x} = 0.058$ and $s_n = 0.2160/\sqrt{n}$. With a fixed amount invested in stocks over an n-year period, the probability of a negative excess return would be

$$\Pr\left(t < \frac{0 - .058}{.2160/\sqrt{n}}\right).$$

Some illustrations using various values of n are shown in Table 1 below. While these illustrations do not relate directly to our eventual application of the Kelly criterion, they do inform us of the relative risk characteristics of stocks vs. T-bills over various periods of time.

TABLE 1

Number of years n	Probability of negative excess return
2	.38
3	.35
5	.29
10	.21
15	.16
20	.13
25	.10
30	.08
35	.06
40	.05

ESTIMATING THE KELLY CRITERION VALUE OF f^* FOR LONG-TERM INVESTMENT IN S&P 500 STOCKS

Suppose we have an initial amount of investment capital X_0 and we now want to determine the optimal "wager-fraction" f^* to invest each year in S&P 500

stocks. Using an unaltered normal curve for our probability distribution is inadequate for two reasons: first, the normal distribution allows for unboundedly large annual excess percentage declines/advances in stocks (unrealistic on both counts); secondly, as inferred by the comment following the proof of Theorem 2, the Kelly criterion will not yield a meaningful $f^* > 0$ if the probability distribution $F(s)$ suggests a negatively infinite lower limit of the integral

$$\int_a^\infty \log(1 + fs)\, dF(s).$$

For the above reasons we estimate using a quasi-normal probability distribution $N(s)$ with mean excess return $\mu = .058$ and $\sigma = .2160$ as we had for the years 1926 to 1984. The distribution is described below in (4) and Figure 3. We define the excess return variable s to be meaningful on the interval $A \le s \le B$, where $A = \mu - 3\sigma = -0.590$ and $B = \mu + 3\sigma = 0.706$, the maximum permissible annual excess percentage changes that are assumed may occur. There are two special constants to be determined, α and h.

$$N(s) = \begin{cases} h + \dfrac{1}{\sqrt{2\pi\alpha^2}} e^{-(s-\mu)^2/2\alpha^2}, & A \le s \le B \\ 0, & s < A \\ 0, & s > B. \end{cases} \tag{4}$$

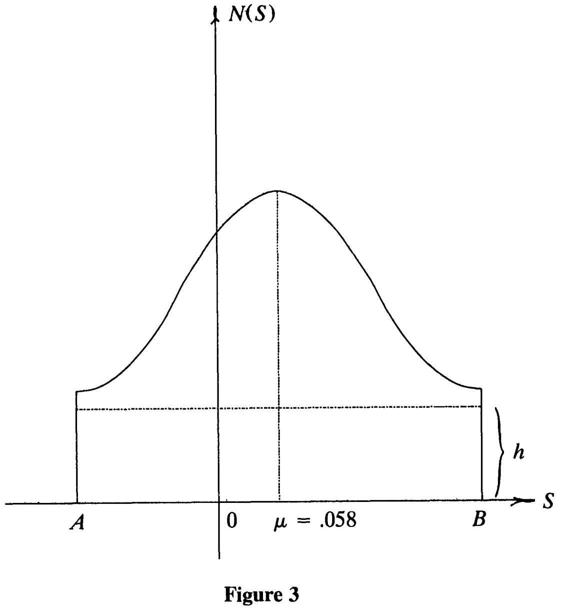

Figure 3

Calculations were accomplished on an Apple IIe microcomputer. All integrations were approximated with Simpson's Rule using $n = 1000$ and $\pi = 3.1415926535$. The value of h had to be chosen so that $\int_A^B N(s)\, ds = 1$ and we found that $h = (1 - .997006378)/(B - A)$ is the necessary correction term for "chopping off the tails" from the standard normal curve. Simultaneously we also wanted the probability distribution model in (4) to have a standard deviation of

$\sigma = .2160$ (to agree with the historical variance rate of excess return on stocks) where $\sigma^2 = \int_A^B s^2 N(s)\,ds - \mu^2$. To achieve this the value of the constant α was numerically calculated to be $\alpha = .2183$. With these adjustments, the distribution $N(s)$ has a mean of .058 and a standard deviation of .2160 as required.

We now want to find the value of f, where $0 < f < -1/A$, such that the following integral is a maximum:

$$G(f) = \int_A^B \log(1 + fs)\,dN(s)$$
$$= \int_A^B (\log(1 + fs))\left[h + \frac{1}{\sqrt{2\pi\alpha^2}} e^{-(s-\mu)^2/2\alpha^2}\right] ds. \tag{5}$$

This time the integration that would be involved in setting $G'(f) = 0$ is non-elementary and cannot be done explicitly. Numerical work on a microcomputer was performed and we found that the maximum value of $G(f)$ occurs when $f^* = 1.17$ and the growth coefficient $G(f^*) = .0350444711$. The mean of the distribution is positive. Also, differentiating $G(f)$ with respect to f and examining the terms in the integrand, we find that

$$\lim_{f \to (-1/A)^-} G'(f) = -\infty;$$

so the uniqueness of f^* is guaranteed by Theorem 2.

Thus, taking into account the time value of money (but neglecting transaction fees and taxes), each year the Kelly-optimal investor should be willing to invest up to 100% of his/her resources in a diversified portfolio of S & P 500 stocks if no margin is permitted. But maximal average real growth will occur (should margin at the T-bill rate be available) if one invests 117% times current resources. Thus the long-term investor, each year, should be fully invested plus borrow to invest an additional 17% above available resources so that continued investments will achieve (asymptotically) maximal average growth relative to T-bills. (In the real world where margin costs exceed T-bill rates, if the extra costs are included in the computations, this percentage would be *somewhat less*.)

It would be interesting to know if $G(f) = 0$ on the interval $(0, -1/A)$ because —if so—then we would have some idea of the "chaotic ruin point" f_c, or the point beyond which margin becomes excessive and thus leads to inevitable ruin (i.e., loss relative to T-bills with a probability of 1). Direct examination of the limit

$$L = \lim_{f \to (-1/A)^-} \int_A^B \log(1 + fs) N(s)\,ds$$

is difficult, but we can obtain an upper bound. With

$$M = \operatorname{Max}(N(s)) = h + \frac{1}{\sqrt{2\pi\alpha^2}} \quad \text{on } [A, B],$$

then

$$L \le \lim_{f \to (-1/A)^-} \int_A^B (\log(1 + fs)) M\,ds$$
$$= M \lim_{f \to (-1/A)^-} \left[\left(s + \frac{1}{f}\right)\log(1 + fs) - s\right]_A^B$$
$$= M\left[A - B + (B - A)\log\left(1 - \frac{B}{A}\right)\right] = -0.51 < 0.$$

Thus $G(f) = 0$ has a unique solution $f_c \in (0, -1/A)$. Because the slope of the curve $G(f)$ versus f is very steep near $-1/A$, it becomes numerically difficult to locate f_c with great accuracy. Computer runs show this value to be *very close* to $-1/A$; in fact, $f_c = 1.69^+$. Thus for a hypothetically immortal investor continually wagering an amount greater than 1.7 times current resources, ruin is certain. Thus excessive use of margin is undesirable.

Before dashing out to become fully invested in stocks for a year, for a lifetime, or for all eternity, there are a few caveats that should be emphasized. Losses (relative to T-bills) *are possible* over the short-term. The mathematically inclined investor would do well to consider the tenable risks implied by Table 1 so that one has some measure of the likelihood that over some finite period stocks will underperform relatively "risk-free" T-bill earnings rates. The Kelly criterion does not address this issue.

Finally, it can be argued that the somewhat artificially constructed probability distribution $N(s)$ may not be fully taking into account: (i) recent expanded stock market volatility caused by program trading and the internationalization of financial markets, and/or (ii) some of the particularly disastrous exogenous events that might occur (such as a cataclysmic earthquake or a massive global recession). The numerical results we have obtained must be interpreted in light of the limitations inherent in any applied probabilistic model.

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A Simple Proof of Tychonoff's Theorem Via Nets

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1. INTRODUCTION. The Tychonoff theorem, a central theorem of point-set topology, states that the product of any family of compact spaces is compact. The current textbook literature contains three standard proofs of this theorem, all of which may be found in the classic text of Kelley [**8**]: the proof using Alexander's subbase theorem [**8**, Ch. 5, Th. 6, Th. 13]; the Bourbaki proof using ultrafilters [**8**, pp. 143–144]; and (at least implicitly) the proof using universal nets [**8**, p. 81, Ex. J]. Of these, the Bourbaki proof is the most popular; it can be presented very briefly without explicit mention of the theory of filters (cf. [**5**], [**8**]). However, it is difficult to motivate without a thorough study of filters. (See Munkres [**10**, pp. 229–234] for a very thoughtful elementary motivation of the Bourbaki proof.)

The aim of this note is to present a simple proof of Tychonoff's theorem (new, so far as I know) using only the basic theory of nets together with a straightforward application of Zorn's lemma.

For the convenience of readers who may not be familiar with the net theory of convergence in topological spaces, the next section summarizes the facts we need. The paper concludes with a few brief comments on the literature.

2. OUTLINE OF THE THEORY OF NETS. The topology of a metric space M is described by the sequences in M. In particular, M is compact provided that every sequence of points in M has a subsequence that converges in M. But one must generalize the notion of sequence to get a theory of convergence that is adequate for arbitrary topological spaces. The modern theory of generalized sequences, or *nets*, is due to Kelley [**6**]. Everything we need is proved in his book [**8**].

A *directed set* is a partially ordered set $(A, \leq)$ such that, given α and $\beta \in A$, there is some $\gamma \in A$ with $\alpha, \beta \leq \gamma$.

Example 1. The positive integers $\mathbb{N}$, directed by the usual order.

Example 2. Let X be a topological space, $p \in X$, and let $\mathcal{N}_p$ be the set of all neighborhoods of the point p. For $U, V \in \mathcal{N}_p$ let $U \leq V$ mean $V \subseteq U$. Then $W = U \cap V \geq U, V$; we say that $\mathcal{N}_p$ is "directed by reverse inclusion".

A *net* in a topological space X is a function x: $A \to X$, where A is any directed set. One says that the net x is based on A. Useful notation: write $x(\alpha)$ as x_α, and denote the net x by $\{x_\alpha$: $\alpha \in A\}$. This notation makes nets resemble sequences; of course a sequence is simply a net based on the directed set $\mathbb{N}$.

The net $\{x_\alpha$: $\alpha \in A\}$ *converges* to a point $p \in X$ provided that, given any neighborhood U of p, there is some $\alpha \in A$ such that, for all $\beta \geq \alpha$, $x_\beta \in U$. (The

limit p is unique if X is a Hausdorff space.) Easy consequence: a subset S of X is closed if and only if the limit of any convergent net of points of S is also in S. This shows that nets are indeed adequate to describe the topology of X.

A point $q \in X$ is a *cluster point* of the net $\{y_\alpha: \alpha \in A\}$ provided that, given any neighborhood U of q and any $\alpha \in A$, there is some $\beta \geq \alpha$ with $y_\beta \in U$. Example: given a *sequence*, suppose that q is a limit of some subsequence; then q is a cluster point of the original sequence.

The most subtle concept in the theory is that of a *subnet*. First, consider two directed sets A and B. A map $\phi: B \to A$ is *cofinal* provided that, given $\alpha \in A$, there exists $\beta \in B$ so that, for every $\beta' \geq \beta$, we have $\phi(\beta') \geq \alpha$. Now let $x = \{x_\alpha: \alpha \in A\}$ be a net in X based on A. If $\phi: B \to A$ is a cofinal map, then the composition $x \circ \phi = \{x_{\phi(\beta)}: \beta \in B\}$ is a net based on B; we say that $x \circ \phi$ is a *subnet* of the net x.

The following result is important because it relates cluster points to subnets.

Proposition. *A point p in X is a cluster point of a net x if and only if there is a subnet of x which converges to p.*

Finally, we require the characterization of compactness in terms of nets.

Theorem. *A topological space X is compact if and only if every net in X has a subnet which converges in X. Equivalently, every net in X has a cluster point.*

3. PROOF OF TYCHONOFF'S THEOREM. Let $\{X_i\}_{i \in I}$ be an indexed family of compact topological spaces. We may assume that these spaces are all non-empty. Recall that the product $\prod_{i \in I} X_i = X$ consists of all functions f defined on the index set I, such that, for each $i \in I$, $f(i) \in X_i$. A *basic neighborhood* N of f in the product topology is determined by a finite subset $F \subseteq I$, together with neighborhoods U_j of $f(j)$ in X_j for each $j \in F$; N consists of all $h \in X$ such that, for all $j \in F$, $h(j) \in U_j$. It will be convenient to say that N is *supported* on F, and to write $N = N\{U_j : j \in F\}$.

By a *partially defined member* g of the product X we mean a function g with domain $J \subseteq I$, such that, for all $i \in J$, $g(i) \in X_i$. (That is, $g \in \prod_{i \in J} X_i$.)

Let $\{f_\alpha: \alpha \in A\}$ be a net in the product space X. Suppose that g, with domain $J \subseteq I$, is a partially defined member of X. Then we say that g is a *partial cluster point* of the given net provided that, given $\alpha \in A$, for every finite set $F \subseteq J$ and every basic neighborhood $N\{U_j: j \in F\}$ of g in $\prod_{i \in J} X_i$, there exists $\beta \in A$, $\beta \geq \alpha$, such that, for all $j \in F$, $f_\beta(j) \in U_j$. (In other words, g is a cluster point in $\prod_{i \in J} X_i$ of the net $\{f_\alpha \restriction J: \alpha \in A\}$.) If g has domain $J = I$, then g is a cluster point in X of the net $\{f_\alpha: \alpha \in A\}$. Our aim is to show the existence of such a g, using Zorn's lemma.

To this end, let $\mathscr{P}$ be the set of all partial cluster points of the given net $\{f_\alpha: \alpha \in A\}$. Note that $\mathscr{P}$ is non-empty because the empty function $\varnothing \in \mathscr{P}$. Partially order $\mathscr{P}$ by inclusion (extension of functions). That is, $g_1 \subseteq g_2$ provided that the domain of g_1 is contained in that of g_2, and g_2 agrees with g_1 on their common domain.

Suppose that $\mathscr{L} = \{g_\lambda: \lambda \in \Lambda\}$ is a linearly ordered subset of $\mathscr{P}$. Define $g_0 = \bigcup_{\lambda \in \Lambda} g_\lambda$. Then g_0 is a partially defined member of X, because any two members of $\mathscr{L}$ agree on their common domain. Moreover $g_0 \in \mathscr{P}$, i.e. g_0 is a partial cluster point of the net $\{f_\alpha: \alpha \in A\}$. This is immediate from the fact that every basic neighborhood of g_0 has finite support F, and so F is contained in the domain of g_λ for some $\lambda \in \Lambda$, and this g_λ is a partial cluster point. Accordingly

$g_0 \in \mathscr{P}$ and g_0 is an upper bound for $\mathscr{L}$. Thus $\mathscr{P}$ satisfies the hypothesis of Zorn's lemma.

Therefore $\mathscr{P}$ contains a maximal member g. We assert that the domain J of g is all of I. If this is not the case choose $k \in I \setminus J$. Now g is a cluster point in $\prod_{i \in J} X_i$ of the net $\{f_\alpha \restriction J\colon \alpha \in A\}$ and therefore g is the limit of some subnet $\{f_{\varphi(\beta)} \restriction J\colon \beta \in B\}$. Moreover, since X_k is compact and non-empty, the net $\{f_{\varphi(\beta)}(k)\colon \beta \in B\}$ has a cluster point $p \in X_k$. Define a function h with domain $J \cup \{k\}$ by setting $h = g$ on J and $h(k) = p$. Then it is clear that h is a partial cluster point of the net $\{f_\alpha\colon \alpha \in A\}$, so that $h \in \mathscr{P}$ and h is strictly larger than g. This contradicts the maximality of g in $\mathscr{P}$. Hence the domain of g is I, g is a cluster point of the net $\{f_\alpha\colon \alpha \in A\}$, and the proof that X is compact is done.

4. COMMENTS ON THE LITERATURE. Tychonoff [**12**] originally proved that an arbitrary product of compact intervals is compact. The general theorem is due to Čech [**4**, p. 830]. The "Bourbaki" ultrafilter proof is given by H. Cartan [**3**]. A form of the "universal net" proof is in Tukey's thesis [**11**, p. 36, p. 75]; the modern version is Kelley's [**6**].

All proofs of the general Tychonoff theorem involve some form of the axiom of choice: this follows from Kelley's well-known result [**7**]. In [**9**] P. Loeb carefully discusses the role of the axiom of choice, and presents a fairly straightforward proof of Tychonoff's theorem which avoids the axiom of choice in certain special cases.

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Optimal Strategies for a Generalized "Scissors, Paper, and Stone" Game

David C. Fisher and Jennifer Ryan

In the game of *Scissors, Paper, and Stone*, two players together chant "one, two, three" (see Figure 1). On the count of three, they independently select either "Scissors" (shown by a "V" formed with the index and middle fingers), "Paper" (shown by extending all fingers) or "Stone" (shown by a clenched fist). If both players pick the same object, the game is tied. Otherwise, a player picking Scissors beats a player picking Paper (Scissors "cut" Paper), but loses to a player picking Stone (Stone "smashes" Scissors). A player picking Paper beats a player picking Stone (Paper "smothers" Stone). Williams [5] gives two similar games "for older children" (see Figure 2) which use five objects instead of three.[1]

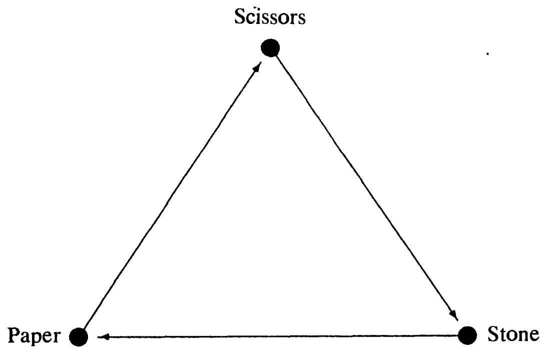

Figure 1. Graphical Model of the "Scissors, Paper and Stone" Game. When the players pick different objects, the winner is the one who picks the object at the head of the arc connecting the two objects. This shows that Scissors beats Paper, Paper beats Stone, and Stone beats Scissors.

Can these three games be generalized to games with any number of objects? For each pair of objects, choose one object to be the winner. This information can be represented as a directed graph with a node for each object and an arc between each pair of nodes pointing toward the winner. Directed graphs with an arc between each pair of nodes are called *tournaments* (see Moon [3] for a comprehensive introduction to tournaments). These games will be called *Tournament Games* (see Figure 3).

[1]Williams also reports that a game similar to Scissors, Paper and Stone is played in China where Humans eat Chickens, Chickens eat Worms, and Worms eat Humans.

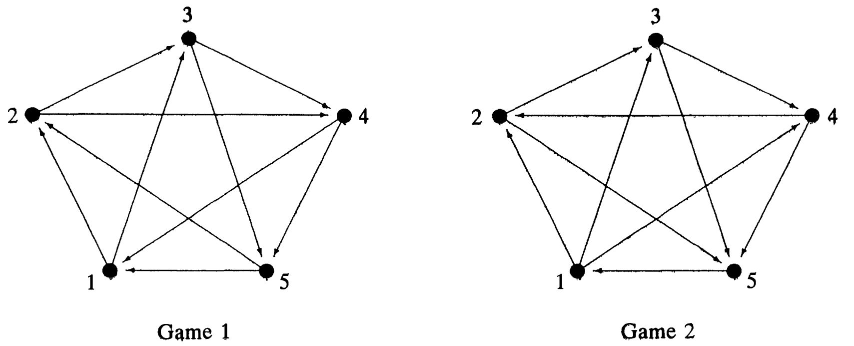

Figure 2. Two Games on Five Objects. In both these games, two players simultaneously choose one of five objects. If the objects are the same, the game is a tie. Otherwise, the player picking the object at the head of the arc wins.

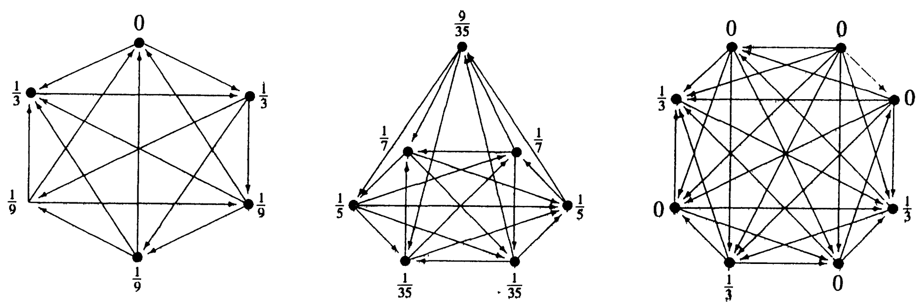

Figure 3. Tournament Games on Various Tournaments. In each of these games, two players simultaneously choose one of the nodes. If the nodes are the same, the game is a tie. Otherwise, the player picking the node at the head of the arc connecting the selected node wins. The nodes are labelled with the probability of playing that node in an optimal strategy. Note that in each game, the number of nodes with a nonzero probability is odd.

As with Scissors, Paper and Stone, tournament games are played many times. In each round, two players each pick an object without knowing the other player's selection. If both pick the same object, the game is tied. Otherwise, the player picking the winning object is declared the winner.

A natural question is "What are the optimal strategies for tournament games?" This article investigates these optimal strategies. In particular, the optimal strategy is shown to be unique. Interestingly, this optimal strategy always uses an odd number of nodes.

1. FINDING THE OPTIMAL STRATEGY. What are the optimal strategies for *Scissors, Paper and Stone*? Clearly, optimal strategies must use more than one object. For example, if one player (Player A) always played Scissors, the other player (Player B) could always win by playing Stone. Similarly, any deterministic strategy (for example, picking Scissors, then Paper, then Stone, and repeating) would allow B to predict A's next play.

Optimal strategies must be random in nature. Assume in each round, the loser pays the winner \$1 with no money exchanged for ties. Let p_1, p_2 and p_3 be the probabilities that Player A picks Scissors, Paper and Stone, respectively. Then

$p_3 - p_2$ is A's average winnings if B plays Scissors, $p_1 - p_3$ if B plays Paper, and $p_2 - p_1$ if B plays Stone. Since B will no doubt play to minimize A's average winnings, A wants to maximize $\min(p_3 - p_2, p_1 - p_3, p_2 - p_1)$ subject to $p_1 + p_2 + p_3 = 1$ and $p_1, p_2, p_3 \geq 0$. This maximum occurs when $p_1 = p_2 = p_3 = 1/3$. So Player A's optimal strategy is to pick each object one third of the time. The average winnings are then 0 which is not surprising since B can do just as well as A by adopting the same strategy.

What are the optimal strategies for the games in Figure 2? Each object in Game 1 beats the two objects immediately counterclockwise from it. By symmetry, an optimal strategy is to pick each object with probability 1/5. In Game 2, 5 beats 2, 3, and 4; 2, 3, and 4 beat 1; and 1 beats 5. This can be thought of as Scissors, Paper, and Stone where 5 is Scissors, 2, 3 and 4 are three types of Paper, and 1 is Stone. Thus 5 is picked 1/3 of the time , 2, 3 and 4 are together picked 1/3 of the time, and 1 is picked 1/3 of the time. Since 4 beats 3, 3 beats 2, and 2 beats 4, these objects are each picked 1/9 of the time. It is somewhat counterintuitive that even though 1 beats only one object, it is played more often than 2, 3 or 4 which each beat two other objects.

How does one find optimal strategies for tournament games on an arbitrary tournament? A strategy can be specified by a nonnegative vector indicating the probability of playing each node. For example, $\mathbf{p} = (1/2, 1/2, 0, \dots, 0)$ is the strategy that only plays nodes 1 and 2 each one half of the time. This could be implemented by flipping a fair coin with node 1 played if it is heads and node 2 played if it is tails. It would *not* be a good idea to simply alternate between node 1 and node 2 since this could be easily outwitted. Of course, if $\mathbf{p} = (p_1, p_2, \dots, p_n)$ is a strategy vector, then $p_1 + p_2 + \cdots + p_n = 1$ and $p_i \geq 0$ for all i. If we let $\mathbf{1}$ and $\mathbf{0}$ denote the vectors (of an appropriate length) whose components are all one and zero, respectively, then these constraints can be written as $\mathbf{1}^T\mathbf{p} = 1$ and $\mathbf{p} \geq \mathbf{0}$.

Tournament games are two person zero-sum matrix games (see Dresher [**1**]). Information about the game's outcome can be recorded in a matrix. For a tournament T on n nodes, let the *payoff matrix* of T, $K(T)$, be the $n \times n$ matrix whose ij element is

$$k_{ij} = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } (i, j) \text{ is an arc} \\ -1 & \text{if } (j, i) \text{ is an arc.} \end{cases}$$

For example, the payoff matrix for the first game in Figure 3 (labeling the nodes clockwise from the top) is

$$K(T) = \begin{bmatrix} 0 & 1 & -1 & -1 & -1 & 1 \\ -1 & 0 & 1 & 1 & 1 & -1 \\ 1 & -1 & 0 & 1 & -1 & 1 \\ 1 & -1 & -1 & 0 & 1 & 1 \\ 1 & -1 & 1 & -1 & 0 & 1 \\ -1 & 1 & -1 & -1 & -1 & 0 \end{bmatrix}.$$

If Player A plays strategy $\mathbf{p}$ and Player B plays node i, then A's average winnings are $(K(T)\mathbf{p})_i$. Thus, $\min_{i \in T}(K(T)\mathbf{p})_i$ is the average winnings when A plays strategy $\mathbf{p}$ and B plays a node minimizing A's winnings. Thus, Player A wants to find

$$v \equiv \max_{\substack{\mathbf{p} \geq \mathbf{0} \\ \mathbf{1}^T\mathbf{p} = 1}} \left(\min_i \left(K(T)\mathbf{p} \right)_i \right). \tag{1}$$

The maximum value in (1), v, is called the value of the game.[2] Since Player B can always make the expected winnings equal to 0 by using Player A's strategy, we have $v = 0$ for all tournaments. Knowing the optimal value, (1) can be simplified. Optimal strategies are any vector, $\mathbf{p}$, satisfying this system:

$$\begin{aligned} K(T)\mathbf{p} &\geq \mathbf{0} \\ \mathbf{p} &\geq \mathbf{0} \\ \mathbf{1}^T\mathbf{p} &= 1. \end{aligned} \tag{2}$$

2. UNIQUENESS OF THE OPTIMAL STRATEGY. In general, two person zero-sum games have many optimal strategies. However, each game in Figures 1, 2 and 3, has a unique optimal strategy. Further, these strategies each use (i.e., picks with a positive probability) an odd number of nodes. Do these observations hold for all tournaments? This section will show that they do.

Lemma 1. *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *be optimal strategies for a tournament game on a tournament*, T. *Then* $q_i > 0$ *implies* $(K(T)\mathbf{p})_i = 0$.

Proof: Since $\mathbf{p}$ and $\mathbf{q}$ satisfy (2), $K(T)\mathbf{q} \geq \mathbf{0}$ and $\mathbf{p} \geq \mathbf{0}$. Thus, $\mathbf{p}^T K(T)\mathbf{q} \geq 0$. Similarly, $\mathbf{q}^T K(T)\mathbf{p} \geq 0$. However, $\mathbf{p}^T K(T)\mathbf{q} = -\mathbf{q}^T K(T)\mathbf{p}$ because $K(T)^T = -K(T)$. Since $K(T)\mathbf{q} \geq \mathbf{0}$ and $\mathbf{p} \geq \mathbf{0}$, the result follows. □

What can be concluded from Lemma 1? If Player A plays an optimal strategy $\mathbf{p}$, then no matter what Player B plays, the best B can do is make A's average winnings equal to zero. Thus, some nodes will make A's average winnings zero, while others will make A's average winning positive. Lemma 1 says that the nodes used in an optimal strategy must be selected from those that make A's average winnings zero.

Let $\mathbf{p} = (p_1, p_2, \ldots, p_n)^T$ be an optimal strategy for a tournament game played as a tournament, T. Let S be the subtournament of T on the nodes where $p_i > 0$. Note that $K(S)$ is the submatrix of $K(T)$ restricted to the rows and columns corresponding to the nodes of S. Using Lemma 1 with $\mathbf{p} = \mathbf{q}$ puts an interesting condition on $K(S)$: $K(S)\mathbf{p}_S = \mathbf{0}$ where $\mathbf{p}_S$ is $\mathbf{p}$ restricted to the nodes of S.

Thus, optimal strategies for tournament games are always played on a subtournament satisfying a special property. Namely, its payoff matrix has a strictly positive null vector. We shall call such subtournaments *positive tournaments*.

Definition. A tournament, T, is *positive* if there is a positive vector, $\mathbf{p}$ with $K(T)\mathbf{p} = \mathbf{0}$.

If we can identify some properties of positive tournaments, the search for optimal strategies will be simplified. Corollary 1 below shows that positive tournaments must have an odd number of nodes. The tournament on one node is trivially a positive tournament. While there are 2 tournaments on three nodes, only one, the 3-cycle, is positive. Of the 12 tournaments on five nodes, the 2 chosen by

[2] Optimal strategies for tournament games on a tournament, T, can be efficiently found on a computer with a linear programming package. A good way to do this is to maximize v subject to $K(T)\mathbf{p} \geq v\mathbf{1}$, $\mathbf{1}^T\mathbf{p} = \mathbf{1}$, and $p \geq \mathbf{0}$.

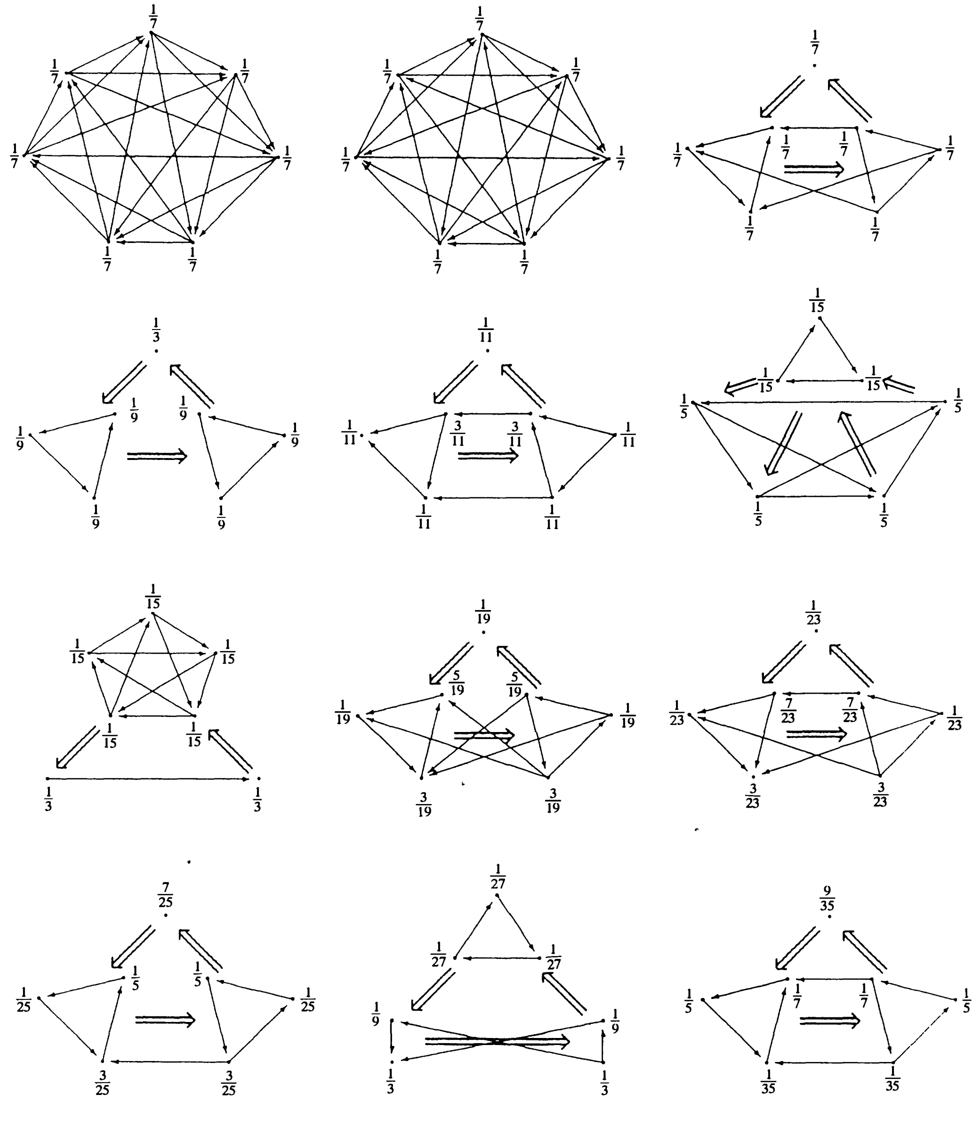

Figure 4. Positive Tournaments on 7 Nodes. The large arrows indicate the direction of arcs that are not explicitly shown (e.g., the lower right tournament is identical to the middle tournament of Figure 3). Next to each node is the probability of picking that node in the optimal strategy for the tournament game. These are the only 7-node tournament games for which the optimal strategy uses all 7 nodes.

Williams (see Figure 2) are the only positive ones. There are 12 positive tournaments (out of 456) on seven nodes. These are shown in Figure 4. There are 792 positive tournaments on nine nodes (out of 191,536 tournaments) and 886,288 positive tournaments on eleven nodes (out of 903,753,248 tournaments).

Moon [**3**] gives a formula (due to Davis) for the number of tournaments on n nodes. Theorem 1 gives an analogous formula for the number of positive tournaments on n nodes. Its proof (based on Burnside's Lemma) can be found in Fisher and Ryan [**2**]. Table 1 illustrates the use of Theorem 1 in finding the number of positive tournaments on 7 nodes.

TABLE 1. Theorem 1 is used to count the number of 7-node positive tournaments. Since $\frac{2^{15}}{5040} + \frac{2^7}{72} + \frac{2^3}{10} + \frac{2^5}{18} + \frac{2^3}{7} = 12$, there are twelve positive tournaments on 7 nodes. This verifies that the 7-node positive tournaments given in Figure 5 are the only ones.

d_1, d_3, d_5, d_7	$f(d_1, d_3, d_5, d_7)$	Summand
7, 0, 0, 0	$1 + \frac{1}{2}[7(-3 + 7 \cdot 1)] = 15$	$\frac{2^{15}}{7!1^7} = \frac{2^{15}}{5040}$
4, 1, 0, 0	$1 + \frac{1}{2}[4(-3 + 4 \cdot 1 + 1 \cdot 1) + 1(-3 + 4 \cdot 1 + 1 \cdot 3)] = 7$	$\frac{2^7}{4!1^4 1!3^1} = \frac{2^7}{72}$
2, 0, 1, 0	$1 + \frac{1}{2}[2(-3 + 2 \cdot 1 + 1 \cdot 1) + 1(-3 + 2 \cdot 1 + 1 \cdot 5)] = 3$	$\frac{2^3}{2!1^2 1!5^1} = \frac{2^3}{10}$
1, 2, 0, 0	$1 + \frac{1}{2}[1(-3 + 1 \cdot 1 + 2 \cdot 1) + 2(-3 + 1 \cdot 1 + 2 \cdot 3)] = 5$	$\frac{2^5}{1!1^1 2!3^2} = \frac{2^5}{18}$
0, 0, 0, 1	$1 + \frac{1}{2}[1(-3 + 1 \cdot 7)] = 3$	$\frac{2^3}{1!7^1} = \frac{2^3}{7}$

Theorem 1. *Let n be an odd number. Then the number of n-node positive tournaments is*

$$\sum \frac{2^{f(d_1, d_3, \ldots, d_n)}}{d_1! 1^{d_1} d_3! 2^{d_3} \cdots d_n! n^{d_n}}.$$

where $f(d_1, d_3, \ldots, d_n) = 1 + (1/2)\Sigma_{k=1,3,\ldots,n} d_k(-3 + d_1 \gcd(k, 1) + d_3 \gcd(k, 3) + \cdots + d_n \gcd(k, n))$ and the outer summation is over all nonnegative $d_1, d_3, \ldots, d_n$ with $d_1 + 3d_3 + 5d_5 + \cdots + nd_n = n$.

We now know that the optimal strategy is always played on a positive subtournament. What more can be said about positive tournaments? Information about positive tournaments will come from the structure of the payoff matrix. Theorem 2 can be generalized to any skew-symmetric matrix whose off-diagonal elements are odd integers.

Theorem 2. *Let T be a tournament on n nodes. Then*

$$\operatorname{rank}(K(T)) = \begin{cases} n & \text{if } n \text{ is even} \\ n - 1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof: Since the only zeroes in $K(T)$ lie on the diagonal, each nonzero term in the expansion of $\det(K(T))$ is in the form $k_{1j_1} k_{2j_2} \cdots k_{nj_n}$ where $i \neq j_i$ for $i = 1, 2, \ldots, n$. Thus, the number of nonzero terms equals the number of derangements (permutations where no element maps to itself) of $1, 2, \ldots, n$ (see Roberts [4] for a derivation):

$$D_n \equiv n!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right). \tag{3}$$

If n is even, then D_n is an odd number. Since every nonzero term in the expansion is either 1 or -1, $\det(K(T))$ is an odd number. Thus, $\det(K(T)) \neq 0$ and $\operatorname{rank}(K(T)) = n$.

Since $K(T)$ is skew-symmetric, we have that

$$\det(K(T)) = \det\left(K(T)^T\right) = \det(-K(T)) = (-1)^n \det(K(T)).$$

Thus if n is odd, $\det(K(T)) = 0$ and so $\operatorname{rank}(K(T)) < n$. Also any $(n-1) \times (n-1)$ principle submatrix of $K(T)$ is the payoff matrix of an even tournament. So rank $(K(T)) = n - 1$. $\square$

Since $K(T)\mathbf{p} = \mathbf{0}$ has a nonzero solution only if $\operatorname{rank}(K(T)) < n$, we have the following corollary.

Corollary 1. *Positive tournaments have an odd number of nodes.*

We now know that the optimal strategy will always be played on an odd number of nodes. If Player A is playing an optimal strategy, must Player B play the same one? In other words, is the optimal strategy unique? Theorem 3 shows that each tournament contains a unique positive subtournament that "beats" all other nodes.

Theorem 3. *The tournament game on an n node tournament T has a unique optimal strategy*, $\mathbf{p}$, *such that* $p_i > 0$ *on a positive subtournament* (*which must have an odd number of nodes*).

Proof: Let $\mathbf{p} = (p_1, p_2, \ldots, p_n)^T$ and $\mathbf{q} = (q_1, q_2, \ldots, q_n)^T$ be two solutions to (2). Let S be the subtournament of T on those nodes where either $p_i > 0$ or $q_i > 0$ (or both). Since both $\mathbf{p}$ and $\mathbf{q}$ are solutions to (2), Lemma 1 gives that the constraints corresponding to the nodes of S hold with equality. Hence, if $\mathbf{p}_S$ and $\mathbf{q}_S$ are the subvectors of $\mathbf{p}$ and $\mathbf{q}$ restricted to the nodes of S, then $K(S)\mathbf{p}_S = K(S)\mathbf{q}_S = \mathbf{0}$ Since $\mathbf{p}_S \neq \mathbf{0}$ and by Theorem 2, the null space of $K(S)$ has dimension at most one, $\mathbf{p}_S = \alpha \mathbf{q}_S$ for some nonzero constant α. Since $\mathbf{1}^T\mathbf{p}_S = \mathbf{1}^T\mathbf{q}_S = 1$, $\mathbf{p}_S = \mathbf{q}_S$ and hence $\mathbf{p} = \mathbf{q}$.

Since $K(S)p_S = \mathbf{0}$ and $p_S > \mathbf{0}$, S is a positive tournament. $\square$

Theorem 3 shows that exactly one of these infinite number of possibilities is true for any tournament game:

- There is one node that beats all others.
- There is a 3-cycle that beats all other nodes at least 2 out of 3 times (as in the third tournament in Figure 3).
- There is a regular subtournament on 5 nodes (like Game 1 in Figure 2) that beats all other nodes at least 3 out of 5 times.
- There is a subtournament like Game 2 in Figure 2 that beats all other nodes a majority of the time (e.g., in the first game in Figure 3, the top node is beaten 2/3 of the time by the optimal strategy).
- There is a positive subtournament on seven nodes (one of the tournaments in Figure 4) that beats all other nodes a majority of the time.
- There is a positive subtournament on nine nodes that beats all other nodes a majority of the time.
- Etc.

Those interested in a more comprehensive exposition on tournament games should read [**2**].

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Original Song of the Simple Group War

Oh, what are the orders of all simple groups?
I speak of the honest ones, not of the loops.
It seems that old Burnside their orders has guessed
Except for the cyclic ones, even the rest.

CHORUS: Finding all groups that are simple is no simple task.

Groups made up with permutes will produce some more;
For A is simple if n exceeds 4.
Then, there was Sir Matthew who came into view
Exhibiting groups of an order quite new.

Still others have come on to study this thing.
Of Artin and Chevalley now we shall sing.
With matrices finite they made quite a list
The question is: Could there be others they've missed?

Suzuki and Ree then maintained it's the case
That these methods had not reached the end of the chase.
They wrote down some matrices, just four by four.
That made up a simple group. Why not make more?

And then came the opus of Thompson and Feit
Which shed on the problem remarkable light.
A group, when the order won't factor by two
Is cyclic or solvable. That's what is true.

Suzuki and Ree had caused eyebrows to raise,
But the theoriticians they just couldn't faze.
Their groups were not new: if you added a twist,
You could get them from old ones with a flick of the wrist.

Still, some hardy souls felt a thorn in their side.
For the five groups of Mathieu all reason defied;
Not A, not twisted, and not Chevalley,
They called them sporadic and filed them away.

Are Mathieu groups creatures of heaven or hell?
Zvonimir Janko determined to tell.
He found out that nobody wanted to know;
The masters had missed 1 7 5 5 6 0.

The floodgates were opened! New groups were the rage!
(And twelve or more sprouted, to greet the new age.)
By Janko and Conway and Fischer and Held
McLaughlin, Suzuki, and Higman, and Sims.

No doubt you have noted the last lines don't rhyme.
Well, that is, quite simply, a sign of the times.
There's chaos, not order, among simple groups:
And maybe we'd better go back to the loops.

Originally appeared in *The American Mathematical Monthly*, 80 (1973), 1028. Sung to the tune of "Sweet Betsy From Pike." For additional verses see p. 945.

A Simple Example of Non-unique Factorization in Integral Domains

Scott T. Chapman

Let $\mathbb{Z}$, $\mathbb{Z}^*$, $\mathbb{Q}$, $\mathbb{Q}^*$, $\mathbb{R}$ and $\mathbb{C}$ represent the integers, the nonnegative integers, the rationals, the nonnegative rationals, the reals, and the complex numbers respectively. In a traditional abstract algebra course, the study of unique factorization domains (UFDs) plays a central role. The usual example of an integral domain presented to undergraduate students where unique factorization of elements into products of irreducible elements fails is the ring of algebraic integers $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$. In this domain 6 can be factored as the products

$$6 = 2 \cdot 3 \quad \text{and} \quad 6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

and an extended argument (usually using norms) is required to show that none of the elements 2, 3, $1 + \sqrt{-5}$, or $1 - \sqrt{-5}$ are units and that neither 2 nor 3 is an associate of either $(1 + \sqrt{-5})$ or $(1 - \sqrt{-5})$. Using the fact that the quadratic field $\mathbb{Q}[\sqrt{-5}]$ has class number 2 and the results of **[1]**, the following weaker factorization property of the integral domain $\mathbb{Z}[\sqrt{-5}]$ can be proved: if $\alpha_1, \dots, \alpha_s$, $\beta_1, \dots, \beta_t$ are irreducible elements of $\mathbb{Z}[\sqrt{-5}]$ such that $\alpha_1 \cdots \alpha_s = \beta_1 \cdots \beta_t$ then $s = t$. An integral domain in general which satisfies this property is known as a *half-factorial domain* (or HFD, see **[2]** or **[3]** for more information on such domains). The purpose of this note is to consider some alternate examples to the traditional one mentioned above in which the unique factorization property breaks down in a much more obvious manner. In addition, we will be able to use these domains to show in a simple manner that a finite set of elements in a general integral domain need not have a greatest common divisor.

Let R be a commutative ring and S any abelian monoid (we will consider the monoid operation here to be $+$). Set

$$R[X;S] = \left\{ \sum_{i=0}^{n} r_i X^{s_i} \,\middle|\, n \in \mathbb{Z}^*, \text{ and for } i \text{ with } 0 \le i \le n,\ r_i \in R \text{ and } s_i \in S \right\}.$$

The set $R[X;S]$ when supplied with the usual polynomial type addition and multiplication is commonly known as the semigroup ring of R over S (see Gilmer **[5]** as a general reference to semigroup rings). Notice that when $S \subset \mathbb{Z}^*$, $R[X;S]$ can be viewed as a subring of $R[X]$. If K is any field and S the submonoid of $\mathbb{Z}^*$ generated by 2 and 3 then notice that

$$K[X;S] = \left\{ \sum_{i=0}^{n} f_i X^{i} \,\middle|\, n \in \mathbb{Z}^*,\ f_i \in K \text{ for each } 0 \le i \le n, \text{ and } f_1 = 0 \right\}.$$

$K[X;S]$ can also be viewed as the extension of the field K by the indeterminates X^2, X^3 ($K[X^2, X^3]$). Elementary arguments show that the only units in $K[X;S]$ are the nonzero elements of K, and that the elements X^2 and X^3 are both irreducible. In this integral domain we have

$$X^6 = X^2 \cdot X^2 \cdot X^2 \quad \text{and} \quad X^6 = X^3 \cdot X^3$$

and a product of 2 irreducibles can be written as a product of 3 irreducibles. Hence $K[X;S]$ is neither a UFD nor an HFD.

Let n and m be positive integers such that $n < m$ and n does not divide m. By the correct choice of the monoid $S \subset \mathbb{Z}^*$, one can produce examples of elements for which factorizations into products of irreducibles can be produced of varying lengths. For instance, let $S = \{z_1 n + z_2 m | z_1, z_2 \in \mathbb{Z}^*\}$ be the submonoid of $\mathbb{Z}^*$ generated by n and m. For K any field we have that

$$K[X;S] = \left\{ \sum_{i=0}^{k} f_i X^i \middle| k \in \mathbb{Z}^*, f_i \in K \text{ for each } 0 \le i \le n, \right.$$
$$\left. \text{and } f_j = 0 \text{ for each } j \notin S \right\}.$$

Again, notice that $K[X;S]$ is equivalent to the extension $K[X^n, X^m]$ and the elements X^n and X^m are irreducible in $K[X;S]$. Hence

$$X^{mn} = \underbrace{X^n \cdots X^n}_{m \text{ times}} \quad \text{and} \quad X^{mn} = \underbrace{X^m \cdots X^m}_{n \text{ times}}$$

and a product of n irreducibles can be written as a product of m irreducibles (this example has appeared recently in [**4**]).

Now, consider the elements X^5 and X^6 in $K[X^2, X^3]$. We claim that a greatest common divisor of these elements in $K[X^2, X^3]$ does not exist. To see this, suppose that $d(X)$ is a greatest common divisor of these elements and let $\deg(d(X))$ represent the degree of $d(X)$ when viewed as a polynomial. Clearly $\deg(d(X)) \le 5$, and since no polynomial of degree 5 divides X^6, $\deg(d(X)) < 5$. Since X^2 and X^3 divide both X^5 and X^6, X^2 and X^3 divide $d(X)$. Hence $\deg(d(X)) \ge 3$. Since X^2 is not a proper divisor of any polynomial of degree 3 in $K[X^2, X^3]$, $\deg(d(X)) > 3$. Since X^3 is not a proper divisor of any polynomial of degree 4 in $K[X^2, X^3]$, degree of $d(X) \ge 5$, a contradiction.

In closing we note that the papers [**2**], [**3**], and [**4**] contain many interesting examples of how the UFD property can fail in an integral domain. Two of these examples which are of interest since their proofs rely on elementary techniques are: (i) the semigroup ring $\mathbb{C}[X;\mathbb{Q}^*]$ is an integral domain with no irreducible elements, and (ii) the subring

$$\mathbb{R} + X\mathbb{C}[X] = \left\{ \sum_{i=0}^{n} f_i X^i \in \mathbb{C}[X] \middle| f_i \in \mathbb{C} \text{ for } i \text{ with } 0 \le i \le n \text{ and } f_0 \in \mathbb{R} \right\}$$

of $\mathbb{C}[X]$ is an HFD which is not a UFD.

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Completion of Song on the Simple Group War
by Scott C. Radtke
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Modern Algebra Class Spring 1991

Wait! Don't give up, else all was for naught.
To order this chaos, a leader is sought.
A man with a vision for an overall plan.
Gorenstein's got this outlined, let's make him the man.

With more help from Thompson's fundamental technique,
The army of researchers, better weapons they seek.
They need some new insight, another approach.
Like a team in the field, they needed a coach.

Then Fischer came forward and geometrically preached,
Add insight from Aschbacher and the problem was breached.
Things now happened quickly, the beast has been tamed.
Classification's now imminent, only details remained.

Some details were dramatic, like the Monster of Griess.
A sporadic group that'll make you look twice.
In fact it's so big that it's given wide berth.
It's order is greater than the atoms of Earth.

When the battles were over, the War came to an end.
Commander-in-Chief Gorenstein, an announcement did send.
We've found all finite simple groups, we sure earned our wages.
The proof that it's true is some 10,000 pages.

Final Chorus: Finding all groups that are simple
is finished at last.

Bernoulli Numbers and Exact Covering Systems

John Beebee

In "Stirling's Series and Bernoulli Numbers" [**1**], professors Deeba and Rodriguez prove the following recurrence for the Bernoulli numbers B_m:

$$B_m = \frac{1}{n(1-n^m)} \sum_{k=0}^{m-1} n^k \binom{m}{k} B_k \sum_{j=1}^{n-1} j^{m-k}, \tag{1}$$

which is true for any positive integer m and any positive integer $n > 1$. I show there are infinitely many more such recurrences, but they are all characterized in the following theorem.

Theorem 1. *The set of arithmetic progressions*

$$A = \{b_j(\mathrm{mod}\, a_j) \colon 1 \le j \le n\}$$

is an exact covering system with $b_1 = 0$ and $0 \le b_j < a_j$ if and only if $\Sigma_{j=1}^{n} a_j^{-1} = 1$ and

$$B_m = \frac{1}{1 - \Sigma_{j=1}^{n} a_i^{m-1}} \sum_{k=0}^{m-1} \binom{m}{k} B_k \sum_{j=2}^{n} a_j^{m-1} \left(\frac{b_j}{a_j}\right)^{m-k} \tag{2}$$

for every positive integer m and any positive integer $n > 1$.

Let $b(\mathrm{mod}\, a)$ be the arithmetic progression $\{n = b + \alpha a \colon \alpha \in Z\}$. An exact covering system is a set A of (disjoint) AP's such that each integer belongs to exactly one AP. For example: $B = \{0(\mathrm{mod}\, n), 1(\mathrm{mod}\, n), \dots, (n-1)(\mathrm{mod}\, n)\}$ is an exact covering system, and when we substitute it into (2) we get (1). But, for example, $\{0(\mathrm{mod}\, 2), 1(\mathrm{mod}\, 4), 3(\mathrm{mod}\, 8), \dots, (2^{n-2}-1)(\mathrm{mod}\, 2^{n-1}), (2^{n-1}-1)(\mathrm{mod}\, 2^{n-1})\}$ is also an exact covering system with $n - AP$'s, and there is a superabundance of other examples [**2**]. If the offsets b_j of an exact covering system are chosen so that $0 \le b_j < a_j$ then exactly one offset is equal to zero. It will be assumed to be b_1. A finite set of disjoint AP's that covers the non-negative integers, or indeed the integers from 1 to $\mathrm{lcm}\{a_j \colon 1 \le j \le n\}$, automatically covers all of the integers. For any exact cover, $\Sigma_{j=1}^{n} a_j^{-1} = 1$. See [**3**].

In "A New Approach to Bernoulli Polynomials" [**4**], D. H. Lehmer proves that the n-th Bernoulli polynomial $B_n(t)$ is the unique monic polynomial of degree n which satisfies Raabe's multiplication identity

$$\frac{1}{n} \sum_{k=0}^{m-1} B_m\left(t + \frac{k}{n}\right) = n^{-m} B_m(nt).$$

I will use the following generalization of Raabe's identity to prove Theorem 1. Aviezri Fraenkel proved the case $t = 0$ in [**5**] and [**6**], and supplied the idea for the

general case. To see that it is a generalization of Raabe's identity, substitute exact cover B into (3).

Lemma. *For any number t,*

$$B_m(t) = \sum_{j=1}^{n} a_j^{m-1} B_m\left(\frac{t+b_j}{a_j}\right) \tag{3}$$

for every non-negative m if and only if $A = \{b_j (\mathrm{mod}\ a_j) : 1 \le j \le n\}$ is an exact covering system, with $0 \le b_j < a_j$.

Proof: Suppose $B_m(t) = \sum_{j=1}^{n} a_j^{m-1} B_m((t+b_j)/a_j)$ for $m \ge 0$ and $n > 1$. Recall that

$$\frac{xe^{tx}}{e^x - 1} = \sum_{k=0}^{\infty} \frac{x^k}{k!} B_k(t), \quad |x| \le 2\pi. \tag{4}$$

Then

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{x^k}{k!} B_k(t) &= \sum_{k=0}^{\infty} \frac{x^k}{k!}\left(\sum_{j=1}^{n} a_j^{k-1} B_k\left(\frac{t+b_j}{a_j}\right)\right)\\ &= \sum_{j=1}^{n} \frac{1}{a_j} \sum_{k=0}^{\infty} \frac{(xa_j)^k}{k!} B_k\left(\frac{t+b_j}{a_j}\right).\end{aligned}$$

By applying (4) to both sides of the above equation we get

$$\begin{aligned}\frac{xe^{tx}}{e^x - 1} &= \sum_{j=1}^{n} \frac{1}{a_j} \frac{(xa_j) e^{\frac{t+b_j}{a_j}(xa_j)}}{e^{(xa_j)} - 1}\\ &= \sum_{j=1}^{n} \frac{xe^{(t+b_j)x}}{e^{(xa_j)} - 1}.\end{aligned}$$

Divide both sides of the latter by xe^{tx} to get

$$\frac{1}{e^x - 1} = \sum_{j=1}^{n} \frac{e^{b_j x}}{e^{a_j x} - 1}.$$

Let $y = e^x$. Then

$$\frac{1}{1 - y} = \sum_{j=1}^{n} \frac{y^{b_j}}{1 - y^{a_j}}.$$

But the last equation can be interpreted as an equality between generating functions for the non-negative integers,

$$1 + y + y^2 + \cdots = \sum_{j=1}^{n} y^{b_j}(1 + y^{a_j} + y^{2a_j} + \cdots). \tag{5}$$

Thus each non-negative integer is expressed exactly once as $b_j + \alpha a_j$. Hence A is an exact cover with $0 \le b_j < a_j$.

For the converse, assume A is an exact cover. Then (5) holds, and we can reverse the steps in the above proof.

This lemma will now be used to prove Theorem 1.

Proof: Suppose (2) is true for every positive integer m. Recall that

$$B_m(t) = \sum_{k=0}^{m} \binom{m}{k} t^{m-k} B_k.$$

Then

$$\begin{aligned} B_m &= a_1^{m-1} B_m + \sum_{k=0}^{m} \binom{m}{k} B_k \sum_{j=2}^{n} a_j^{m-1} \left(\frac{b_j}{a_j} \right)^{m-k} \\ &= a_1^{m-1} B_m + \sum_{j=2}^{n} a_j^{m-1} \sum_{k=0}^{m} \binom{m}{k} \left(\frac{b_j}{a_j} \right)^{m-k} B_k \\ &= a_1^{m-1} B_m + \sum_{j=2}^{n} a_j^{m-1} B_m \left(\frac{b_j}{a_j} \right) \\ &= \sum_{j=1}^{n} a_j^{m-1} B_m \left(\frac{b_j}{a_j} \right). \end{aligned}$$

If we also have $\sum_{j=1}^{n} a_j^{-1} = 1$, then $B_0 = \sum_{j=1}^{n} a_j^{-1} B_0(b_j/a_j)$, so by the lemma, with $t = 0$, A is an exact cover.

The converse is proved by reversing the steps in the above proof.

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Mathematics is the art of giving the same name to different things.

—*Poincaré*

Picture Puzzle

What could this group be talking about?

(*See page* 969.)

> I learnt to associate mathematics, whether of yesterday or today, not just with definitions and theorems, algorithms and proofs, still less with masses of formulae, but with the creative minds of real people . . . I was taught to ask, not just what they achieved, but what they tried, and . . . the way they thought, even the errors they made.
>
> —*L. Young*

Collaborating editors: *David F. Appleyard, Paul T. Bateman, Bruce C. Berndt, Duane M. Broline, Barry W. Brunson, Frank S. Cater, Gulbank D. Chakerian, Underwood Dudley, Gerald A. Edgar, Michael A. Filaseta, Ira M. Gessel, Richard A. Gibbs, Douglas A. Hensley, John R. Isbell, Mourad E. H. Ismail, Murray Klamkin, Daniel J. Kleitman, Frederick W. Luttmann, Frank B. Miles, Richard Pfiefer, Stephen L. Portnoy, J. O. Shallit, John Henry Steelman, Kenneth B. Stolarsky, Douglas B. Tyler, Daniel Ullman, Edward T. H. Wang, and William E. Watkins.*

Answer to Picture Puzzle

(*on page* 949)

What except group theory? They are three of the world's greatest group theorists, Walter Feit, John Thompson, and Daniel Gorenstein.

Thanks

The Problems Section could not function without the efforts of many people, including our many referees. Each year we thank those who have contributed their time and talents. Thanks for your help.

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LETTERS

We Don't Seem To Get It

I believe the commentary in the June-July, 1992 issue of the *Monthly*, while well-intentioned (I know, I know) misses the main point. The spirit of the note is that we, in many of our public reports—specifically *Moving Beyond Myths*, are creating the perception that our undergraduate teaching is in trouble—more trouble than that of other departments. You clearly indicate that this perception makes you uncomfortable. Good! That's precisely the point. *Moving Beyond Myths* simply describes the reality of undergraduate mathematics instruction. The point is not whether we are (or are perceived to be) doing a better or worse job than our colleagues in other departments. The point is we are not doing as well as can and should.

If we are not willing to recognize and discuss our deficiencies then nothing will change. We are not politicians attempting to convince people that they should feel good about the economy despite high unemployment rates. The point of *Moving Beyond Myths* is to make us sufficiently uncomfortable so that we will act to change the reality, not simply the perception.

Moreover, despite the reports and increasingly distressing data, there is an enormous inertia and complacency in the system. Let me cite only one example. In our 1990 report, Gail Young and I pointed out that there are now more students taking advanced mathematics courses in non-mathematics departments than in mathematics departments. This fact was borne out most recently in the CBMS report showing only 7% of undergraduate mathematics enrollment beyond calculus (down from 9% five years ago and 11% 20 years ago). In response to our report, Gail and I received 290 letters—only six from mathematics faculty.

We don't seem to get it. Undergraduate mathematics instruction needs to change—for the good of our students and our profession. If we have to be made uncomfortable in order to spur change—so be it.

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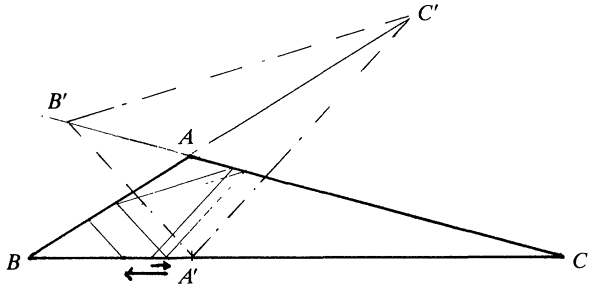
Right Triangle Orbits

The existence of a periodic orbit for a billiard ball in an arbitrary right triangle, posed as problem in Michael D. Boshernitzan, "Billiards and rational periodic directions in polygons," *Monthly* 99, No. 6, 522–529, is exhibited (as "purely periodic orbit") in my book *Plane Geometry and its Groups* (Holden-Day, San Francisco, 1967), Fig. 7–10 and the formula 5 lines down. The problem (periodic,

not purely periodic) for a general obtuse triangle can be treated using the directions of the orthic triangle. This needs continuity arguments that were outside the scope of my book. For this line of approach, I am enclosing a picture which in this case has to stand for rather more than 1000 words.

The billiard ball problem restricted to closed Jordan paths has been studied extensively by R. Sturm, Maxima und Minima in der elementaren Geometrie, Teubner, Leipzig-Berlin, 1910 (based on the same author's paper in Crelle's Journal 1884).



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Although the study of the history of mathematics has an intrinsic appeal of its own, its chief raison d'être is surely the illumination of mathematics itself. For example, the gradual unfolding of the integral concept—from the volume computations of Archimedes to the intuitive integrals of Newton and Leibniz and finally the definitions of Cauchy, Riemann and Lebesgue—cannot fail to promote a more mature appreciation of modern theories of integration.

—C. H. Edwards

PROBLEMS AND SOLUTIONS

Edited by:
Richard T. Bumby, Fred Kochman and Douglas B. West

Proposed problems should be sent to the MONTHLY PROBLEMS address given on the inside front cover. Please include solutions, relevant references, etc. Three copies are requested.

Solutions of published problems should arrive before May 31, 1993 at the MONTHLY PROBLEMS address given on the inside front cover. Solutions should be typed with double spacing, including the problem number and the solver's name and mailing address. Two copies suffice. A self-addressed postcard or label should be included if an acknowledgment is desired.

*An asterisk (*) after the number of a problem, or part of a problem, indicates that no solution is currently available. Partial solutions will be useful in such cases. Otherwise, the published solution is likely to be based on a solution which is complete and correct. Of course, an elegant partial solution or a method leading to a more general result is always useful and welcome. In addition, references to other appearances of MONTHLY problems or to solutions of these problems in the literature are also solicited.*

PROBLEMS

10265. *Proposed by Bjorn Poonen (student), University of California, Berkeley, CA.*

Let $a_1, \ldots, a_n, b_1, \ldots, b_n, \alpha$ be real numbers with $b_1, \ldots, b_n$ and α all positive. Prove

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i a_j}{(b_i + b_j)^{\alpha}} \geq 0.$$

10266. *Proposed by Daniel B. Shapiro and Patrick Rabau, The Ohio State University, Columbus, OH.*

Let L/K be finite algebraic extension of fields, and let T: $L \to L$ be a K-linear map. Then T will be said to have "Property G" if, for each polynomial $f \in K[x]$, the set of roots of f which lie in L are permuted by T.

(a) If K is infinite, show that T with property G must be a K-automorphism of L.

(b) Determine all examples of T with property G which are not K-automorphisms of L.

(c) What happens if L/K is an infinite algebraic extension.

10267. *Proposed by Lenny Jones and Mike Seyfried, Shippensburg University, Shippensburg, PA; and Stephen Schroer, Mercersburg Academy, Mercersburg, PA.*

Find all pairs of positive integers $\langle n, k \rangle$ such that the set of all kth powers of elements of the symmetric group S_n on n things is a proper subgroup of S_n.

10268. *Proposed by Ondrej Šuch (student), Queens University, Kingston, Ontario, Canada.*

Define a sequence $\langle a_n \rangle$ for $n \in \mathbb{N}$ by

$$a_0 = 3 \qquad a_1 = 0 \qquad a_2 = 2$$
$$a_{n+3} = a_{n+1} + a_n \quad (n \in \mathbb{N}).$$

If p is a prime, show that $p | a_p$.

10269. *Proposed by D. M. Bloom, Brooklyn College, CUNY, Brooklyn, NY.*

Prove that there is constant $K < 1$ with the following property. Let $\mathscr{G}$ be a regular $(2m + 1)$-gon inscribed in the unit circle, and let any point $P \in \mathscr{G}$ be given, then there are distinct vertices V_0 and V_1 of $\mathscr{G}$, such that

$$|d(P, V_0) - d(P, V_1)| \leq K/m.$$

10270. *Proposed by Marian Deaconescu, University of Timişoara, Timişoara, Romania.*

Prove that a finite group G has the property

$$N_G(H)/C_G(H) \cong \mathrm{Aut}(H)$$

for all subgroups H if and only if G is isomorphic to one of the groups S_n for $n \leq 3$.

10271. *Proposed by Victor I. Kostin, Institute of Mathematics, Novosibirsk, Russia.*

Let A be a skew-hermitian N by N matrix with N distinct eigenvalues. Let b be a column vector with nonzero projections on each eigenvector of A. Prove that all eigenvectors of the $(N + 1)$ by $(N + 1)$ matrix

$$\begin{bmatrix} A & b \\ -b^* & -1 \end{bmatrix}$$

have negative real parts.

10272. *Proposed by J. Marshall Ash and Leonid Krop, DePaul University, Chicago, IL.*

Show that $\sqrt[n]{2} + \sqrt[n]{3}$ is irrational for $n = 2$, 3, and 4 and find the minimal polynomials that these quantities satisfy.

10273. *Proposed by Jesús Ferrer, Universidad de Valencia, Burjasot, Spain.*

Let $\langle \mathscr{U}_n \rangle$ be a sequence of distinct ultrafilters on the set $\mathbb{N}$ of non-negative integers.

(a) Show that there is a sequence of disjoint sets $\langle A_k \rangle$ such that each A_k is an element of some $\mathscr{U}_n$.

(b) Show that there is $M \subset \mathbb{N}$ such that

$$\{n \in \mathbb{N} : M \in \mathscr{U}_n\} \quad \text{and} \quad \{n \in \mathbb{N} : M \notin \mathscr{U}_n\}$$

are both infinite.

NOTES

(10267) For each n, if e_n is the exponent of S_n (so that σ^{e_n} is the identity for all $\sigma \in S_n$), then the condition on k depends only on its residue class modulo e_n. The pairs $\langle n, 0 \rangle$ lead to the subgroup consisting only of the identity, which is not considered "proper." Thus, it suffices to determine the pairs of positive integers $\langle n, k \rangle$ with $k < e_n$ which satisfy the stated condition. **(10268)** There are examples of integers $n \geq 2$ with $n | a_n$ which are not prime. Contributions to a theory of such examples, or numerical results, are solicited. **(10269)** Problem A-5 on the 1989 Putnam examination asked a similar question with an upper bound of the form $1/m - A/m^3$. **(10270)** The basic notation of group theory is covered in introductory texts on Abstract Algebra. When H is a subgroup of G, $N_G(H) = \{x \in G : xyx^{-1} \in H \text{ for all } y \in H\}$ is called the "normalizer" of H in G and $C_G(H) = \{x \in G : xyx^{-1} = y \text{ for all } y \in H\}$ is called the "centralizer" of H in G. When $H = G$, the action $y \mapsto xyx^{-1}$ is generally referred to as an "inner automorphism," but there appears to be no convenient name for the object $N_G(H)/C_G(H)$ associated with a pair of groups $H \subset G$. **(10271)** Here M^* denotes the complex conjugate transpose (one of the two meanings of the word "adjoint" mentioned in problem 10205) of the matrix M, and A "skew-hermitian" means that $A^* = -A$. **(10272)** The authors suggested that the problem should also contain an explicit invitation to generalize the irrationality result obtained here. It should be sufficient to remind readers that such information is *always* welcome. **(10273)** A "filter" $\mathscr{F}$ on a set X is a proper collection of subsets of X closed under intersection with the property that if $A \in \mathscr{F}$ and $A \subset B$ then $B \in \mathscr{F}$. A filter which is not contained in any other filter is called "ultrafilter." Filters were introduced as an approach to convergence (see J. L. Kelley, *General Topology*), and have become a major tool in Model Theory (see C. C. Chang and H. J. Keisler, *Model Theory*). Further development of related ideas can be found in L. Gillman and M. Jerison, *Rings of Continuous Functions*.

SOLUTIONS

0-1-Matrices with Line-Sums Equal to 2

E3419 [1991, 55]. *Proposed by Marcin E. Kuczma, University of Warsaw, Warsaw, Poland.*

Let $F(n)$ be the number of n by n matrices with entries 0 or 1 and row and column sums equal to 2. Let $f(n) = n^{1/2}F(n)(n!)^{-2}$. Prove that $\lim_{n \to \infty} f(n)$ exists and has a value between 0 and 1.

Solution I submitted independently by about half of the respondents. The limit exists and has the value $(\pi e)^{-1/2}$. One can see this by first showing that the

generating function $1 + \sum_{n\geq 1} f(n)x^n n^{-1/2}$ equals $e^{-x/2}(1-x)^{-1/2}$. Then Darboux's Lemma applied to this function yields

$$f(n) = (n/e)^{1/2}\binom{n-1/2}{n} + O(n^{-1}),$$

so the claimed result then follows from Stirling's formula. The generating function is derived in numerous references, including [**1**, **4**, **7**, **8**]. Darboux's Lemma is stated and explained in [**9**].

Solution II by Richard Stong, University of California, Los Angeles, CA. We give a combinatorial derivation of the generating function for $F(n)\cdot(n!)^{-2}$. Given a matrix A of the type described, we define a 2-regular labeled multigraph whose vertices are the rows R_i of A. We draw an edge from R_i to R_j for each k with $A_{ik} = A_{jk} = 1$. (Thus, if rows i and j are the same, we have a double edge between R_i and R_j.) The resulting graph is a disjoint union of cycles, each of length at least 2. Let $c_i(A)$ count the cycles in the resulting graph by length, with $c'(A) = \sum_{i\geq 3} c_i(A)$ and $c(A) = c_2(A) + c'(A)$. Each of the $2^{c'(A)}$ orientations of the longer cycles yields a distinct derangement π with cycles of the same lengths as in the graph; we use the analogous notation to count cycles in permutations. If $D(A)$ is the collection of derangements arising from the matrix A, we have $\sum_{\pi\in D(A)} 2^{-c'(\pi)} = 1$.

If matrices A and A' differ only by a column permutation, then the resulting graphs are the same, and $D(A) = D(A')$. Furthermore, $D(A)$ determines the cycles of the graph, so if $D(A) = D(A')$, then A and A' differ only by a column permutation. Also, columns that produce a double edge (2-cycle) in the graph are identical, and interchanging them does not change A at all. Therefore, the number of matrices A that correspond to a particular derangement set $D(A)$ is $n!/2^{c_2(\pi)}$, for any $\pi \in D(A)$.

Let $D_{n,i} = \{\pi \in D_n : c_2(\pi) = i\}$, and let $\mathscr{A}$ denote the set of legal matrices. Then we have $F(N) = \sum_{A\in\mathscr{A}} 1 = \sum_{A\in\mathscr{A}}\sum_{\pi\in D(A)} 2^{-c'(\pi)} = n!\sum_{\pi\in D_n} 2^{-c(\pi)}$.

Consider a generating function for permutations defined as follows:

$$\Phi(y, x_1, x_2, \ldots) = 1 + \sum_{n\geqslant 1}\sum_{\pi\in S_n} \frac{y^n}{n!}\prod_{i=1}^{n} x_i^{c_i(\pi)}.$$

As is well known, we can re-express Φ as $\prod_{k\geq 0} e^{x_k y^k/k}$, because the contributions to the coefficient of $y^n/n!$ in the product of the expansions of the exponentials are $n!/(\prod k^{a_k} a_k!)$ for those choices of $\{a_k\}$ where $\sum k a_k = n$, and this is precisely the number of permutations with $c_k(\pi) = a_k$ for all k.

Since $F(n) = n!\sum_{\pi\in D_n} 2^{-c(\pi)}$, we obtain $F(n)/(n!)^2$ as the coefficient of y^n in $\Phi(y; 0, 1/2, 1/2, \ldots)$. Hence $\sum_{n\geq 0} F(n)y^n/(n!)^2 = \prod_{k\geq 2} e^{y^k/(2k)} = e^{-y/2}\exp(\sum_{k\geq 1} y^k/2k) = e^{-y/2}(1-y)^{-1/2}$. The asymptotic behavior of the generating function is obtained as in solution I.

Solution III by Nicolau C. Saldanha and Carlos Tomei, Pontifícia Universidade Católica, Rio de Janeiro, Brazil. We first obtain a recursive description of $F(n)$. Let $G(n)$ be the number of n by n matrices of 0's and 1's having exactly one nonzero entry in the first row and column and exactly two 1's in the remaining row and columns. For the matrices counted by $F(n)$, there are $\binom{n}{2}$ ways to place the 1's in the first row and $n-1$ ways to place the second 1 in the first column chosen, after which we have $G(n-1)$ ways to fill in the remaining 1's, so $F(n) = \binom{n}{2}(n-1)G(n-1)$. On the other hand, there are $F(n-1)$ matrices counted by

$G(n)$ that have a 1 in the upper left corner and $(n-1)^2G(n-1)$ that do not, so $G(n) = F(n-1) + (n-1)^2G(n-1)$. Together, this yields

$$F(n+1) = \binom{n+1}{2}\left[2F(n) + nF(n-1)\right].$$

Setting $b_n = 2(n+2)F(n+2)/(n+2)!^2$, the recurrence becomes $b_n = b_{n-1} + b_{n-2}/(2n)$ with $b_{-1} = 0$ and $b_0 = b_1 = 1$.

Editorial Comment. Note that $F(n)$ may be interpreted as the number of 2-regular labeled bipartite graphs with $2n$ vertices (and specified bipartition), which explains its interest in **[3]**. Other methods to estimate $F(n)$ occur in **[2, 3, 5]**. Many respondents derived a recurrence, much as in solution III above (the recurrence can also be obtained directly without the auxiliary $G(n)$), and then massaged it directly without generating functions to bound the limit in the desired range. The references given use a variety of methods, and they are highly instructive and highly recommended. The reader interested in asymptotics of generating functions should also examine the "transfer lemma" methods explained in **[6]**.

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Solved also by R. A. Agnew, D. Brown (Canada), D. Callan, R. J. Chapman (Great Britain), R. High, R. B. Israel (Canada), J. H. van Lint (The Netherlands), S. G. Penrice, C. Rousseau, A. Tissier (France), Central Michigan University Problem Group, National Security Agency Problems Group, and the proposer.

A Finite Radius of Convergence

6651 [1991, 169]. *Proposed by Richard L. Bishop and Lee A. Rubel, University of Illinois at Urbana-Champaign.*

Prove that the differential equation

$$z^4 \frac{d^2W}{dz^2} = W^4$$

has no non-constant entire solutions, but that, for every $R > 0$, it does have a non-constant solution analytic in $\{z : |z| < R\}$.

Solution I by Robert B. Israel, University of British Columbia, Vancouver, B.C., Canada. It is clear that any solution analytic in a neighborhood of 0 satisfies $W(0) = 0$. If W is not constant, $W(z) = z^n v(z)$ for some positive integer n and

function v analytic in a neighborhood of 0 with $v(0) \neq 0$. The differential equation becomes

$$n(n-1)z^{n+2}v + 2nz^{n+3}v' + z^{n+4}v'' = z^{4n}v^4.$$

But since $4n > n+2$, the only way to balance Taylor coefficients of z^{n+2} is to have $n = 1$, i.e. $W'(0) \neq 0$.

Note that the differential equation has a scaling symmetry: If $W(z)$ is a solution, then so is $U(z) = c^{-2}W(c^3 z)$ for any nonzero complex constant c (on the appropriate domain). Therefore we can conclude

(a) If there is a non-constant entire solution, then there is one with $W'(0) = 1$;

(b) If there is a non-constant solution that is analytic in a neighborhood of 0, then for any $R > 0$ there is a non-constant solution analytic in $\{z : |z| < R\}$.

I will show that there is a solution with $W'(0) = 1$ that is analytic in $\{z : |z| < R\}$ if $0 < R < 27/128$, but that no solution with $W'(0) = 1$ can be analytic in $\{z : |z| < 5\}$.

The existence result may be obtained from the Contraction Mapping Theorem. Taking $K = 4/3$ and $0 < R < 27/128$, let $\mathbf{X}$ be the complete metric space of analytic functions f on $\mathbf{D} = \{z\colon |z| < R\}$ such that $|f(z)| \le K|z|$, with the metric $d(f, g) = \sup\{|f(z) - g(z)|\colon z \in \mathbf{D}\}$. This choice of K and R yields the largest permissible R. For $f \in \mathbf{X}$, define

$$\Phi f(z) = z + \int_0^z (z - \zeta)\frac{f(\zeta)^4}{\zeta^4}\,d\zeta.$$

It is easily verified than any function fixed by Φ must satisfy $z^4 f''(z) = f(z)^4$ and $f'(0) = 1$.

Note that Φf is analytic in $\mathbf{D}$, with

$$|\Phi f(z)| \le |z| + \int_0^{|z|}(|z| - t)K^4\,dt = |z| + \frac{K^4|z|^2}{2} \le K|z|$$

if $1 + K^4R/2 \le K$ (which is true for our choices of K and R). Thus Φ maps $\mathbf{X}$ into itself. For $f, g \in \mathbf{X}$ we have

$$\begin{aligned}|f(z)^4 - g(z)^4| &= |f(z) - g(z)|\,|f(z)^3 + f(z)^2g(z) + f(z)g(z)^2 + g(z)^3|\\ &\le 4K^3|f(z) - g(z)|\,|z|^3 \le 4K^3\,d(f,g)\left|\frac{z^4}{R}\right|\end{aligned}$$

(using Schwarz's Lemma in the last step), so that

$$\begin{aligned}|\Phi f(z) - \Phi g(z)| &\le \frac{4K^3}{R}\,d(f,g)\int_0^{|z|}(|z| - t)\,dt\\ &= \frac{2K^3}{R}\,d(f,g)|z|^2 \le 2K^3R\,d(f,g)\end{aligned}$$

i.e. $d(\Phi f, \Phi g) \le 2K^3R\,d(f, g)$. If $2K^3R < 1$ (which is true for our choices), Φ is a strict contraction on $\mathbf{X}$, and therefore it has a fixed point $W(z)$.

Now suppose W is a solution in a neighborhood of the real interval $[0, R)$ with $W'(0) = 1$. We have $W'' \ge 0$ on this interval, so $W'(z) \ge 1$ and $W(z) \ge z$ there. From the differential equation we get $W'' \ge 1$, so $W'(z) \ge 1 + z$ and $W(z) \ge z + z^2/2$ on the interval. Now let $P(z) = W'(z) - W'(z)^{3/2}/3$. I claim that $P(z) > 0$ on $[0, R)$. If this is true, then $(d/dz)W^{-1/2} = -W^{-3/2}W'/2 < -1/6$ on $(0, R)$. Since (if $R > 2$) $W(2) \ge 4$, this would force $W^{-1/2}$ to hit 0 before $z = 5$. The conclusion is that we must have $R < 5$.

Proof of claim: Since $P(0) = 1 > 0$, is this were false there would be some $t \in (0, R)$ with $P(t) = 0$ and $P'(t) \le 0$. Now $0 \ge P'(t) = W''(t) - \frac{1}{2}W(t)^{1/2}W'(t) = t^{-4}W(t)^4 - \frac{1}{6}W(t)^2$, which would mean $W(t) \le t^2/\sqrt{6}$, contradicting the inequality $W(z) \ge z + z^2/2$ on our interval.

Solution II by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. If we substitute the power series $\sum_{k=0}^{\infty} a_k z^k$ for $W(z)$, it immediately follows that $a_0 = 0$. Furthermore, for $k \ge 2$,

$$k(k-1)a_k = \sum \{a_p a_q a_r a_s : p, q, r, s > 0; p + q + r + s = k + 2\}. \quad (1)$$

We conclude that all coefficients are uniquely determined by a_1 and are positive if a_1 is positive. It is also seen from (1) that a_k is a monomial of degree $3k - 2$ in a_1.

By induction we will prove: if $a_1 = 2$ then $a_k > 2$ for all $k > 1$. The number of terms in the summation in (1) is equal to $\binom{k+1}{3}$, so

$$k(k-1)a_k > \binom{k+1}{3}2^4,$$

which clearly implies $a_k > 2$ if $k > 1$. We conclude that the radius of convergence is at most 1 in this case.

On the other hand if $a_1 = \frac{1}{2}$ then we have $a_1 + a_2 + \cdots + a_{k-1} < 1 - 1/(k-1)$ for $k - 1 \ge 3$. This is easily checked for $k - 1 = 3$, and with induction it follows from (1) that

$$k(k-1)a_k \le (a_1 + a_2 + \cdots + a_{k-1})^4 < 1;$$

hence, $a_k < 1/(k-1)/k$. So in this case the radius of convergence is at least 1, and thus the radius of convergence of every non-constant solution is nonzero.

Editorial Comment. Let $W(z)$ be a formal solution $W'(0) = 1$. Richard Stong showed the Taylor coefficients of W are dominated by those of an explicitly constructed D satisfying the polynomial equation $D^4 - z^2D + z^3 = 0$. Therefore D, and hence W, has non-zero radius of convergence. $D(z)$ satisfies a recurring similar to (1) with the factor $n(n-1)$ replaced by 1.

S. G. Merzlyakov solved the same problem for the more general equation $z^rW'' = W^k$, where r and k are integers with $r \ge 1$, $k \ge 3$, and $k \ge r$. His method was similar to Solution I. He also noted the same result is valid for

$$z^rW^{(l)} = W^k,$$

where

$$k(l-1) < 4 \le k(l-1)$$

and

$$2 \le l \le k - 1.$$

Several solvers noted that the simple expression

$$W(z) = -\frac{\sqrt[3]{6}}{3}z^{2/3}$$

satisfies the differential equation. However, this function has a branch point at $z = 0$, and hence cannot be analytic in a disk centered at that point.

Solved also by L. N. Howard, T. M. McDonald, S. G. Merzlyakov (Russia), R. Stong, D. B. Tyler, and the proposers. One incorrect solution was received.

6653 [1991, 273]. *Proposed by D. K. Lee, University of Ulsan, Korea, and B. A. Murray, University of Newcastle-upon-Tyne, UK.*

For non-negative real a evaluate the integral

$$I(a) = \int_0^\pi \psi(\theta) \sin\theta \, d\theta,$$

where $\psi(\theta) = \arccos(\{a - \cos\theta\}/\sqrt{1 + a^2 - 2a\cos\theta}),\ 0 \le \psi(\theta) \le \pi$.

Solution I by Douglas B. Tyler, California State University—Dominguez Hills, Carson, CA. We prove

$$I(a) = \begin{cases} \pi(1 - a/2) & \text{if } 0 \le a \le 1 \\ \pi/(2a) & \text{if } 1 \le a. \end{cases}$$

One quickly checks that $\psi(\theta) = \arccos(-\cos\theta) = \pi - \theta$ when $a = 0$ and $\psi(\theta) = \arccos(\sin(\theta/2)) = (\pi - \theta)/2$ when $a = 1$. Thus integrating by parts gives

$$2I(1) = I(0) = \int_0^\pi (\pi - \theta)\sin\theta \, d\theta = -(\pi - \theta)\cos\theta\Big|_0^\pi + \int_0^\pi - \cos\theta \, d\theta = \pi.$$

From here on we assume that $a > 0$ and $a \neq 1$. Integration by parts yields

$$I(a) = \int_0^\pi - \psi(\theta)\, d(\cos\theta) = -(\psi(\theta)\cos\theta)\Big|_0^\pi + \int_0^\pi \frac{\cos\theta(a\cos\theta - 1)}{1 + a^2 - 2a\cos\theta}\, d\theta \quad (1)$$

$$= \psi(\pi) + \psi(0) + \int_0^\pi \left(\frac{\cos\theta}{-2} + \frac{1 - a^2}{4a} + \frac{a^4 - 1}{4a} \cdot \frac{1}{1 + a^2 - 2a\cos\theta} \right) d\theta$$

$$= \arccos\left(\frac{a - 1}{|a - 1|} \right) + \frac{\pi(1 - a^2)}{4a} + \frac{a^4 - 1}{4a} \int_0^\pi \frac{d\theta}{1 + a^2 - 2a\cos\theta}. \quad (2)$$

The Weierstrass transformation, $z = \tan(\theta/2)$, applies to the remaining integral. One has $\cos\theta = (1 - z^2)/(1 + z^2)$ and $d\theta = 2\,dz/(1 + z^2)$. Thus

$$\int_0^\pi \frac{d\theta}{1 + a^2 - 2a\cos\theta} = \int_0^\infty \frac{2\,dz}{(1 + a^2)(1 + z^2) - 2a(1 - z^2)}$$

$$= \int_0^\infty \frac{2\,dz}{(1 + a)^2 z^2 + (1 - a)^2}$$

$$= \frac{2}{1 - a^2} \arctan\left(\frac{1 + a}{1 - a} z \right)\Big|_0^\infty = \frac{\pi}{|a^2 - 1|}.$$

Now, substitute this result into (2). A separate analysis of $a > 1$ and $a < 1$ then leads to the stated value for $I(a)$.

Solution II by Richard Stong, University of California, Los Angeles, CA. The formula for $I(a)$ given in Solution I may be obtained using integrals in the complex plane as follows.

First note that $\psi(\theta) = \mathscr{I}(\log(a - e^{-i\theta}))$. Then

$$I(a)\frac{1}{2}\mathscr{I}\left(\frac{1}{i}\int_0^{\pi}\log(a - e^{i\theta})(e^{i\theta} - e^{-i\theta})\,d\theta\right),$$
$$= \frac{1}{2}\mathscr{I}\left(\int_1^{e^{-i\pi}}\log(a - z)(z^{-2} - 1)\,dz\right), \tag{3}$$

where the integral in (3) is taken around a clockwise arc on the lower half of the unit circle. Now, for $a > 0$, integration by parts allows us to discover that

$$I(a) = -\frac{1}{2}\mathscr{I}\left(\left[(z - a + z^{-1} - a^{-1})\log(a - z) - z + a^{-1}\log z\right]_1^{e^{i\pi}}\right), \tag{4}$$
$$= \frac{1}{2}(2 - a - a^{-1})\arg(a - 1) + \frac{\pi}{2a}.$$

where $\arg(a - 1)$ is π if $a < 1$ and is 0 if $a > 1$; if $a = 1$, then the first term is zero and the ambiguity in $\arg(a - 1)$ is irrelevant. This gives the value for $I(a)$ stated at the start of Solution I.

If $a = 0$, a different application of integration by parts yields

$$I(0) = -\frac{1}{2}\mathscr{I}\left((z + z^{-1})\log(-z) - z + \frac{1}{z}\Bigg|_1^{e^{i\pi}}\right) = \pi$$

as asserted.

Editorial comment. Readers supplied 17 essentially different solutions to this problem. Ian McGee and Cecil Rousseau (jointly) and Jean Anglesio provided elementary solutions similar to Solution I; eight additional solvers began this way, but evaluated the integral in (1) or (2) variously, using the substitution $z = \tan(\theta/2)$ and the partial fractions, contour integration along the unit circle and the residue theorem, Fourier cosine series, the change of variable $u = \psi(\theta)$ and inverse functions for ψ, or tables of integrals.

Using Fourier cosine series or contour integration and residue theory, four other solvers first showed that

$$\frac{d}{da}I(a) = \begin{cases} -\pi/2 & \text{if } 0 < a < 1 \\ -\pi/(2a^2) & \text{if } a > 1 \end{cases}$$

and then applied the Fundamental Theorem of Calculus.

The remaining nine solutions employed changes of variable, inverse-trigonometric identities, integrations by parts, partial fractions, Chebyshev polynomials of the second kind, the geometric series, the Euler beta function, reversal of integration order in iterated integrals, Cauchy's theorem, direct contour integration and the computer algebra system Macsyma in a variety of ways to obtain their results.

Arsène Fiegel displayed an infinite series for the answer when $a > 1$; equating this series to $\pi/2a$, one obtains the interesting formula

$$\sum_{k=1}^{\infty}\frac{(2k)!}{(k!)^2}\left(\frac{a}{a^2 + 1}\right)^{2k} = \frac{2}{a^2 - 1} \qquad (a > 1).$$

T. McCoy, Kim McInturff and Douglas B. Tyler extended the result to negative a, obtaining

$$I(a) = \begin{cases} \pi(1 - a/2) & \text{if } -1 \le a < 0 \\ 2\pi + \pi/(2a) & \text{if } a \le -1. \end{cases}$$

This result may also be obtained from (4) for $a \neq 0$.

When $a > 1$, $\psi(\theta) = \arctan(\sin\theta/(a - \cos\theta))$ and $I(a)$ with ψ in this form is known. (See I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products*, prepared by A. Jeffrey, Academic Press, 1980.)

Solved also by J. Anglesio (France), S.-J. Bang (Korea), C. Burger (Germany), R. J. Chapman (U.K.), P. Deiermann, M. Dresevic (Yugoslavia), A. Fiegel (France), W. Gao, H. Lipman, O. P. Lossers (The Netherlands), T. L. McCoy, I. McGee (Canada), & C. Rousseau, K. McInturff, R. Richberg (Germany), N. S. Thornber, Anchorage Math Solutions Group, National Security Agency Problems Group, and Western Maryland College Problems group. A partial solution ($a > 1$ only) was given by M. L. Glasser.

Nonsingular Magic Matrices

E 3440 [1991, 437]. *Proposed by William P. Wardlaw, U.S. Naval Academy, Annapolis, MD.*

Let A be a 3 by 3 magic matrix with real elements; i.e., there is a nonzero real number s such that each row of A sums to s, each column of A sums to s, the main diagonal of A sums to s, and the counter-diagonal of A sums to s.

(i) Show that if A is also nonsingular, then A^{-1} is magic.

(ii) Show that A has the form

$$\begin{pmatrix} s/3+u & s/3-u+v & s/3-v \\ s/3-u-v & s/3 & s/3+u+v \\ s/3+v & s/3+u-v & s/3-u \end{pmatrix},$$

where u and v are arbitrary, and nonsingular if and only if $v^2 \neq u^2$.

Solution by Jim Hartman, The College of Wooster, Wooster, OH.

(i) $$\text{Let } X = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Note that the conditions for A to be a magic matrix with line-sum s may be expressed by the following four properties:

$$\begin{aligned} &1)\ AX = sX, \\ &2)\ A^T X = sX, \\ &3)\ \operatorname{tr}(A) = s, \\ &4)\ \operatorname{tr}(EA) = s. \end{aligned}$$

Now, suppose A is nonsingular. Multiplying both sides of 1) by $s^{-1}A^{-1}$, we get $A^{-1}X = s^{-1}X$. Similarly, multiplying both sides of 2) by $s^{-1}(A^T)^{-1}$, we get $(A^{-1})^T X = (A^T)^{-1}X = s^{-1}X$. Note that s is an eigenvalue for A. If α and β are the others, then $s = \operatorname{tr}(A) = s + \alpha + \beta$, which yields $\alpha = -\beta$. Thus the eigenvalues for A^{-1} are s^{-1}, β^{-1} and $-\beta^{-1}$. Hence $\operatorname{tr}(A^{-1}) = s^{-1}$. Finally note that if A is magic with line-sum s, so is EA. Hence $\operatorname{tr}(EA^{-1}) = \operatorname{tr}(A^{-1}E) = \operatorname{tr}(A^{-1}E^{-1}) = \operatorname{tr}((EA)^{-1}) = s^{-1}$. Thus A^{-1} is magic with line-sum s^{-1}.

(ii) If we add the sum of the middle column, the sum of the middle row, the sum of the main diagonal, and the sum of the counter-diagonal, we get $4s$. But this is the sum of all the entries in A plus $3a_{22}$. Since the sum of all the entries in A is $3s$, we conclude that $a_{22} = s/3$. Letting $u = a_{11} - s/3$ and $v = a_{31} - s/3$, we can use the magic properties of A to see that A has the desired form.

Now $\det(A) = 3s(v^2 - u^2)$. As $s \neq 0$, we conclude that A is nonsingular if and only if $v^2 \neq u^2$.

Editorial comment. Most solvers observed that i) follows from the formula in ii) by a straightforward computation of A^{-1}, with several solvers using computer algebra software to assist in the calculations. John P. Robertson noted that this formula has appeared previously, citing Martin Gardner, *Riddles of the Sphinx*, New Mathematical Library, MAA, 1987, p. 137, and Maurice Kraitchik, *Mathematical Recreations*, 2nd edition, Dover, 1953, p. 148. John D. Eggers used this formula and induction to get a formula for A^n. His formula shows that if A is a 3 by 3 magic matrix with nonzero line-sum, then A^n is magic if and only if n is odd or A is singular. This result is a slight extension of Proposition 4.1 in Arno van den Essen, "Magic squares and linear algebra," this MONTHLY, 97 (1990), 60–62.

Several solvers noted that i) is false for 4 by 4 magic matrices. H. Turner Laquer provided the rather nice counter-example

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 2 \\ 2 & 2 & -1 & -1 \end{pmatrix}.$$

Interested readers are also directed to James E. Ward III, "Vector spaces of magic squares," *Math. Mag.*, 53 (1980), 108–111 for related results on magic matrices.

Solved also by 57 readers and the proposer.

REVIVALS

Irrational Series

E 2923 [1982, 63; 1985, 736]. *Proposed by P. Erdös, Hungarian Academy of Sciences, Budapest, Hungary and Claudia Spiro, University of Illinois, Urbana, IL.*

Let $1 < a_1 < a_2 < \cdots$ be an infinite sequence of integers. Prove that

$$\sum_{n=1}^{\infty} 2^{a_n}/a_n!$$

is irrational.

Note: Professor Wolfgang Walter of Universität Karlsruhe has pointed out to us that the solution published in 1985 contains a serious gap. Here is a correct solution.

Composite solution by Peter B. Borwein, Dalhousie University, Halifax, Nova Scotia, Canada; Michael Golomb, Purdue University, West Lafayette, IN; O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands; and John P. Robertson, Berwyn, PA. Let $\nu(n)$ denote the sum of the digits (i.e., the

number of ones) in the binary expansion of the positive integer n. Then an elementary theorem of Legendre asserts that $n! = 2^{n-\nu(n)}\beta(n)$, where $\beta(n)$ is odd. Clearly, $\beta(n)|\beta(n+1)$.

Let γ be the sum of the infinite series of the problem. Then $\gamma = \sum_{n=1}^{\infty}\delta_n 2^n/n!$, where $\{\delta_n\}_{n=1}^{\infty}$ is a sequence of zeros and ones in which one occurs infinitely often. Suppose $\gamma = h/k$, where h and k are positive integers. Put $k = 2^s t$, where t is odd. Let N be a power of 2 greater than $\max(t, 2^{s+2})$. Since t is an odd number less than N, it follows that $t|\beta(N)$ and so $2^s\beta(N)\gamma = \beta(N)h/t$ is an integer. Also, if

$$u_N = 2^s\beta(N)\sum_{n=1}^{N}\delta_n 2^n/n! = 2^s\sum_{n=1}^{N}\delta_n 2^{\nu(n)}\beta(N)/\beta(n),$$

then u_N is an integer, since $\beta(N)/\beta(n)$ is an integer for $n = 1, 2, \dots, N$. On the other hand, since we have assumed $\nu(N) = 1$, we have

$$\begin{aligned}
2^s\beta(N)\gamma - u_N &= 2^s\beta(N)\sum_{n=N+1}^{\infty}\delta_n 2^n/n! \\
&= 2^{s-N+1}N!\sum_{n=N+1}^{\infty}\delta_n 2^n/n! \\
&= 2^{s+2}\sum_{n=N+1}^{\infty}\delta_n 2^{n-N-1}\prod_{j=N+1}^{n} j^{-1} \\
&< \frac{2^{s+2}}{N+1}\sum_{n=N+1}^{\infty}\left(\frac{2}{N+2}\right)^{n-N-1} \\
&= \frac{2^{s+2}}{N+1}\,\frac{N+2}{N} < \frac{2^{s+3}}{N+1} < 1.
\end{aligned}$$

Thus $2^s\beta(N)\gamma - u_N$ is both a difference of two integers and a positive number less than one. This contradiction shows that the assumption $\gamma = h/k$ is untenable.

A slight modification of the above proof gives the more general result that

$$\sum_{n=1}^{\infty}(2/m)^{a_n}/a_n!$$

is irrational for any given positive integer m. The modification consists in multiplying the infinite series by $2^s\beta(N)m^N$ instead of by $2^s\beta(N)$.

Editorial comment. The flawed solution published in 1985 attempted unsuccessfully to prove the more general result that $\sum r^{a_n}/a_n!$ is irrational for any fixed positive rational number r. While this result may very well be true, the editors do not know how to prove it.

The gap in the 1985 solution occurs in the very last line. The displayed formula just preceding it shows that e^x is a rational number whose numerator is divisible by the prime number p. If e^x were actually an integer, then we could conclude that $p \leq e^x$, which was an essential step in the argument. However, it does not seem possible to prove that e^x is an integer in the context of the solution.

Solved also by R. Breusch, E. Butler, F. Dodd, G. Ehrlich, S. M. Gagola, Jr., M. F. Kruelle, L. M. Levine, J. M. Stark, K. L. Stellmacher, University of South Alabama Problem Group, and each of the two proposers.

Collaborating editors: *David F. Appleyard, Paul T. Bateman, Bruce C. Berndt, Duane M. Broline, Barry W. Brunson, Frank S. Cater, Gulbank D. Chakerian, Underwood Dudley, Gerald A. Edgar, Michael A. Filaseta, Ira M. Gessel, Richard A. Gibbs, Douglas A. Hensley, John R. Isbell, Mourad E. H. Ismail, Murray Klamkin, Daniel J. Kleitman, Frederick W. Luttmann, Frank B. Miles, Richard Pfiefer, Stephen L. Portnoy, J. O. Shallit, John Henry Steelman, Kenneth B. Stolarsky, Douglas B. Tyler, Daniel Ullman, Edward T. H. Wang, and William E. Watkins.*

Answer to Picture Puzzle

(*on page* 949)

What except group theory? They are three of the world's greatest group theorists, Walter Feit, John Thompson, and Daniel Gorenstein.

Thanks

The Problems Section could not function without the efforts of many people, including our many referees. Each year we thank those who have contributed their time and talents. Thanks for your help.

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REVIEWS

Edited by **Darrell Haile**
Indiana University, Bloomington, IN 47405

Numbers, by Ebbinghaus, Hermes, Hirzebruch, Koecher, Mainzer, Neukirch, Prestel and Remmert, Springer-Verlag, New York, 1990, xviii + 391 pp.

Reviewed by **T. Y. Lam**

The idea of writing a book on numbers is not new, but the idea of having eight authors team-write such a book perhaps is. *Numbers* is the product of the collaborative efforts of eight German authors and two editors, all well-known mathematicians. The original German edition appeared in 1983. The present English version, translated from the 1988 second German edition, is included in the "Readings in Mathematics" subseries of Springer's *Graduate Texts in Mathematics.*

A book on numbers can be many things, so perhaps I should first point out what this book is *not*. From the title *Numbers*, one might conjure up images of prime numbers, perfect numbers, pythagorean triples, diophantine equations, Fermat's Last Theorem, magic squares and the like. If you are trying to find information on any of these things, however, you'll be largely disappointed. In short, *Numbers* is not about number theory in the traditional sense. Rather, it is about *number systems*, namely, the systems of integers, rationals, real and p-adic numbers, complex and hypercomplex numbers, infinitesimals, cardinal and ordinal numbers, and finally, numbers and games. The goal of the book is to give a panoramic view of the development of the theory of number systems through time, whereby the reader will gain a broad perspective of that part of the mathematical culture engendered by the concept of (all kinds of) numbers. It has been said that numbers and figures are the"two wings of mathematics"; if this is so, the book under review would be relevant to a very large part of mathematical culture indeed.

A well-educated student in mathematics would no doubt have learned something about most of the number systems mentioned above. However, in the traditional undergraduate education, one learns about number systems in bits and pieces, and usually only as the need arises. Thus, we learn about $\mathbb{Z}$ and $\mathbb{Q}$ in a beginning algebra course, $\mathbb{R}$ and $\mathbb{C}$ in introductory courses on real and complex analysis, and cardinal, ordinal numbers perhaps in a first course on set theory. The average abstract algebra teacher would mention the quaternions as one (and most probably the only!) example of a noncommutative division ring, thereupon promptly abandoning the subject. Undergraduates from a strong department may have the good fortune to learn a bit more about hypercomplex numbers and p-adic numbers. But the theory of infinitesimals? It probably won't be taught in a department unless one of the professors is a card-carrying member of the school of nonstandard analysis. In graduate school, with our desire to write a thesis in minimum time dominating all else, we specialize all too quickly into our chosen mathematical nook, and have little time left to engage in the study of other

branches of mathematics beyond our own expertise. Thus, with the exception of those whose fields of specialty have to do with numbers, the average mathematician may know little more about number systems than was taught to him or her in undergraduate days.

Yet the number concept is a theme that has tied together different branches of mathematics for several millenia. Every student and practitioner of the science of mathematics would do well to become critically informed about the number systems—not only their technical, but also their cultural, historical, and epistemological aspects. In this context, *Numbers* makes excellent reading. Starting virtually with hieroglyphs for numbers from ancient Egypt, the fourteen chapters of the book contributed by the eight authors guide us systematically through the evolution of the number concept, from $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, to quaternions and octonions, to Cantor's transfinite numbers and Gödel's Incompleteness Theorem, to Robinson's hyperreal numbers and Conway's numbers and games. Richly textured with historical details and quotations from original sources, the book unfolds a wonderful pageant of events, ideas, viewpoints, controversies, failures and triumphs, foresights, hindsights and oversights which surround the "long march" of the concept of numbers. It is a lively story about a lively culture which is, in the words of the editor, "meant to entertain as well as to inform."

Is a complete theory of the irrational numbers to be found in Book V of Euclid's "Elements"? Or could it be true that propositions such as $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$ were never really fully proved before late 19th century? The exchange between Lipschitz and Dedekind on this point is thought-provoking. Cardano used complex numbers as early as 1545 to solve quadratic equations: was it a stroke of genius, or simply a matter of the end justifying the means? Leibniz was perhaps speaking more as a theologist than a mathematician when he referred to the complex numbers as a "subtle and wonderful refuge of the divine spirit". But how about Euler, who openly conceded that "square roots of negative numbers cannot be reckoned among the possible numbers", but was ever so remarkably adept at using the complex numbers in his great calculations? It took a Gauss to give the complex numbers their complete franchise in mathematics, but nowadays complex numbers are almost second nature to physicists, and to engineers in aeronautics, network analysis and communications sciences. Had Hamilton known about the concept of a $\mathbb{C}$-vector space (and its dimension), would he not have saved all the time he spent in finding a multiplication on $\mathbb{R}^3$ extending the complex multiplication in the plane? Should Hamilton have sole credit for discovering the quaternions, since, after all, Euler had discovered the 4-square identity almost a hundred years earlier, and Gauss, in 1819, had even set down explicitly (alas, in another unpublished paper!) the rule for composing two quadruples over the reals? The assessment of the role of quaternions in science makes for another point of controversy. Should we believe Thomas Hill who proclaimed that in the quaternions "there is as much real promise of benefit to mankind as in any event of Victoria's reign", or should we believe Lord Kelvin in whose opinion the quaternions "have been an unmixed evil to those who have touched them in any way"? The intriguing pathway along which the evolution of number systems took its course is naturally not without a few surprises. After Weierstrass so brilliantly laid the modern foundations for the theory of limits via ε's and δ's, it would certainly seem that those infamous "infinitesimals" used heuristically for two hundred years in calculus had been banished forever from rigorous mathematics. Who would have guessed that, another hundred years later, like a phoenix rising from its

ashes, these discredited infinitesimals would make a most spectacular comeback in the new subject of nonstandard analysis?

Readers interested in the discussion of these and related issues in a historical context will be amply rewarded by reading the book under review. But, unlike many other "popular" works written earlier on the subject, *Numbers* is intended to be a book on serious mathematics as well. Theorems are not only explicitly stated, but in most cases also carefully proved. This includes, for instance, the Fundamental Theorem of Algebra (with a survey of Gauss' four proofs), and such gems as Frobenius' Theorem on finite-dimensional associative real division algebras, the Gelfand-Mazur Theorem on commutative Banach division algebras, and the Kervaire-Milnor (1,2,4,8)-Theorem (the latter assuming Bott's Periodicity Theorem). As the editor says, this book is not for the faint-hearted: readers are expected to have pencil and paper in hand, to work through the mathematics presented. But the book has eminently succeeded in maintaining a fine balance between history and mathematics; open-minded readers stand to profit from the authors' expert treatment of both.

If I am allowed to do a little nitpicking, I must point out that there are quite a few typographical errors in the book, many occurring, unfortunately, in the names of mathematicians. Readers who are finicky about accuracy of names would probably not enjoy seeing spellings such as "Appolonius", "Grassman", "Malcav", or "Michael Stiefel" (not to mention "Adam's Theorem" about vector fields on spheres). A biographical entry such as "Benjamin Peirce (1809–1932)" would be sufficiently suspect to prompt the reader to check into another source (the former President of Harvard died in 1880). But it would be sad if a trusting reader quotes from the book the entry "Isaac Newton (1643–1727)" or the entry "Richard Dedekind (1831–1896)" (Newton was born in 1642, and Dedekind lived until 1916!). The list goes on; however, I am confident that all of these small nuisances will be eradicated from a future edition.

I believe a reviewer's job also includes making constructive suggestions if possible. On this front, I felt that some space in the book should have been devoted to a careful account of how the study of number systems led to the birth of modern abstract algebra. For instance, the study of algebraic integers and their factorizations led to Kummer's "ideal numbers", and subsequently to Dedekind's concept of an ideal. In the hands of E. Noether and W. Krull, this blossomed into modern-day commutative algebra. The axiomatization of the basic laws governing numbers led to the abstract notion of a field, upon which Steinitz created the modern theory of algebraic and transcendental field extensions. Modular arithmetic heralded the theory of finite fields, and The Fundamental Theorem of Algebra prompted the notion of an algebraically closed field. An effort to abstract the key properties of the real numbers led Artin and Schreier to their discovery of the theory of formally real fields and real-closed fields. We don't have to look far to see that a large part of 20th century abstract algebra had its deep roots in the number systems. However, in *Numbers*, this fact seemed to have been largely ignored. In this reviewer's opinion, another chapter detailing the vital role played by number systems in the creation and development of modern abstract algebra would have been a most fitting addition to the book, providing a crucial link from the past to the present.

In all other aspects, I found *Numbers* to be a thoroughly researched and superbly written opus. It tells an epic story with clarity, taste, and new insight. For amateurs and professionals alike, this is a wonderful book to read to develop an

understanding and appreciation of our mathematical heritage. It deserves to be a "must" for every college library.

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Galois Theory, by Joseph Rotman. Universitext, Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, Hong Kong, 1990, 155 × 233 mm, xii + 108 pp, ISBN 0-387-97305-2; ISBN 3-540-97305-2.

Reviewed by **Jean-Pierre Tignol**

Mention Galois theory to a mathematician, and you will get a shrewd nod. Just as algebra is quintessential mathematics, Galois theory is quintessential algebra. The reasons for this are manifold. First, Galois theory is at the same time the crowning achievement of the (algebraic) theory of equations, which was another name for algebra until the middle of the nineteenth century, and the cradle of group theory. In addition, it yields the solution of several time-honored problems, determining necessary and sufficient conditions for an equation to be solvable by radicals or for a regular polygon to be constructible by ruler and compass. Moreover, Galois theory lies at the foundation of several important branches of mathematics which sprouted during the nineteenth century, such as algebraic number theory or algebraic geometry (not to mention (modern) algebra), which have been very productive and are still very active. It has therefore become an unavoidable part of the standard mathematics curriculum. Finally, it is also quite appealing to the student because it mixes in a very efficient way such basic notions as groups and fields, giving very quickly an impression of depth; the romantic figure of Evariste Galois, impetuous teen-ager angrily arguing with his examiners and killed at 20 in a duel, may also contribute to the fascination.

Fascination there is; as Emil Artin once wrote[1]:

> Since my mathematical youth, I have been under the spell of the classical theory of Galois. This charm has forced me to return to it again and again, and to try to find new ways to prove these fundamental theorems.

Therefore, it comes as no surprise that the literature on Galois theory is very extensive. While it was still regarded as an advanced topic at the turn of the century, Galois theory worked its way into more and more elementary textbooks during the first decades of this century. Since its "modern" treatment in Van der Waerden's epoch-making *Moderne Algebra*, generation after generation of algebraists re-worked the theory, shifting the emphasis from polynomials to field extensions to group actions on fields to rings of endomorphisms to étale algebras, while the "fundamental theorem", setting up a one-to-one correspondence be-

[1] p. 380 in *Collected Papers* (S. Lang and J. Tate, eds.), Addison-Wesley, Reading, Mass. 1965.

tween intermediate fields of certain extensions and subgroups of their associated Galois groups remained the central result of the theory. Of course, radically new point of views are not frequent, but each year brings its yield of new books proposing new variations on the old Galois theme. No doubt that a true connoisseur could tell a mellow Artin 1942 (old, but gold!) from a mature Kaplansky 1969 or a Bourbaki nouveau.

The 1990 vintage, as represented by Rotman's book, is distinguished and sinewy. From the definition of a commutative ring to the fundamental theorem to solvability of equations by radicals in 65 pages, 80 theorems and 106 exercises. The exposition, which follows the now classical tradition of Artin's "Galois theory", is quite efficient, packing much material in a limited number of pages. True, the author resorts to a practice that some may deem unfair, consigning to exercises some details of proofs, but at least he does not slur over these details. Of course, no one would expect an encyclopedic treatise on field theory in 65 pages. Thus, as the author himself points out in the introduction (perhaps to lay his scruples to rest), a number of subjects have not found their place in the text, notably those relating to infinite extensions, such as transcendence degree or algebraic closure (but a nice algebraic proof of the fundamental theorem of algebra is included). The greatest asset of this book is its nice selection of topics, focusing on the fundamental theorem of Galois theory and its application to solvability of equations by radicals, but pausing to make excursions to finite fields or to work out explicitly some illuminating examples. The style is no-nonsense (except for exercise 106), crisp but not hurried.

The final 40 pages consist of appendices discussing group theory (to the extent it is used in the main part of the book), ruler and compass constructions (in more detail than usual) and old-fashioned Galois theory. This latter appendix deserves special mention, since it is not customary for a textbook of this size and scope to include such a detailed sketch of the historical motivations behind the theory it describes. One can only agree with the author when he wonders how such thoughts occurred to Galois in the late 1820's, and be grateful to him for providing his readers with material for an answer.

Some points are not above criticism, however. Abel and Ruffini would perhaps be surprised to see that their result on the insolvability of the general equation of degree 5 is interpreted as the existence of a particular equation of degree 5 with rational coefficients which is not solvable by radicals. But here is something more worrisome: would Gauss have become a mathematician, had he read Rotman's book[2] when he was 19? It has been told[3] that one of his earliest mathematical achievements, which was crucial in his decision to devote his life to mathematics, was the solution by radicals of the equation which yields the p-th roots of unity, for any prime p (and most notably for $p = 17$, where Gauss' solution shows that the regular polygon with 17 sides can be constructed by ruler and compass). Now, according to Rotman (p. 34), these equations are solvable by radicals *by definition*.

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[2]or almost any of the modern books on Galois theory, for that matter.

[3]see for instance p. 870 in: M. Kline, *Mathematical Thought from Ancient to Modern Times*, Oxford Univ. Press, New York, 1972.

TELEGRAPHIC REVIEWS

Edited by

Arnold Ostebee and Paul Zorn

with the assistance of

the Mathematics Departments of Carleton, Macalester, and St. Olaf Colleges

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Finite Mathematics, T. *Finite Mathematics with Applications, Second Edition.* David E. Zitarelli, Raymond F. Coughlin. Saunders College, 1992, xxiv + 641 pp, $40 net. [ISBN: 0-03-055864-6] New for this edition: 20% more problems, now divided into standard exercises, applications with references to the appropriate literature, and cumulative exercises. More word problems. Incorporates use of programmable/graphing calculators. (*First Edition*, TR, August-September 1989.) MC

Discrete Mathematics, T(14-15: 1). *Discrete Mathematics.* Melvin Hausner. Saunders College, 1992, xv + 720 pp, $41 net. [ISBN: 0-03-003278-4] Topics covered include logic, foundations, algorithms, combinatorics, graphs, trees, Boolean algebras, number theory, grammars and automata, Turing machines. Every chapter ends with a summary of key results and definitions. Answers to odd-numbered exercises in back. LC

Discrete Mathematics, T(13). *2000 Solved Problems in Discrete Mathematics.* Seymour Lipschutz, Marc Lars Lipson. Schaum's Solved Prob. Ser. McGraw-Hill, 1992, v + 404 pp, $16.95 (P). [ISBN: 0-07-038031-7] Short computational and theoretical problems, mainly elementary, on standard topics plus languages, grammars, and automata; ordered sets and lattices; Boolean algebra; and more theoretical treatment of algebraic structures. A few algorithms (e.g., on graph theory), few applications (except on logic circuits), no references. JPH

Linear Algebra, S(15-17). *Schaum's Outline of Theory and Problems of Linear Algebra, Second Edition.* Seymour Lipschutz. McGraw-Hill, 1991, vii + 453 pp, $12.95 (P). [ISBN: 0-07-038007-4] Main changes from the 1968 *First Edition* have been made for pedagogical reasons with little change in content. Some topics, such as elementary matrices and LU factorization, are now treated in the text rather than introduced only in problems. Index. JS

Algebra, T(18: 1), S, P. *The Cohomology of Groups.* Leonard Evens. Math. Mono. Clarendon Pr, 1991, xii + 159 pp, $39.95. [ISBN: 0-19-853580-5] For full comprehension and appreciation of results, the reader should have familiarity with homological algebra, finite group theory, commutative algebra. After foundations are developed discussion includes wreath products, the norm map, spectral sequences, variety theory. Exercises, references, index. JS

Calculus, T(13). *Calculus for Advanced Placement.* N.M. Haralambis. J Weston Walch, 1991, 223 pp, $9.95 (P) [ISBN: 0-8251-1879-4]; *Solution Guide,* vi + 161 pp, $13.95 (P). [ISBN: 0-8251-1880-8] The author's premise seems to be that advanced placement classes don't need all the detail of the current breed of calculus texts, but

only a concise statement of each concept and technique, followed by exercises much like those on AP tests. If author is right, the book is well written; concept of what a calculus course should be challenges all the 900+ page texts on the market today. TAV

Calculus, T(13: 1). *Calculus for the Management, Life, and Social Sciences, Third Edition.* Bernard Kolman, Charles G. Denlinger. Harcourt Brace Jovanovich, 1992, xv + 671 pp, $40. [ISBN: 0-15-505785-5] One semester version of calculus for non-mathematics majors. Each chapter includes brief review of ideas, supplemental exercises, and chapter test. (*First Edition*, TR, November 1981; *Second Edition*, TR, January 1989.) AD

Calculus, T(13-14: 1-4), L. *Calculus, Third Edition.* Dennis G. Zill. PWS-Kent, 1992, xxii + 1187 pp. [ISBN: 0-534-92793-9] Substantial revision of *Second Edition*. Changes include new material, rearrangement and rewriting of text, and new problems (especially applications and ones dealing with graphing calculators and computers). A "four-color" text. KS

Complex Analysis, T(16: 2). *An Introduction to Complex Function Theory.* Bruce P. Palka. Undergrad. Texts in Math. Springer-Verlag, 1991, xvii + 559 pp, $39. [ISBN: 0-387-97427-X] Theoretically-oriented text intended for mathematically talented students. Covers standard topics, but in more detail than most texts. Extra detail means some standard topics might not be reached in a semester. MPR

Differential Equations, T(16-17: 2). *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem.* Kenneth R. Meyer, Glen R. Hall. Appl. Math. Sci., V. 90. Springer-Verlag, 1992, xii + 292 pp, $49.80. [ISBN: 0-387-97637-X] Nice introductory text on Hamiltonian systems using the N-body problem as the central example. Focuses mainly on analytical aspects, but concludes with a chapter on twist maps and invariant curves. MPR

Dynamical Systems, S(18), P. *Chaotic Transport in Dynamical Systems.* Stephen Wiggins. Interdiscip. Appl. Math., V. 2. Springer-Verlag, 1992, xiii + 301 pp, $39.95. [ISBN: 0-387-97522-5] Dynamicists partition phase space of mathematical models into regions of qualitatively different motions. Boundaries of these regions need not be impermeable; the system may oscillate between regions or evolve through several. "Transport" is the author's term for passing from one region to another. He devises a mathematical treatment of it within classical dynamical systems and offers many physical examples. 88 exercises. SK

Functional Analysis, P. *Radon Integrals.* Bernd Anger, Claude Portenier. Progress in Math., V. 103. Birkhäuser, 1992, 332 pp, $75. [ISBN: 0-8176-3630-7] A unified approach to both the integration and set-theoretical aspects of measure theory based on two concepts: that of a regular linear functional on a function cone, and that of an upper functional as an abstract version of an upper integral. Radon integrals on an arbitrary Hausdorff space are introduced as regular linear functionals on a cone of lower-semicontinuous functions. DH

Analysis, S(18), P, L. *Measures and Differential Equations in Infinite-Dimensional Space.* Yu. L. Dalecky, S.V. Fomin. Math. & Its Applic., V. 76. Kluwer Academic, 1991, xv + 337 pp, $126. [ISBN: 0-7923-1517-0] Measures and quasimeasures, Gaussian measures on Hilbert space, Radon measures in linear topological space, differentiable measures and distributions, evolution differential equations, integration in path space, probabilistic representations of solutions of parabolic equations. Background needed in functional analysis, measure theory, probability theory, theory of partial differential equations. KS

Differential Geometry, P. *Feuilletages: Études géométriques.* Claude Godbillon. Progress in Math., V. 98. Birkhäuser, 1991, xiii + 474 pp, $138. [ISBN: 0-8176-2638-7] A comprehensive introduction to theory of foliations, in five long chapters, most with extensive appendices. Begins with basic definitions and standard examples; proceeds to research problems, recent results. Includes 75-page bibliography, organized chronologically from 1944 to 1989. In French. PZ

General Topology, T(16-18: 3). *An Introduction to Topology and Homotopy.* Allan J. Sieradski. PWS-Kent, 1992, xiii + 479 pp. [ISBN: 0-534-92960-5] First half appropriate for a one-semester introductory topology course. Introduces concept of topological space via metric spaces. Second half introduces groups, categories, and CW-complexes, and covers the fundamental group, homotopies, covering spaces, fibrations, and the classification of 2-surfaces. Written in a somewhat sophis-

ticated style. Full of nice diagrams; very good exercise sets. MC

Algebraic Topology, P. *Topological Classification of Integrable Systems.* Ed: A.T. Fomenko. Adv. in Soviet Math., V. 6. AMS, 1991, vi + 345 pp, $180. [ISBN: 08218-4105-X] Eleven papers on such topics as topological invariants of integral systems, Morse type theory for integrals of generic systems, characterization of topological equivalence of integral systems, computation of invariants, and computer studies. Note price. MPR

Systems Theory, P. *Self-Organization, Emerging Properties, and Learning.* Ed: Agnessa Babloyantz. NATO ASI Ser. B, V. 260. Plenum Pr, 1991, xix + 300 pp, $85. [ISBN: 0-306-43930-1] Papers from a 1990 NATO workshop in Texas on self-organizing systems—systems of units which organize themselves into structures or actions to produce properties not possessed by the individual units. Three major themes: self-organization and dynamics of networks of interacting elements; experimental and theoretical modelling of networks of neurons; role of dynamical attractors in cognitive learning. RM

Stochastic Processes, P. *Excursions of Markov Processes.* Robert M. Blumenthal. Prob. & Its Applic. Birkhäuser, 1992, xi + 275 pp, $64.50. [ISBN: 0-8176-3575-0] Given a set in the state space of a Markov process, an excursion is the portion of the path of the process between successive meetings with the set. Appropriate measures on the portions allow them to be treated in the same way that Levy measure describes the jumps in a process with independent increments. Presumes strong background in measure-theoretic stochastic processes. A useful reference for the specialist. TAV

Stochastic Processes, P, L. *Random Walks, Brownian Motion, and Interacting Particle Systems: A Festschrift in Honor of Frank Spitzer.* Eds: Rick Durrett, Harry Kesten. Progress in Prob., V. 28. Birkhäuser, 1991, xii + 455 pp, $68. [ISBN: 0-8176-3509-2] Twenty papers from a conference honoring one of the experts on random walks, plus five reprints of seminal papers by Spitzer. BC

Elementary Statistics, S(14-15). *Dictionary/Outline of Basic Statistics.* John E. Freund, Frank J. Williams. Dover, 1991, ix + 195 pp, $6.95 (P). [ISBN: 0-486-66796-0] Unabridged, slightly corrected (but not updated) republication of a 1966 McGraw-Hill work (TR, February 1968). Divided into two parts: a dictionary of statistical terms, and an outline of statistical formulas. RSK

Statistical Methods, S(17), C, P. *Meta-Analysis by the Confidence Profile Method: The Statistical Synthesis of Evidence.* David M. Eddy, Vic Hasselblad, Ross Shachter. Stat. Model. & Dec. Sci. IBM PC Software. Academic Pr, 1992, vii + 428 pp, $59.95. [ISBN: 0-12-230620-1] Describes new set of meta-analytic methods known as Confidence Profile Method (CPM), a set of quantitative techniques for interpreting results of individual experiments, exploring effects of biases, adjusting experiments for factors that affect comparability, and combining evidence from multiple sources. Includes problems which illustrate methodological issues, formulation of analytical problems, the mathematics of the CPM, solutions of specific problems using CPM, and issues that arise in applications. Most examples are drawn from medicine. Software included with book. KB

Statistical Methods, T(17: 1, 2). *Analysis of Variance in Experimental Design.* Harold R. Lindman. Texts in Stat. Springer-Verlag, 1992, ix + 531 pp, $49.95. [ISBN: 0-387-97571-3] Non-theoretical presentation of the usual designs, including thorough discussions of assumptions, expected mean squares, comparison procedures, robustness, and variance estimates. Also includes chapters on designs with quantitative factors, multivariate analysis of variance, analysis of covariance, and the general linear model (with proofs in an appendix). Other appendices describe SAS and SPSS procedures for the analysis of variance. RSK

Statistical Methods, T(15-18: 1, 2). *Introduction to Reliability Analysis: Probability Models and Statistical Methods.* Shelemyahu Zacks. Texts in Stat. Springer-Verlag, 1992, xiii + 212 pp, $39.50. [ISBN: 0-387-97718-X] Outgrowth of a workshop on statistical methods of reliability analysis for engineers. Stresses methodology and illustrative applications, not theoretical development. Topics include system effectiveness; reliability of composite and repairable systems; graphical analysis of life data; estimation of life distributions; Bayesian reliability estimation; testing and acceptance procedures. Exposition is very readable

with many examples and exercises. MK

Statistical Methods, S(18), P. *Statistical Inference: Theory and Practice.* Eds: Tadeusz Bromek, Elżbieta Pleszczyńska. Theory & Decision Lib.: Ser. B, V. 17. Kluwer Academic, 1991, ix + 311 pp, $165. [ISBN: 0-7923-0718-6] Main feature is the presentation of examples of statistical inference applied to practical problems (e.g., paternity proving) where the intricacies of the problems require more than the ready-made theoretical schemes described earlier can handle. Purpose is to "convince the reader that it is indeed necessary to treat each practical problem individually, and to maintain a constant cooperation between statisticians and specialists in the field in question." Note price! RSK

Elementary Computer Science, P. *The New Hacker's Dictionary.* Ed: Eric S. Raymond. MIT Pr, 1991, xx + 433 pp, $10.95 (P). [ISBN: 0-262-68069-6; 0-262-18145-2] A very humorous book which defines a number of slang terms drawn from computer science, engineering, and programming. Includes many terms used by hackers to describe the programming process. Not a technical book, but an enjoyable and leisurely read for anyone in computer science. Highly recommended as bedtime reading. GMS

Computer Systems, P, L. *OSI: A Model for Computer Communications Standards.* Uyless Black. Prentice Hall, 1991, xvi + 528 pp. [ISBN: 0-13-637133-7] An examination and explanation of the Open Systems Interconnection Model for data communication. Contains a discussion of the many layers of this model, and a description of many, many X.abc standards. JAS

Theory of Computation, P. *Complexity Theory of Real Functions.* Ker-I Ko. Prog. in Theoret. Comput. Sci. Birkhäuser, 1991, viii + 309 pp, $49.50. [ISBN: 0-8176-3586-6] Includes a review of the fundamental notions and detailed discussions in the computational complexity of real functions in the model of discrete complexity theory. Applied NP-completeness theory to prove lower bounds for basic numerical operations, such as maximization and integration. DH

Artificial Intelligence, P. *Neural Networks for Perception, Volumes 1 & 2.* Ed: Harry Wechsler. Academic Pr, 1992. *Volume 1: Human and Machine Perception,* xxi + 520 pp, $59.95 [ISBN: 0-12-741251-4]; *Volume 2: Computation, Learning, and Architectures,* xix + 363 pp, $49.95. [ISBN: 0-12-741252-2] Collection of papers on relationships between human perception and recent research in neural networks. Goal is better understanding of human perception and building systems that model perception to perform useful tasks. RM

Applications (Economics), P. *Nonlinear Dynamics, Chaos, and Instability: Statistical Theory and Economic Evidence.* William A. Brock, David A. Hsieh, Blake LeBaron. MIT Pr, 1991, xv + 328 pp, $32.50. [ISBN: 0-262-02329-6] Exposition, with proofs, of the Brock–Dechert–Scheinkman test, a statistical method for identifying non-linearities in seemingly random time series of economic data. SK

Applications (Physical Science), P. *Time's Arrow: The Origins of Thermodynamic Behavior.* Michael C. Mackey. Springer-Verlag, 1992, xv + 175 pp, $49. [ISBN: 0 387-97702-3] The author tackles no small problem, but makes a frontal assault on one of the perplexing questions of physical science: how do we reconcile the increase of entropy with the known reversability of all the laws of microscopic physics? He identifies as the core of his work the proof that for there to be a global evolution of the entropy to its maximal value of zero (the strong form of the second law of thermodynamics), it is necessary and sufficient that the system have a property known as exactness. Alas, he then explains why this raises as many questions as it answers. AWR

Applications (Physics), S(18), P. *The Schrödinger Equation.* F.A. Berezin, M.A. Shubin. Math. & Its Applic., V. 66. Kluwer Academic, 1991, xviii + 555 pp, $249. [ISBN: 0-7923-1218-X] A mathematical approach to Schrödinger's equation. Very thorough, very rigorous, but nothing new. Note the price. MPR

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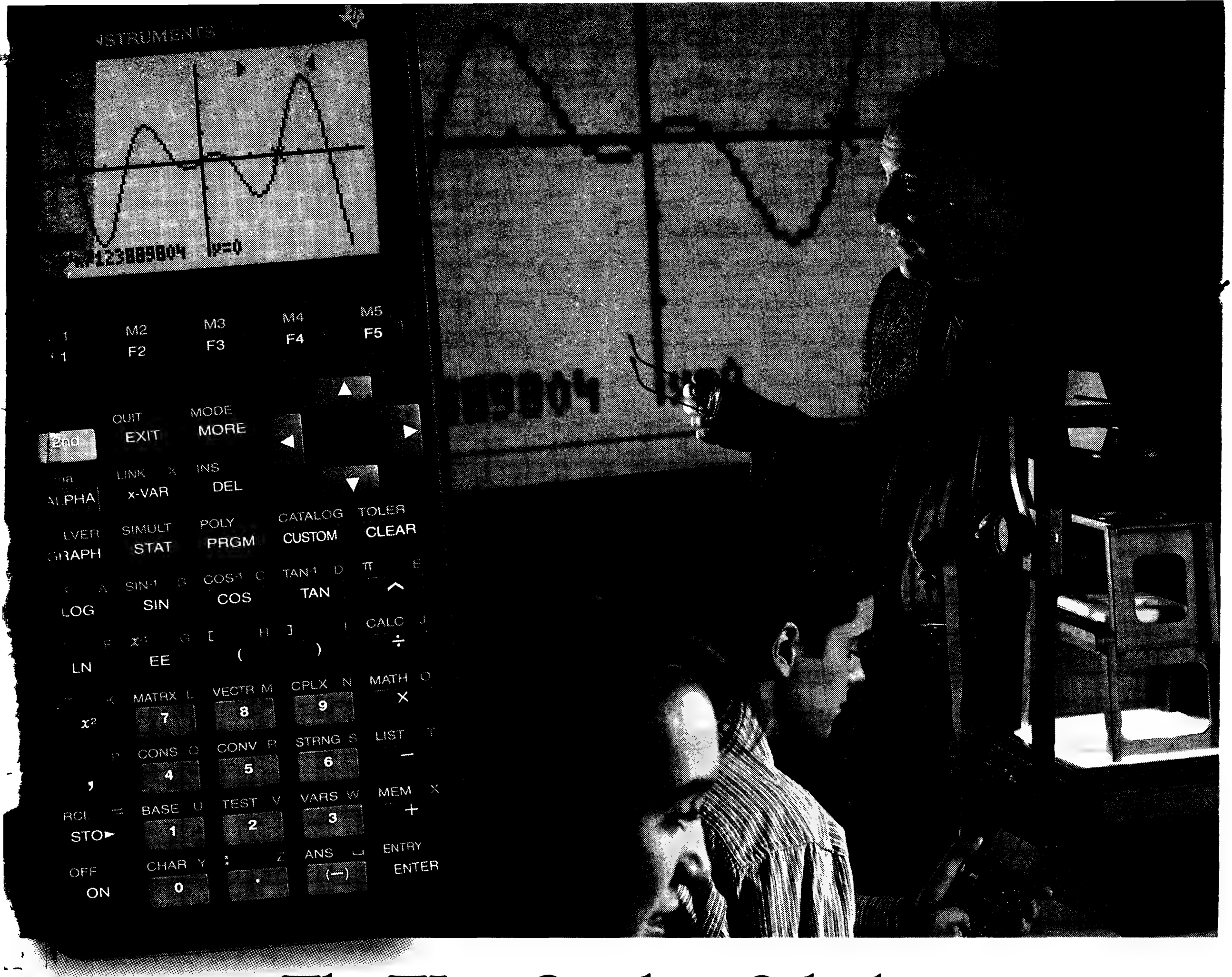

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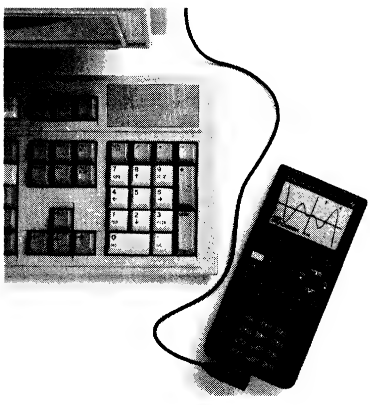

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Technology in Mathematics Teaching (TMT '93)
A Bridge Between Teaching and Learning
Friday to Monday 17-20 September 1993
The University of Birmingham, England

This is the European edition of the sixth annual international conference in the series *Technology in Collegiate Mathematics* and the first time it has come to Europe.

The structure of the programme provides for those involved in the teaching of mathematics at every level primary through university. There will be a diversity of themes, both educational and technological, and opportunities for talks, workshops, research reports, symposia, and discussion groups.

It is being hosted by the School of Education in conjunction with Computers in Teaching Initiative Centre for Mathematics and Statistics (CTICMS), and will take place at the University of Birmingham, UK. Colleagues from the United States are most welcome to participate.

Other highlights of the conference:
- There will be a special theme workshop on Technology in Undergraduate University Mathematics running throughout.
- There will be a full social programme during the Conference; accompanying non-participants are welcome.

The three strands running throughout TMT '93 are:
Strand 1: The mathematical content of teaching and learning environments
Strand 2: Technology as a resource for the teacher
Strand 3: Hands-on interaction between learners and technology

There will be invited lectures (45 minutes), reports (20 minutes), "hands-on" computer and graphing calculator workshops (90 minutes), and poster sessions.

Colleagues desiring to present a lecture, report, poster, or conduct a "hands-on" workshop should contact Bert Waits and Frank Demana, TMT '93, Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, OH 43210, no later than 22 January 1993 for additional details. E-mail: waitsb@mps.ohio-state.edu

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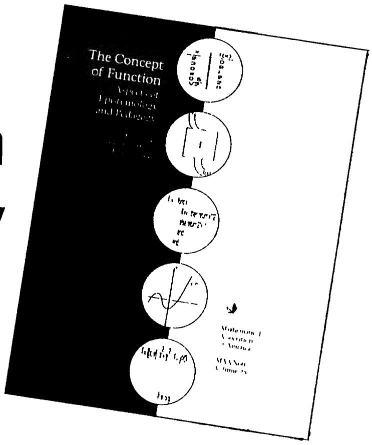
The Concept of Function Aspects of Epistemology and Pedagogy



Guershon Harel and Ed Dubinsky, Editors

The contributors of this volume probe the idea of what it means to learn the concept of function and how instruction, based on research, could assist teachers in finding ways of helping their students understand this all-important mathematical concept.

The concept of function is one that will appear again and again in a student's mathematics training. Arithmetic in the early grades, algebra in junior high school, and transformational geometry in high school are all largely based on the idea of function. Moreover, people involved in calculus reform know that understanding the idea of function is an indispensable part of the background students need to understand calculus. As mathematical education is being renewed and reformed throughout the world, this movement requires that we learn more about the concept of function both from epistemological and pedagogical points of view.

There are several major themes that emerge in the pages of this volume. They are theoretical perspectives of development of the function concept, theory-based teaching experiments, conceptions held by students and teachers, and the use of pedagogical software. The volume begins with a summary and overview of the subject and is followed by a brief glossary of terms.

The development of the papers presented in the volume began with a conference held in West Lafayette, Indiana in October 1990 with the support of Purdue University and the Exxon Foundation. This volume is, however, much more than just a conference proceedings. It is a truly cooperative writing effort by a group of dedicated researchers and educators.

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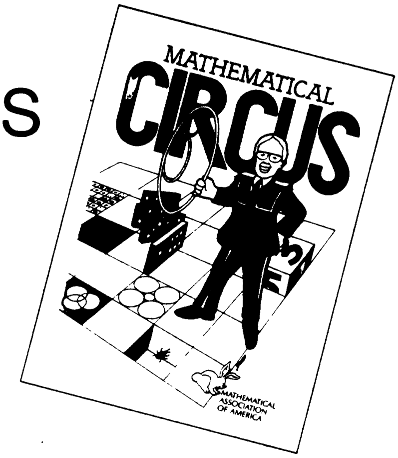

Drawn from Martin Gardner's "Mathematical Games" column in SCIENTIFIC AMERICAN.

A circus suggests fun and enjoyment and there is plenty of both to be found here. The book should certainly be in the school library. It will also be a valuable resource for the teacher.

The Mathematical Gazette

His puzzles exercise the mind and not only fascinate puzzle fanatics but are also capable of amusing and intriguing serious professional mathematicians, scientists, and astronomers.

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